Research Article

Characteristic-based Level Set Method for Motion by Mean Curvature Analysis

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Abstract

This paper presents a characteristic-based scheme for solving the level set equation in arbitrary two-dimensional domain. The characteristic-based approach is used to derive a level set equation in two dimensions for solving the evolving interface problems with zero level set along their interfaces. An explicit finite volume method is employed to discretize the characteristic level set equation. The scheme is used to study motion by mean curvature where the interface moves in the normal direction with the velocity proportional to its curvature. Accuracy and robustness of the proposed method are evaluated via test cases with prescribed velocity fields and its curvature on structured triangular grids. The predicted results are compared with those in literatures.

Keywords: Characteristic Level Set Equation; Explicit Scheme; Finite Volume Method; Mean Curvature Analysis.

1. Introduction

A difficulty in analyzing the multiphase or multifluid flows is that the internal interface position and shape must be determined as a part of the solution of the field equations. Flows near the interfaces often have strong vortical components because of the related sharp gradients in the fluid properties. To predict the flow phenomena accurately, we precisely tracked the interface in both time and space.The interface normal vectors and curvatures also needed to be approximated through a highly accurate numerical scheme in order to model the interface accurately. Numerically, there are two sources of error in the calculation of the surface tension: 1) the discretization of the surface tension, and 2) the approximation of the interface curvature [1].

Many researchers have proposed various techniques for capturing or tracking the interface topology. Typical methods are the front tracking method [2,3], the level set method [4,5], and the volume-of-fluid method [6,7]. In the past years, the level set method has gained popularity in describing the location and motion of advancing interfaces due to its simplicity. The basic idea is to determine the position of the front interface by a distinct iso-value of a scalar function, namely the so called level set function. The applications of the level set method for moving boundaries and interfaces problems exist in many fields such as the crystal growth, multiphase flows, multifluid flows, front propagations, fluid-structural interactions, curvature-driven flow, etc. In general, the interface is represented by a zero level contour of a signed distance function. It is desirable to maintain the level set function, $\phi = \phi(\mathbf{x}, t)$ as a signed distance

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function during the interface evolution for accurate interface capturing. The calculation of the interface curvature from the level set method is straightforward but not necessarily accurate [5].

The objective of this work is to develop a robust explicit finite volume element method for solving the characteristic level set equation in two-dimensional triangular grids domain. This paper presents an idea of using the finite volume and finite element schemes to discretize the characteristic level set equation for solving curvature-driven flow problems [8,9] by adding the curvature driven motion term into the level set equation. In this paper, the concept of characteristic-based scheme [10],for approximating the Lagrangian derivatives in time, is used to derive the level set equation. An explicit finite volume method is employed to develop the discretized equations for the spatial domain. The approximation of the gradients at cell faces for determining curvature term is calculated by means of the conventional weighted-residual finite element technique [11]. Robustness and efficiency of the proposed method are examined by analyzing two examples, and comparing with those reported by other researchers.

2. Characteristic Level Set for Motion by Mean Curvature Formulation

Given a smooth front $\partial\Omega$ in Euclidean space R^N,the mean curvature κ_x of $\partial\Omega$ at the point x is the average of the principal curvatures at x. The problem of motion by mean curvature of fronts of R^N can be stated as follows: The evolution of a given initial front $\partial\Omega_0$ such that at every time t, the boundary point of the front moves with a normal speed equal to the mean curvature of the boundary [12]. The level set method is an implicit method for determining the evolution of an interface of fluids. The level set function ϕ (or distance function) such that $|\phi(\mathbf{x},t)| = \min(|\mathbf{x}-\mathbf{x}_I|)$ for all $\mathbf{x}_I \in \partial\Omega_I$ is a passive scalar function that is advected by the flow with the local flow velocity. This paper considers the motion by mean curvature where the interface moves in the normal direction with a velocity proportional to its curvature, i.e., $\mathbf{V} = -b\kappa|\nabla\phi|$ where b is a constant,and the curvature $\kappa = \nabla\cdot(\nabla\phi/|\nabla\phi|)$. For the two-dimensional domain, the level set function can be written as,

$$\frac{\partial\phi}{\partial t} + \mathbf{V}\cdot\nabla\phi - b\kappa|\nabla\phi| = 0 \qquad (1)$$

where $\phi = \phi(\boldsymbol{x},t)$ defines the implicit interface by its zero level set, and is chosen to be positive outside $\Omega\,(\Omega^+)$, negative inside $\Omega\,(\Omega^-)$, and zero on the interface ($\partial\Omega_I$), and $t \in (0,T)$ for $T < \infty$.The velocity field $\mathbf{V} = \mathbf{V}(\mathbf{x}, \nabla\phi(\mathbf{x},t), t)$ can be defined in several ways depending on the applications [4,5]. The initial condition is defined for $\mathbf{x} \in \Omega$ with $\Omega \subset R^2$ and $\Omega = \Omega^+ \cup \Omega^- \cup \partial\Omega_I$ by $\phi(\mathbf{x},0) = \phi_0(\mathbf{x})$.

By following the idea described in [10], Eq. (1) is semi-discretized along the characteristic line. By utilizing the Crank–Nicolson scheme, it can be written in the form,

$$\frac{1}{\Delta t}\left(\phi^{n+1}\Big|_{\mathbf{x}} - \phi^{n}\Big|_{\mathbf{x}-\Delta\mathbf{x}}\right) = \frac{1}{2}\left[(b\kappa|\nabla\phi|)^{n}_{\mathbf{x}-\Delta\mathbf{x}} + (b\kappa|\nabla\phi|)^{n+1}_{\mathbf{x}}\right]. \qquad (2)$$

Then, the local Taylor series expansion in space is applied to the second term on the left-hand side and to the right-hand side terms. The incremental distance $\Delta\mathbf{x}$ along the characteristic path is then approximated by $\Delta\mathbf{x} = \mathbf{V}^{n+1/2}\Delta t$, where $\mathbf{V}^{n+1/2}$ is the average velocity along the characteristic at time $t = n + 1/2$. Finally, Eq. (2) can be written in a fully explicit form as,

$$\frac{1}{\Delta t}(\phi^{n+1} - \phi^{n}) = \Big[-\mathbf{V}\cdot\nabla\phi + b\kappa|\nabla\phi| + \frac{\Delta t}{2}(\nabla\phi\mathbf{V}\cdot\nabla\mathbf{V} + \mathbf{V}\mathbf{V}\cdot\nabla^{2}\phi)\Big]^{n} . \quad (3)$$

By utilizing some vector identities, Eq. (3) can be written preferably in the conservation form for applying the finite volume method as follows,

$$\phi^{n+1} - \phi^{n} = -\Delta t\big[\nabla\cdot(\mathbf{V}\phi) + \phi\nabla\cdot\mathbf{V} - b\kappa|\nabla\phi|\big]^{n} + \frac{(\Delta t)^{2}}{2}\big[\nabla\cdot(\mathbf{V}\nabla\phi\cdot\mathbf{V}) - \mathbf{V}\cdot\nabla\phi\nabla\cdot\mathbf{V}\big]^{n} . \quad (4)$$

3. Finite Volume Scheme

The computational domain is first discretized into a collection of non-overlapping convex polygon control volumes $\Omega_i \in \Omega$ and $i = 1,\ldots,N$, that completely cover the domain such that $\Omega = \cup_{i=1}^{N}\Omega_i$, $\Omega_i \neq 0$, and $\Omega_i \cap \Omega_j = 0$ if $i \neq j$. Eq. (4) is integrated over the control volume Ω_i to obtain

$$\int_{\Omega_i}\left(\phi^{n+1} - \phi^{n}\right)d\mathbf{x} = -\Delta t\int_{\Omega_i}\big[\nabla\cdot(\mathbf{V}\phi) + \phi\nabla\cdot\mathbf{V} - b\kappa|\nabla\phi|\big]^{n}d\mathbf{x} + \frac{(\Delta t)^{2}}{2}\int_{\Omega_i}\big[\nabla\cdot(\mathbf{V}\nabla\phi\cdot\mathbf{V}) - \mathbf{V}\cdot\nabla\phi\nabla\cdot\mathbf{V}\big]^{n}d\mathbf{x} . \quad (5)$$

The divergence theorem is applied to some spatial terms on the right-hand side to yield,

$$\int_{\Omega_i}\phi^{n+1}d\mathbf{x} = \int_{\Omega_i}\phi^{n}d\mathbf{x} - \Delta t\int_{\partial\Omega_i}\hat{\mathbf{n}}_i\cdot\left[\mathbf{V}\phi - \frac{\Delta t}{2}\mathbf{V}\nabla\phi\cdot\mathbf{V}\right]^{n}d\sigma + \Delta t\int_{\partial\Omega_i}\hat{\mathbf{n}}_i\cdot\big[b|\nabla\phi|\nabla\phi/|\nabla\phi|\big]^{n}d\sigma + \Delta t\int_{\Omega_i}\left[\phi\nabla\cdot\mathbf{V} - \frac{\Delta t}{2}\mathbf{V}\cdot\nabla\phi\nabla\cdot\mathbf{V}\right]^{n}d\mathbf{x} . \quad (6)$$

By using the approximation to the cell average of ϕ over Ω_i at time t^n and t^{n+1} [13] for any control volume, we approximated the flux integral over $\partial\Omega_i$ appearing on the right-hand side of Eq. (7) by the summation of fluxes passing through all adjacent cell faces. Finally, a fully explicit formulation for solving a characteristic level set equation is obtained in the form,

$$\phi_i^{n+1} = \phi_i^{n} - \frac{\Delta t}{|\Omega_i|}\sum_{j=1}^{3}|\Gamma_{ij}|\hat{\mathbf{n}}_{ij}\cdot\mathbf{V}_{ij}^{n}\left[\phi_{ij}^{n} - \frac{\Delta t}{2}\left(\mathbf{V}_i^{n}\cdot\nabla\phi_i^{n}\right)\right] + \frac{\Delta t}{|\Omega_i|}b|\nabla\phi_i^{n}|\sum_{j=1}^{3}|\Gamma_{ij}|\hat{\mathbf{n}}_{ij}\cdot\nabla\phi_{ij}^{n}/|\nabla\phi_{ij}^{n}| + \frac{\Delta t}{|\Omega_i|}\left(\phi_i^{n} - \frac{\Delta t}{2}\left(\mathbf{V}_i^{n}\cdot\nabla\phi_i^{n}\right)\right)\sum_{j=1}^{3}|\Gamma_{ij}|\hat{\mathbf{n}}_{ij}\cdot\mathbf{V}_{ij}^{n} . \quad (7)$$

The level set function, ϕ_{ij}^{n}, at cell face at the time step t^n, is approximated by applying the Taylor series expansion in space such that

$\phi_{ij}^n = \phi_i^n + (\mathbf{x}_{ij} - \mathbf{x}_i) \cdot \nabla \phi_i^n$ where $\mathbf{x}_i$ and $\mathbf{x}_{ij}$ are the cell centroid and face centroid locations, respectively. For the opposite direction of velocity, the values of ϕ_{ij}^n may be similarly computed but by using the values from the neighboring control volumes according to the upwinding direction, such that

$$\phi_{ij}^n = \phi_j^n + (\mathbf{x}_{ij} - \mathbf{x}_j) \cdot \nabla \phi_j^n .$$

The boundary conditions for Eq.(7) can be given by

$$\begin{aligned} \phi &= g_D \quad \text{on} \quad \partial\Omega_{\mathrm{D}} \\ \kappa \frac{\partial \phi}{\partial n} &= g_N \quad \text{on} \quad \partial\Omega_{\mathrm{N}} \end{aligned} \tag{8}$$

where $\partial\Omega = \partial\Omega_{\mathrm{D}} \cup \partial\Omega_{\mathrm{N}}$.

A numerical simulation can take into account only a part of the truncation of the domain and, thus, can lead to artificial boundaries. The ghost cells concept is implemented in this paper. The scalar quantity on the Dirichlet boundary condition is given by the user, and the linear extrapolation is used to determine the scalar quantity on the Neumann boundary condition.

The CFL-like stability criterion must be fulfilled in order to ensure the stability of an explicit scheme on a triangular mesh. The permissible time step within each cell is determined from

$$\Delta t = C \min_i \left(\frac{|\Omega_i|}{\max_{j=1,2,3} |\vec{v}_{n,ij}|}, \frac{|\Gamma_i^c|^2}{2\kappa} \right) \tag{9}$$

where $\vec{v}_{n,ij}$ is the scaled normal velocity at Γ_{ij}, Γ_i^c is the characteristic length of cell i, and $0 < C \le 1$. In this paper, all examples are performed with the value of $C = 0.8$.

4. One-Dimensional Numerical Analysis

In this section, a numerical analysis of the one-dimensional homogeneous convection-diffusion equation is presented. For simplicity, the order of accuracy and stability of the explicit numerical scheme given by Eq.(10) will be analyzed on a uniform one-dimensional grid cell, $|\Omega_i| = \Delta x$ .Theone-dimensional homogeneous convection-diffusion equation is

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left(a\phi - b\frac{\partial \phi}{\partial x} \right) = 0 \tag{10}$$

where a is a given velocity. The numerical equation for the ith cell, $|\Omega_i| \in (x_{i-1/2}, x_{i+1/2})$, may be written as

$$\begin{aligned} \phi_i^{n+1} = \phi_i^n &- \frac{a\Delta t}{\Delta x}\left(\phi_{i+1/2}^{n+1/2} - \phi_{i-1/2}^{n+1/2} \right) + \\ &\frac{b\Delta t}{\Delta x}\left(\left.\frac{\partial \phi}{\partial x}\right|_{i+1/2}^n - \left.\frac{\partial \phi}{\partial x}\right|_{i-1/2}^n \right) \end{aligned} \tag{11}$$

The gradient quantities at the cell faces i-1/2 and i+1/2 are calculated by using the one-dimensional linear interpolation function

$$\begin{aligned} \left.\frac{\partial \phi}{\partial x}\right|_{i-1/2}^n &= \frac{1}{\Delta x}\left(\phi_i^n - \phi_{i-1}^n \right) \\ \left.\frac{\partial \phi}{\partial x}\right|_{i+1/2}^n &= \frac{1}{\Delta x}\left(\phi_{i+1}^n - \phi_i^n \right) \end{aligned} . \tag{12}$$

Similarly, the gradient quantity at cell-centered of $|\Omega_i|$ is

$$\left.\frac{\partial \phi}{\partial x}\right|_i^n = \frac{1}{2\Delta x}\left(\phi_{i+1}^n - \phi_{i-1}^n\right). \qquad (13)$$

By substituting these expressions into the right-hand side of Eq.(11), we obtain

$$\begin{aligned}\phi_i^{n+1} = \phi_i^n - \frac{R}{4}\left(\phi_{i+1}^n + 3\phi_i^n - 5\phi_{i-1}^n + \phi_{i-2}^n\right) \\ + \frac{R^2}{4}\left(\phi_{i+1}^n + \phi_i^n - \phi_{i-1}^n + \phi_{i-2}^n\right) \\ + r\left(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n\right)\end{aligned} \qquad (14)$$

where $R = \frac{a\Delta t}{\Delta x}$ and $r = \frac{b\Delta t}{(\Delta x)^2}$, are the cell Courant number and the cell diffusion parameter, respectively. The truncation error analysis by using the Taylor series expansion on Eq.(14) at $(x_n, t^n) = (i, n)$ shows that the accuracy is of the order $O(\Delta t^2, \Delta t \Delta x, \Delta x^2)$.

The discrete Fourier transform is applied to Eq.(14), term by term, in order to analyze the stability of the numerical scheme. Then the amplification factor $G(\theta)$ is calculated as

$$\begin{aligned}G(\theta) = \frac{\hat{\phi}^{n+1}}{\hat{\phi}^n} = \Big[1 - R(\cos(\theta) - 1)^2 - \\ \frac{R^2}{2}\sin^2(\theta) + 2r(\cos(\theta) - 1)\Big] - \\ I(R\sin(\theta))\left[3 - \cos(\theta) - \frac{R}{2}(1 - \cos(\theta))\right]\end{aligned} \qquad (15)$$

where θ is a phase angle. For a stable solution, the modulus of $G(\theta)$ must be bounded for all values of θ ($|G(\theta)| \le 1$). The critical points at $\theta = 0, \pm\pi$ are obtained by differentiating Eq.(15) with respect to θ and setting the derivative to be zero. The values of $|G(\theta)|$ at these points are

$$|G(0)| = 1 \text{ and} \\ |G(\pm\pi)| = |4(R + r) - 1| \qquad (16)$$

For $|G(\pm\pi)|$ to be bounded by one, the numerical scheme is conditionally stable when $0 \le R + r \le 0.5$. Such condition implies that $|G(\theta)| \le 1 + C\Delta t$, where $C > 0$. The numerical scheme satisfies the von Neumann condition. Thus the scheme is stable. However, it should be noted that the stability condition is restrictive. The CFL-like condition as described by Eq.(9), which is obtained by applying the discrete Fourier transform to the convection and diffusion parts separately, is stable with a large time step. Such stability condition has been tested by using many numerical examples as will be presented in the following section.

5. Results

To evaluate the robustness and accuracy of the proposed characteristic level set method, two examples are examined. These examples are used to study motion by mean curvature where the interface moves in the normal direction with a velocity proportional to its curvature. All examples presented in this section are tested using uniform triangular grids. These examples are: (1) the motion by mean curvature of a star shape, and (2) the motion in the normal direction involving mean curvature of a quatrefoil problems.

5.1 Motion by Mean Curvature of a Star Shape

The first benchmark problem is used to examine the capability of the proposed method to calculate the curvature of a star shape [9], where the motion of simply-closed curve collapsing with the speed proportional to the local curvature in a square domain of

$\Omega = (-1,-1)\times(1,1)$. This problem is tested using the same domain size as described in the previous problems. The initial condition of the star shape is given by,

$$\phi(\mathbf{x}) = r - 0.2[\cos(7\tan(y/x)) + 2.5] \qquad (17)$$

where $r = \sqrt{x^2 + y^2}$ and the coefficient of curvature term, b is set to 0.02. The challenging task of this test case is to verify the Grayson's theorem [14], such that all simply closed curves eventually collapse to a round point. Due to the effect of the curvature, the initial star shape will deform to a round-shape. Numerical solutions are presented by using two grids S1 ($h = 1/64$) and S2 ($h = 1/128$). The initial form of the star shape is shown in Fig. 1. Figures 2(a)-(d) show the zero level contour plots obtained from grid S1 at $t = 0.25, 0.5, 1.0$, respectively. Similarly, figures 3(a)-(d) show the solutions obtained from grid S2. These solutions are similar to those reported in [9]. The numerical solution at time $t = 1.0$ also confirms that the Grayson's theorem is numerically verified.

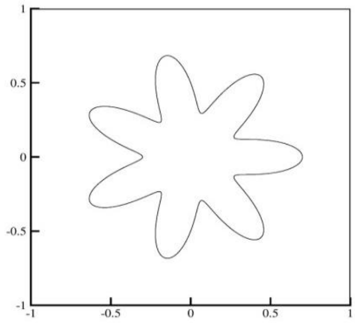
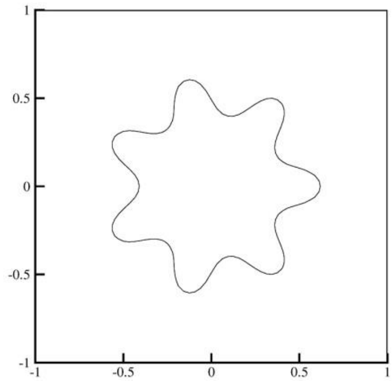
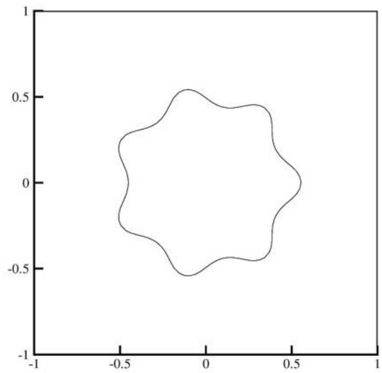


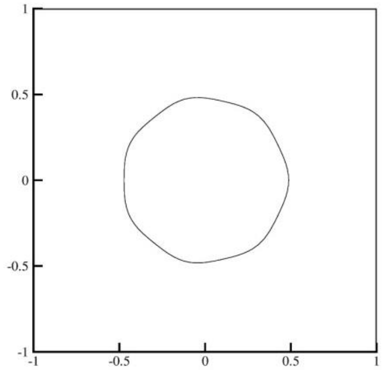
Fig.1. Plots of initial condition of problem 5.1.



(a) $t = 0.25$

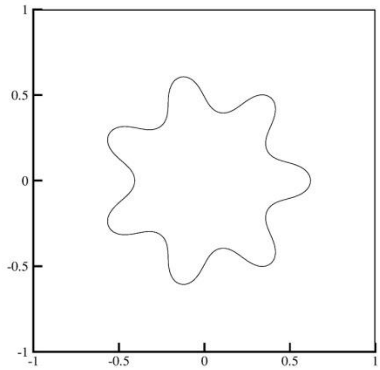


(b) $t = 0.5$

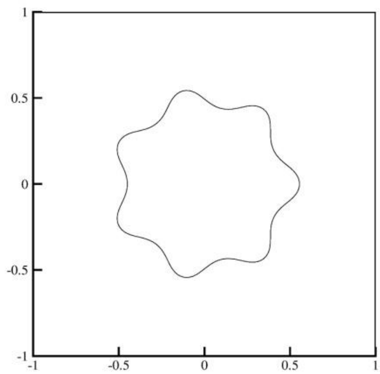


(c) $t = 1.0$

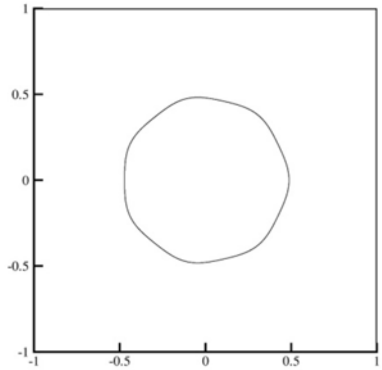
Fig.2. Numerical solutions of grid S1 at three times of problem 5.1.



(a) $t = 0.25$



(b) $t = 0.5$



(c) $t = 1.0$

Fig.3. Numerical solutions of grid S2 at three times of problem 5.1.

5.2 Motion in the normal direction involving mean curvature of a quatrefoil

The next problem is used to examine the capability of the proposed method for calculating the interface normal vector and surface curvature of a quatrefoil in a square domain of $\Omega = (-1,-1) \times (1,1)$. This problem is also tested using the same domain size as described in the previous problems. The initial condition of the quatrefoil is given by [8],

$$\phi(\mathbf{x}) = -1 + \frac{r}{0.6 + 0.4\sin(4\tan^{-1}(y/x))}. \qquad (18)$$

The velocity field in the normal direction is,

$$\mathbf{V} = -\frac{\nabla\phi}{|\nabla\phi|} \qquad (19)$$

with the curvature coefficient of 0.0005. At time $t = 0.2$ the quatrefoil is split into disconnected components.

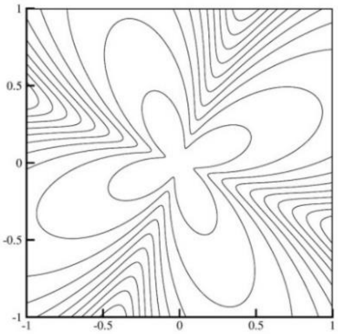
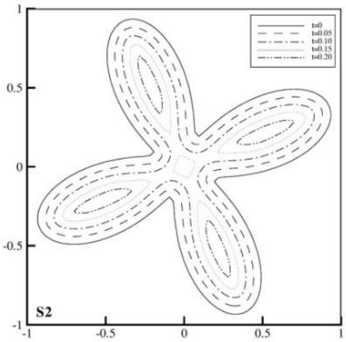
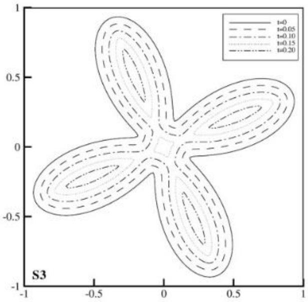


Fig.4. Plots of continuous initial condition for problem 5.2.

The initial form of the quatrefoil is shown in Fig. 4. The topology changes of the quatrefoil at the four time steps of $t = 0.05, 0.1, 0.15,$ and 0.2 computed from using the two grids S1 and S2 are shown in Figs. 5(a)-(d), respectively. The results obtained from these grids are comparable to those presented in [8].



(a) Grid S1



(b) Grid S2

Fig.5. Comparison of numerical solutions ontwo grid sizes of problem 5.2 (continuousinitial condition).

This problem is tested again with the discontinuity initial condition (along the interface of the quatrefoil) of 1 when $\phi > 0$, and 0 otherwise, in order to assess the robustness of the proposed method. The initial form of the binary quatrefoil is shown in Fig. 6. The topology changes of the binary quatrefoil at the four time steps are presented in Figs. 7(a)-(d), respectively. The computed solutions are quite closed to those from the continuous initial condition given by Eq. (9), except that there is one zero level component at the center of the domain. This component, however, does not appear when the calculation starts from the continuous initial condition. The solutions show that the characteristic-based level set method provides high resolution results comparable to other high-resolution schemes [8,9].

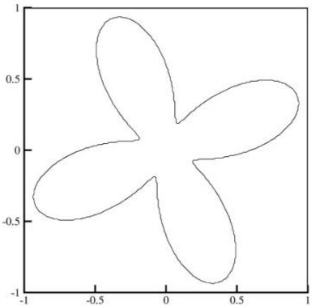
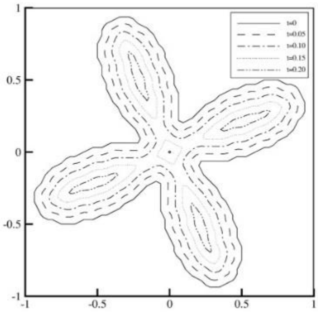
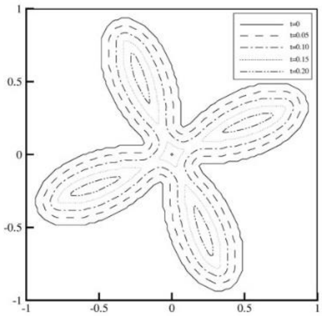


Fig.6. Plots of discontinuous initial conditionof problem 4.2.



(a) Grid S1



(b) Grid S2

Fig. 7. Comparison of numerical solutions on two grid sizes of problem 5.2 (discontinuous initial condition).

6. Conclusion

A fully explicit finite volume method for solving the characteristic-based level set equation in two-dimensional domain is presented. The theoretical formulation of the characteristic level set equation based on the characteristic-based scheme has been explained in details.The finite volume method was applied to derive the discretized equations for the spatial domain. The scheme is used to study motion by mean curvature where the interface moves in the normal direction with a velocity proportional to its curvature. Two numerical examples were used to evaluate the performance and to determine the order of the accuracy of the proposed method. These examples showed that the method provides the second order accurate and converged solution with improved accuracy as the grid is refined.

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