The Binomial Parameter Estimation by Using Weighted Method of Two Bayesian Estimators

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Abstract

This paper proposes a new binomial parameter estimation by using the Weighted method of two Bayesian estimators: Bayesian estimator given Beta(1/2,1/2) prior distribution and Bayesian estimator given Beta(2,2) prior distribution. The used weighted value has the minimum variance of the estimator. In addition, this research compares three parameter estimation methods. These methods are: Bayesian method given Beta(1/2,1/2) prior distribution (B1), Bayesian method given Beta(2,2) prior distribution (B2), and Weighted method of two Bayesian estimators (W). Monte Carlo simulation is used to investigate the behavior of this new binomial parameter estimation based on mean square errors (MSE). Simulation results are as follows: For all sample sizes, the MSE of B1 method is the lowest when $p \le 0.20$ and the MSE of B2 method is the lowest when $0.30 < p < 0.50$. For the sample sizes at least 50, The MSE of W method is the lowest when p is equal to 0.25.

Keywords: Parameter estimation, Binomial distribution, Bayesian estimator

1. Introduction

Population parameter estimation may be conducted either by employing point estimation or interval estimation methods. Point estimation utilizes a single value obtained from samples to provide a population parameter estimation such as the average income estimation for Thais based on the income of sample Thais, or the proportion estimation of Thais using mobile phones based on the proportion of Thai samples using mobile phones. Point estimation may be carried out in many ways, such as by using the ordinary least squares method, the maximum likelihood method, or the Bayesian method.

There exists a conceptual difference between the Bayesian method and the ordinary least squares method and the maximum likelihood method, in that p is generally considered an unknown constant value, while the Bayesian method considers p to be a random variable P. P may be shown in a probability distribution known as a prior distribution, since the distribution is determined prior to the gathering of data. In the course of gathering data, knowledge obtained from the process is used to improve the prior distribution. This improved distribution is known as the posterior distribution [1].

Since Bayesian parameter estimators of binomial distribution vary, depending on the chosen prior distribution function, this researcher has conceptualized a development of binomial parameter estimation by using the Weighted method of two Bayesian estimators. The development of this estimation relies upon the use of combining forecasting [2], employing two Bayesian estimators to come up with a weighted value,which minimizes the estimator's variance.

The objective of this research is to propose a method for binomial parameter estimation by using the Weighted method of two Bayesian estimators. One Bayesian estimator is given a Beta(1/2,1/2) prior distribution and the other Bayesian estimator is given a Beta(2,2) prior distribution. The weighted value used, minimizes the variance of the estimator. This research also compares three binomial distribution parameter estimation methods: the Bayesian method given Beta(1/2,1/2) prior distribution, the Bayesian method given Beta(2,2) prior distribution and the Weighted method of two Bayesian estimators.

2. Methodology

The scope of this research is as follows:

1. Determine the sample sizes used (n) as 10, 20, 30, 50, 100, 200 and 500.

2. Determine altogether 12 parameter values p as 0.01, 0.03, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45 and 0.50.

This research employs Monte Carlo simulation, using program R, version 2.4.0 [3-4]. The micro computer is set to repeat its calculation 10,000 times in each situation. Steps in the research are as follows:

2.1 Data Simulation for the Research

Data is simulated by creating random variable X with binomial distribution parameters n and p, as designated in the research scope. The probability mass function of the binomial distribution may be shown as follows [5].

$$P(X=x) = \binom{n}{x} p^{x}(1-p)^{n-x}$$
$$; x = 0, 1, 2, \ldots, n$$

2.2 Estimation of Parameter p by Using the Three Parameter Estimation Methods

Once the binomial distribution random variable X has been created, the three parameter estimation methods are used to estimate parameter p.

1) The Bayesian Method Given Beta(1/2,1/2) Prior Distribution [6]

The Bayesian estimator for p when given Beta(1/2,1/2) prior distribution is :

$$\hat{p}_1 = \frac{x+0.5}{n+1}$$

and the variance of estimator is :

$$Var(\hat{p}_1) = \frac{(x+0.5)(n-x+0.5)}{(n+1)^2(n+2)}$$

2) The Bayesian Method Given Beta(2,2) Prior Distribution [6]

The Bayesian estimator for p when given Beta(2,2) prior distribution is :

$$\hat{p}_2 = \frac{x+2}{n+4}$$

and the variance of estimator is :

$$Var(\hat{p}_2) = \frac{(x+2)(n-x+2)}{(n+4)^2(n+5)}$$

3) The Weighted Method of Two Bayesian Estimators

The shape of Beta(1/2,1/2) does indicate that the probability that p takes a small value (near zero) or a large value (near one) is greater than a moderate value (near 0.5). Similarly, the shape of Beta(2,2) does indicate that p has greater probability of taking a moderate value (near 0.5) than a small value or a large value. Thus, the prior distribution of Weighted method is the Beta(1/2,1/2) and Beta(2,2) prior distribution. This method was conceptualized by combining the Weighted values where the researcher applied the principle of obtaining a weighted value, which creates minimum variance of the estimator. This approach combines the Bayesian method given Beta(1/2,1/2) prior distribution and the Bayesian method given Beta(2,2) prior distribution as follows:

$$\hat{p}_W = \omega\hat{p}_1 + (1-\omega)\hat{p}_2$$

The estimator variance is :

$$Var(\hat{p}_W) = \omega^2\, Var(\hat{p}_1) + (1-\omega)^2\, Var(\hat{p}_2)$$

The objective of this development is to obtain a weighted value which produces minimum estimator variance. Thus the first derivative is identified with respect to ω and it is set equal to 0.

$$\frac{\partial}{\partial\omega}Var(\hat{p}_W) = \frac{\partial}{\partial\omega}\left[\omega^2 Var(\hat{p}_1) + (1-\omega)^2 Var(\hat{p}_2)\right] = 2\omega Var(\hat{p}_1) - 2(1-\omega)Var(\hat{p}_2) = 0$$

The equation is then solved and the formula for ω is obtained as follows.

$$\hat{\omega} = \frac{Var(\hat{p}_2)}{Var(\hat{p}_1) + Var(\hat{p}_2)}$$

The estimator obtained from the Weighted method of two Bayesian estimators is :

$$\hat{p}_W = \hat{\omega}\hat{p}_1 + (1-\hat{\omega})\hat{p}_2$$

when

$$\hat{p}_1 = \frac{x+0.5}{n+1}$$

$$\hat{p}_2 = \frac{x+2}{n+4}$$

$$Var(\hat{p}_1) = \frac{(x+0.5)(n-x+0.5)}{(n+1)^2(n+2)}$$

$$Var(\hat{p}_2) = \frac{(x+2)(n-x+2)}{(n+4)^2(n+5)}$$

$$\hat{\omega} = \frac{Var(\hat{p}_2)}{Var(\hat{p}_1) + Var(\hat{p}_2)}$$

and the variance of estimator is :

$$Var(\hat{p}_W) = \hat{\omega}^2 Var(\hat{p}_1) + (1-\hat{\omega})^2 Var(\hat{p}_2)$$

2.3 Calculating the Mean Square Error of the Parameter Estimator in Each Method and Comparing.

The estimator of mean square error (MSE) is calculated from:

$$MSE = \frac{\sum_{i=1}^{M}(p-\hat{p}_{ki})^2}{M}$$

when

p	denoted	parameter value
$\hat{p}_{ki}$	denoted	parameter estimator calculated from method k in repetition i
k	denoted	parameter estimation method
M	denoted	the number of repeated times. Here, the repetition is set at 10,000 times

3. Simulation Results

In presenting the findings of this research, for the sake of convenience, the following symbols will be used.

B1	means	the Bayesian method given Beta(1/2,1/2) prior distribution
B2	means	the Bayesian method given Beta(2,2) prior distribution
W	means	the Weighted method of two Bayesian estimators
MSE	means	the mean square error

Details of the research findings are as follows.

When the sample size is equal to 10, 20 and 30:

When the parameter (p) is equal to 0.01, 0.03, 0.05, 0.10, 0.15 and 0.20, the B1 method provides the lowest MSE followed by W and B2, respectively.

When the parameter (p) is equal to 0.25, 0.30, 0.35, 0.40, 0.45 and 0.5, the B2 method provides the lowest MSE followed by W and B1, respectively.

When the sample size is equal to 50, 70, 100, 200 and 500:

When the parameter (p) is equal to 0.01, 0.03, 0.05, 0.10, 0.15 and 0.20, the B1 method provides the lowest MSE followed by W and B2, respectively.

When the parameter (p) is equal to 0.25, the W method provides the lowest MSE followed by B2 and B1, respectively.

When the parameter (p) is equal to 0.30, 0.35, 0.40, 0.45 and 0.50, the B2 method provides the lowest MSE followed by W and B1, respectively.

Graphs depicting the sample size classification of MSE levels appear in Figure 1, while the MSE of parameter estimators appear in Table 1.

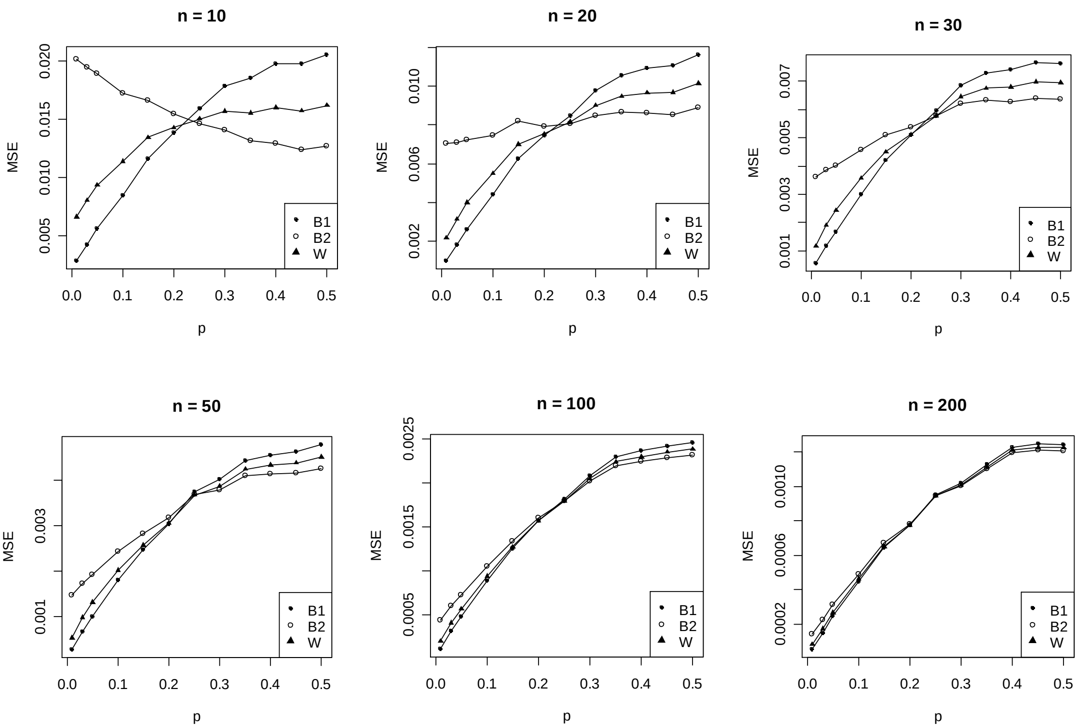

Figure 1 Mean Square Error (MSE), classified by sample sizes

Table 1 The MSE of parameter estimators, classified by sample sizes (n) and parameter values (p)

p	Method	Sample Sizes (n)							
		10	20	30	50	70	100	200	500
0.01	B1	0.002837	0.000985	0.000551	0.000280	0.000193	0.000117	0.000056	0.000021
	B2	0.020152	0.007004	0.003563	0.001481	0.000844	0.000438	0.000142	0.000035
	W	0.006594	0.002135	0.001128	0.000530	0.000343	0.000197	0.000082	0.000026
0.03	B1	0.004248	0.001802	0.001084	0.000638	0.000444	0.000308	0.000145	0.000058
	B2	0.019505	0.007096	0.003723	0.001696	0.001012	0.000598	0.000216	0.000070
	W	0.008076	0.003143	0.001795	0.000956	0.000625	0.000405	0.000169	0.000062
0.05	B1	0.005602	0.002613	0.001739	0.000937	0.000681	0.000499	0.000239	0.000095
	B2	0.019039	0.007269	0.004095	0.001853	0.001181	0.000756	0.000304	0.000105
	W	0.009421	0.004015	0.002498	0.001244	0.000853	0.000589	0.000261	0.000098
0.10	B1	0.008575	0.004419	0.003010	0.001750	0.001265	0.000910	0.000444	0.000181
	B2	0.017442	0.007450	0.004617	0.002367	0.001597	0.001086	0.000486	0.000190
	W	0.011566	0.005512	0.003596	0.001964	0.001377	0.000969	0.000457	0.000184
0.15	B1	0.011603	0.005982	0.004074	0.002471	0.001785	0.001259	0.000625	0.000250
	B2	0.016411	0.007739	0.005022	0.002824	0.001973	0.001346	0.000649	0.000254
	W	0.013323	0.006617	0.004402	0.002578	0.001838	0.001280	0.000631	0.000251
0.20	B1	0.013748	0.007535	0.005163	0.003049	0.002286	0.001546	0.000785	0.000321
	B2	0.015408	0.008139	0.005483	0.003161	0.002369	0.001577	0.000795	0.000322
	W	0.014230	0.007671	0.005216	0.003052	0.002297	0.001545	0.000786	0.000321
0.25	B1	0.016265	0.008969	0.005902	0.003597	0.002549	0.001834	0.000943	0.000379
	B2	0.014800	0.008500	0.005736	0.003547	0.002498	0.001823	0.000938	0.000379
	W	0.015289	0.008604	0.005734	0.003533	0.002501	0.001817	0.000937	0.000378
0.30	B1	0.017859	0.009872	0.006492	0.004058	0.002879	0.002086	0.001052	0.000416
	B2	0.014038	0.008623	0.005954	0.003835	0.002755	0.002023	0.001036	0.000413
	W	0.015725	0.009130	0.006157	0.003918	0.002801	0.002047	0.001042	0.000414
0.35	B1	0.018762	0.010275	0.007108	0.004338	0.003200	0.002227	0.001140	0.000449
	B2	0.013331	0.008449	0.006182	0.003975	0.003007	0.002128	0.001115	0.000446
	W	0.015728	0.009251	0.006591	0.004136	0.003093	0.002172	0.001126	0.000447
0.40	B1	0.019688	0.010891	0.007732	0.004591	0.003369	0.002351	0.001198	0.000478
	B2	0.012876	0.008591	0.006550	0.004149	0.003123	0.002228	0.001166	0.000473
	W	0.015926	0.009634	0.007095	0.004356	0.003239	0.002287	0.001181	0.000476
0.45	B1	0.021131	0.011016	0.007854	0.004688	0.003477	0.002429	0.001244	0.000485
	B2	0.013236	0.008496	0.006558	0.004193	0.003203	0.002294	0.001209	0.000479
	W	0.016788	0.009654	0.007165	0.004429	0.003335	0.002360	0.001226	0.000482
0.50	B1	0.020380	0.011076	0.007755	0.004886	0.003558	0.002471	0.001240	0.000506
	B2	0.012582	0.008480	0.006447	0.004358	0.003276	0.002331	0.001204	0.000500
	W	0.016077	0.009676	0.007063	0.004612	0.003413	0.002399	0.001222	0.000503

4. Conclusions

By comparing the mean square error of parameter estimators obtained from the three estimation methods, the following conclusions can be drawn.

The Bayesian method given Beta (1/2,1/2) prior distribution (B1) provides the lowest MSE when the parameter value (p) is less than or equal to 0.20 for every sample size level (10, 20, 30, 50, 70, 100, 200 and 500).

The Bayesian method given Beta (2,2) prior distribution (B2) provides the lowest MSE when the parameter value (p) is greater than or equal to 0.30 for every sample size level.

The Weighted method of two Bayesian estimators (W) provides the lowest MSE when the parameter value (p) is equal to 0.25 and the sample size is 50 or greater.

5. Recommendations

For all sample sizes, when the parameter value (p) is less than 0.25 or greater than 0.75, the Bayesian method given Beta(1/2,1/2) prior distribution (B1) should be used. However, when the parameter value (p) is equal to 0.25 or 0.75, the Weighted method of two Bayesian estimators (W) should be used. Moreover, in cases when the parameter value (p) is between 0.25 and 0.75, the Bayesian method given Beta(2,2) prior distribution (B2) should be used. This proposal may be summarized as the diagram in Figure 2.

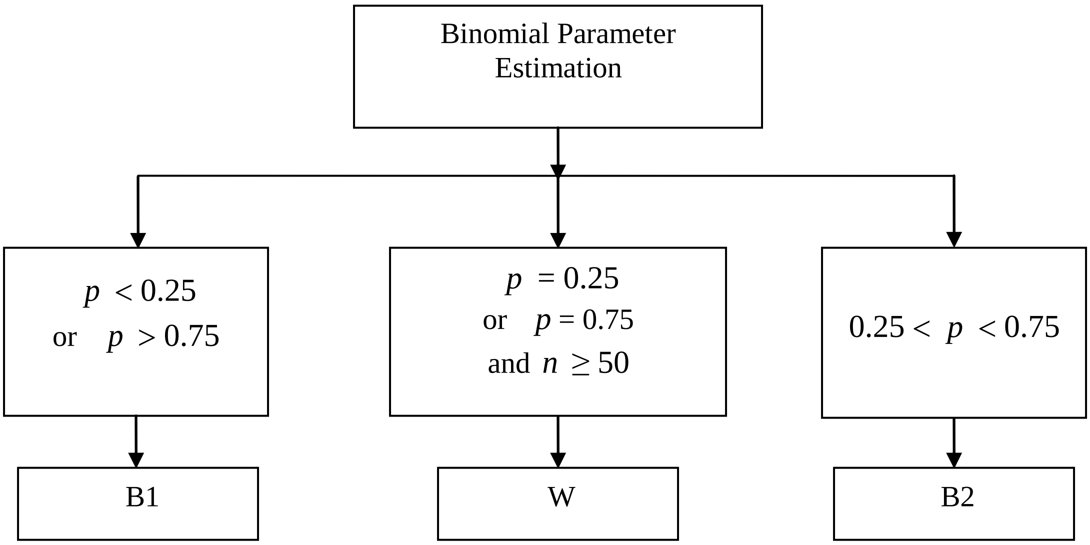

Figure 2 Suitable methods for binomial parameter estimation in different situations

6. Acknowledgements

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