Tail Dependence of Student's t Copula and Double t Copula and Their Effects on Pricing Credit Derivatives

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Abstract

Copula is a marginal-free distribution, used to identify dependence structure of a random vector. In an application that dependence at the extreme is of interest, the Student's t copula and the double t copula can be employed. In two dimensions, the level of extreme dependence can be measured by tail dependence. We define tail dependence at q quantile and compare the tail dependence between the Student's t copula and the double t copula at different quantiles. The effects of tail dependence at q quantile are studied in the context of credit derivative pricing, where dependence at the extreme is a crucial factor.

Keywords: tail dependence, student's t copula, double t copula pricing credit derivatives, q quantile, crucial factor

1. Introduction

Copula is a marginal-free way of describing dependence structure of a random vector. For a continuous random vector, the information that is contained in their copula plus their marginals are equivalent to that of their joint distribution function, as implied in Sklar's theorem (see [4]). In this article, we are interested in the dependence structure at the extreme of the Gaussian copula, the Student's t copula and the double t copula.

Besides a copula, many dependence measures can be devised. The most widely used measurement includes the (linear) correlation, the rank correlation, etc. However, as pointed out in Embrechts, McNeil, and Straumann [2], a correlation and a rank correlation is only one number and does not capture the whole dependence structure. The authors give a lucid example showing that a correlation does not contain information about dependence at the extreme. In fact, two copulas with the same correlation can have a totally different dependent structure at the extreme. When dependence at the extreme is of interest, a measurement called the coefficient of tail dependence or, in short, the tail dependence, is more suitable. It can be shown (see [2]) that the Gaussian copula with correlation of size less than one has zero tail dependence, while the Student's t copula with nonzero correlation possesses strictly positive tail dependence. It implies that the Student's t copula has a stronger dependence structure at the extreme than does the Gaussian copula. In the first half of this article, we take a closer look at the tail dependence between the two copulas. We define tail dependence at q quantile, a relaxed version of the tail dependence. We then observe the measurement at various values of q with respect to the Gaussian copula, the Student's t copula and, in addition, the double t copula.

In recent years, copula models have gained popularity in the financial community. For its simplicity, the Gaussian copula becomes a classical model for applying market co-movement to financial derivative pricing (Vasicek [5] and Elizalde [1]). However, there were some inadequacies in using the Gaussian copula to capture some dependence structures, especially at extreme market conditions. The main reason is that a Gaussian distribution has a narrow tail and, as discussed earlier, the Gaussian copula has no tail dependence. Many turn to other alternative copula models when they advance into the areas that the Gaussian model is not sufficient. When the issue is about the tails, the Student's t copula or, in short, the t copula is a natural choice since it belongs to the same elliptical distribution family as the Gaussian copula but possesses some favorable tail properties. The t distribution has a fatter tail than that of the normal distribution, and the t copula has positive tail dependence. Based on a factor model, Hull and White [3] devised the double t copula, and suggested computational procedures to facilitate computation of credit derivative prices. However, they do not discuss the tail dependence property of the copula. In the second half of this article, we study the effect of tail dependence on credit derivative pricing. The focus is on pricing an instrument called Collateralized Debt Obligation (CDO) with the Gaussian copula, the Student's t copula and the double t copula model.

This article is organized as follows. In Section 2, the three copulas of interest are defined, followed by the definition of tail dependence at q quantile and its values with respect to the three copulas. In Section 3, CDO structure is reviewed, and the effects of the tail dependence at q quantile on its price are discussed. We end with conclusion remarks in Section 4.

2. Copulas and Tail Dependence at *q* Quantile

Suppose random vector $\mathbf{x} = (X_1,\dots,X_n)^T$ has joint distribution F. Assume F continuous throughout. Recall that Sklar's theorem states that, if F has continuous marginals, F_i's, there exists a unique $C(x_1,\dots,x_n)$ such that:

$$F(\mathrm{x}) = F(x_1,\dots,x_n) = C(F_1(x_1),\dots,F_n(x_n)) \quad (1)$$

where C is a distribution in n dimensions with uniform [0, 1] marginals. Therefore, a copula can be defined in at least two ways.

First, define directly C, a continuous distribution function of n dimensions with uniform [0, 1] marginals. Second, define a joint distribution F, and then define C as:

$$C(\mathrm{x}) = C(x_1,\dots,x_n) = F\left(F_1^{-1}(x_1),\dots,F_n^{-1}(x_n)\right) \quad (2)$$

where F_i^{-1} is the quantile function (generalized inverse) of F_i. By the second method, the copula is said to be induced by the joint distribution. The Gaussian copula, the Student's t copula and the double t copula are all defined by the second method. The Gaussian copula, denoted by C^{Ga} is defined as the copula induced by the following multivariate standard normal density function with correlation matrix Σ.

$$f(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}x^t \Sigma x} \quad (3)$$

When the dimension n is 2 with correlation ρ,

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}$$

The Student's t copula, denoted by C^t, is defined as the copula induced by the following multivariate Student's t distribution with ν degrees of freedom and with positive definite correlation matrix matrix Σ.

$$f(x) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^n|\Sigma|}}\left(1+\frac{x^T\Sigma^{-1}x}{\nu}\right)^{-\frac{\nu+n}{2}} \quad (4)$$

When the dimension n is 2 with correlation ρ and degrees of freedom ν:

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{x^2-2\rho xy+y^2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2}$$

The double t copula, denoted by C^{dt} by Hull and White [3] is defined algorithmically as the copula induced by the joint distribution of $\mathbf{X} = (X_1,\dots,X_n)^T$ where:

$$X_i = a_i M + \sqrt{1-a_i^2}\,Z_i\,, \quad \text{for } i = 1,\dots,n \quad (5)$$

where $M, Z_1, \dots, Z_n$ are $n + 1$ independent Student's t random variables with ν degrees of freedom that are scaled to have unit variance and a_i for $i = 1, \dots, n$ are distribution parameters. Observe that the correlation of X_i and X_j is equal to $a_i a_j$.

The tail dependence is defined for a pair of random variable, i.e., it is defined in the case of the dimension n is equal to 2. For a copula $C(x, y)$ of a pair of random variable X and Y, its tail dependence is defined as:

$$\alpha = \lim_{q \to 1} P\left(Y > F_2^{-1}(q) \middle| X > F_1^{-1}(q)\right) = \lim_{q \to 1} \frac{1 - 2q + C(q,q)}{1-q}$$

where F_1 and F_2 are the marginal distribution functions of X and Y, respectively. α can take values in the interval [0, 1]. If $\alpha = 0$, roughly speaking, X and Y are almost independent at their extremes. On the contrary, if $\alpha > 0$, X and Y are significantly dependent at their extremes. It can be shown that, if $\rho < 1$, C^{Ga} has $\alpha = 0$, no matter how high the correlation ρ is. It can also be shown that C^t has $\alpha > 0$ when $\rho > -1$ (see [2]). Therefore, when $-1 \leq \rho < 1$, the behavior C^{Ga} and C^t are totally different at the extreme.

To observe the dependence close to the extreme, we define tail dependence at q quantile:

$$\alpha_q = P\left(Y > F_2^{-1}(q) \middle| X > F_1^{-1}(q)\right) = \frac{1 - 2q + C(q,q)}{1-q}$$

for $0 < q < 1$.

Denote the value of α_q of the Gaussian copula, the Student's t copula, and the double t copula with correlation ρ by $\alpha^{Ga}_{q,p}$, $\alpha^{t}_{q,p}$ and $\alpha^{dt}_{q,p}$, respectively. Table 1 and Figures 1, 2 and 3 show the value of $\alpha^{Ga}_{q,p}$, $\alpha^{t}_{q,p}$ and $\alpha^{dt}_{q,p}$ at various q and ρ. These values are obtained from numerical integrations. Note that we use five degrees of freedom in the Student's t and the double t distribution. Observe that when ρ is equal to 0.1 or 0.5:

$$\alpha^{Ga}_{q,p} < \alpha^{dt}_{q,p} < \alpha^{t}_{q,p} \quad (6)$$

(6) also holds when $\rho = 0.9$ and $q < 0.93$. However, when $\rho = 0.9$ and $q > 0.93$, we have $\alpha^{Ga}_{q,p} < \alpha^{t}_{q,p} < \alpha^{dt}_{q,p}$.

Table 1: Tail dependence at quantile q of various copulas

q	rho=0.1			rho=0.5			rho=0.9		
	Guassian	Student	Double t	Guassian	Student	Double t	Guassian	Student	Double t
0.10	0.9037	0.9095	0.9042	0.9248	0.9303	0.9283	0.8980	0.9684	0.9681
0.20	0.8101	0.8168	0.8122	0.8589	0.8640	0.8618	0.8700	0.9394	0.9360
0.30	0.7175	0.7220	0.7216	0.7954	0.7985	0.7970	0.8511	0.9119	0.9045
0.40	0.6250	0.6359	0.6312	0.7319	0.7329	0.7324	0.8385	0.8843	0.8731
0.50	0.5319	0.5480	0.5399	0.6666	0.6667	0.6667	0.8257	0.8562	0.8416
0.60	0.4375	0.4538	0.4468	0.5978	0.5993	0.5986	0.8052	0.8266	0.8097
0.70	0.3409	0.3514	0.3504	0.5225	0.5297	0.5264	0.7776	0.7946	0.7771
0.80	0.2406	0.2672	0.2488	0.4357	0.4561	0.4471	0.7401	0.7582	0.7441
0.90	0.1334	0.1857	0.1380	0.3240	0.3728	0.3546	0.6828	0.7139	0.7131
0.91	0.1220	0.1774	0.1261	0.3101	0.3633	0.3441	0.6748	0.7133	0.7106
0.92	0.1104	0.1690	0.1139	0.2953	0.3536	0.3333	0.6661	0.7117	0.7084
0.93	0.0986	0.1605	0.1016	0.2796	0.3433	0.3221	0.6564	0.7092	0.7065
0.94	0.0866	0.1518	0.0890	0.2625	0.3326	0.3105	0.6456	0.7039	0.7056
0.95	0.0743	0.1428	0.0760	0.2438	0.3212	0.2984	0.6331	0.6986	0.7046
0.96	0.0616	0.1335	0.0627	0.2229	0.3089	0.2858	0.6185	0.6889	0.7053
0.97	0.0484	0.1235	0.0490	0.1987	0.2952	0.2728	0.6004	0.6735	0.7083
0.98	0.0344	0.1124	0.0347	0.1694	0.2794	0.2597	0.5765	0.6480	0.7152
0.99	0.0193	0.0992	0.0197	0.1294	0.2592	0.2486	0.5387	0.5995	0.7313
1.00	0.0000	0.0686	N/A	0.0000	0.2070	N/A	0.0000	0.4454	N/A

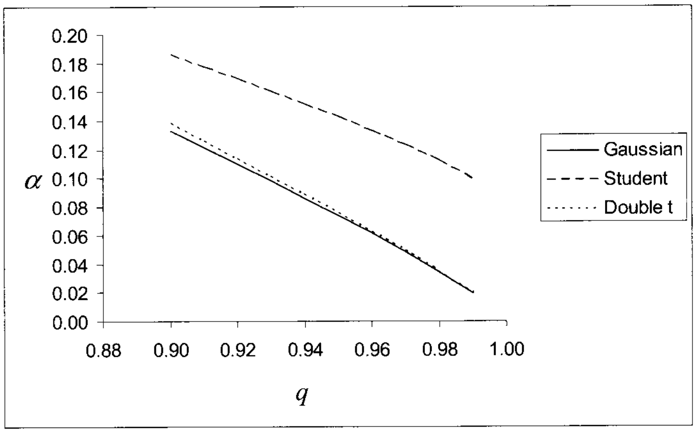

Figure 1: Tail dependence at quantile q of various copulas when $\rho = 0.1$

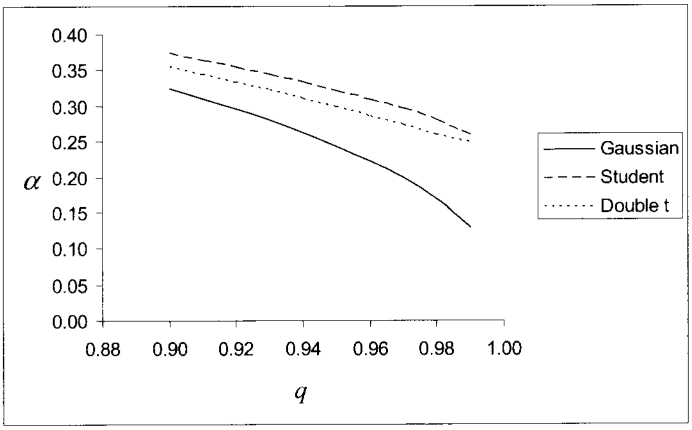

Figure 2: Tail dependence at quantile q of various copulas when $\rho = 0.5$

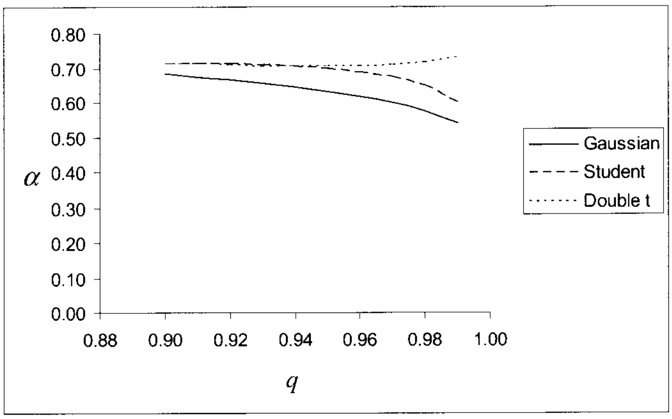

Figure 3: Tail dependence at quantile q of various copulas when $\rho = 0.9$

3. Effects of Tail Dependence at q Quantile on CDO Price

Credit risk constitutes a large portion of risk in a financial institution. A Collateralized Debt Obligation or CDO is a financial instrument that can be used to pass on credit risk from a financial institution to outside parties. The structure of a CDO is described as follows. A financial institution owns a portfolio of defaultable instruments, say bonds. It then passes on the credit or the default risk of the bond portfolio to customers who want to bear risk by issuing securities called CDOs. The securities are divided into layers or tranches. A security in each tranche behaves like a bond with fixed coupon rate where the principal varies according to the defaults in the original portfolio. Figure 4 is simplified from that of Hull and White [3] to show the structure of a CDO. The first tranche in Figure 4 is labeled "first 10% loss, rate 30%" meaning the principal of a security in this tranche is adjusted proportionally to 10% of the bond portfolio principal subject to the first losses resulting from the defaults in the bond portfolio. A CDO customer in the first tranche receives interest of 30% of its outstanding principal during the life of the CDO. A security in the second tranche has principal proportional to 20% of the bond portfolio principal subject to the losses beyond the first 10% losses. A security in the last tranche has principal proportional to the remaining 70% of the principal subject to the losses beyond the first 30%, 10% absorbed by the first tranche and 20% absorbed by the second tranche. For example, suppose the portfolio consists of bonds from 50 names, each with value 1, constituting the bond portfolio principal of 50. If two names default, the first 10% of the principal is reduced by 40%, and, hence, the principal of a security in the first tranche is also reduced by 40%. The interest of 30% is paid on the remaining 60% of the principal. If five names default, the principal of a first tranche security is dissipated, and the holder ceases to receive the interest. If 10 names default, the principal of a second tranche security is reduced by 50%. When the defaults reach 15 names, second tranche holders lose all of their principal. The defaults beyond 15 names will affect the third tranche. The structure of a CDO can have more features, e.g. a recovery rate can be imposed, but, in this article, we assume only this minimal structure just described.

Observe that the first tranche has the highest coupon rate, and the second tranche rate is higher than that of the third tranche. The coupon rate reflects the risk that the holders of the securities in each tranche bear. When defaults occurs, the holders of the first tranche securities will be responsible, so they have more risk than others, resulting in the highest coupon rate. To price a CDO is to determine a coupon rate in each tranche that is considered fair to the holders of securities in that tranche. Our interest is to show the effect of the tail dependence in a CDO price in an over simplistic case, namely the case when there exist only two names in the portfolio of bonds. We then advance to compare the CDO prices in a more realistic setting with respect to the Gaussian copula, the Student's t copula, and the double t copula. Our study is limited to the price of the first tranche only.

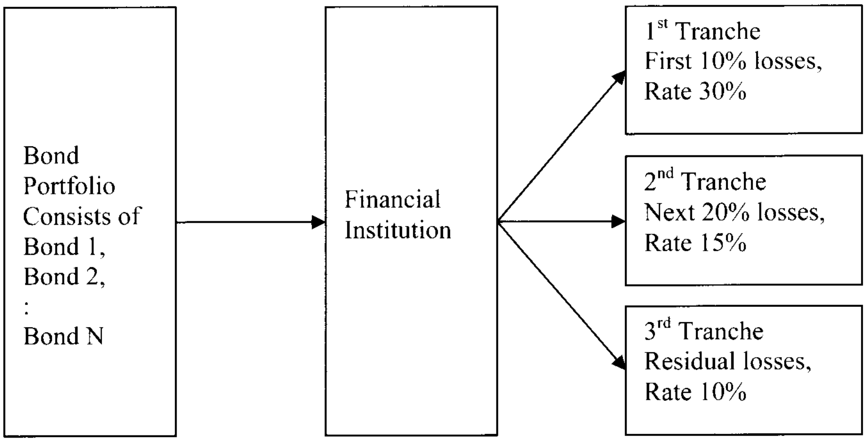

Figure 4: Tail dependence at quantile q of various copulas when $\rho = 0$

As a first step, assume that there exist two names in the bond portfolio, each contribute 50% of the portfolio value. The first tranche of the CDO absorbs the first $100a$% of the losses with coupon rate 100 y% per year. Without loss of generality, assume the starting time is zero and the CDO will mature in one year. Assume that a customer of a CDO pays a principal V upfront. The coupons will be paid k times equally spaced at $t_1, t_2, \dots, t_k = 1$. At maturity, the remaining principal will be repaid to the customer together with the last coupon. Denote the discount rate for t_k by γ_k.

Let T_1 and T_2 be the times to defaults of names 1 and 2, respectively.

Assumptions:

1. T_1 and T_2 be nonnegative random variables identically distributed by continuous distribution function G.
2. The dependence structure of T_1 and T_2 is defined by copula C.

Denote the discount rate for t_k by γ_k. To price a CDO is to determine coupon rate y that makes the principal paid upfront to be equal to the expected present value of the coupon plus the outstanding principal at maturity. Therefore, y is the rate that validates (7).

$$\begin{aligned} V &= \mathrm{E}\left[\sum_{i=1}^{k}\gamma_i\left(\frac{y}{k}\right)V_i + \gamma_k V_k\right] \\ &= \sum_{i=1}^{k}\gamma_i\left(\frac{y}{k}\right)\mathrm{E}[V_i] + \gamma_k \mathrm{E}[V_k] \end{aligned} \tag{7}$$

The analysis can be carried out in two cases, when $a \le 0.5$ and when $a > 0.5$.

In case $a \le 0.5$, at each payment time t_i, $i = 1, \dots, k$, the holder receive the coupon payment only if there is no default. That means T_1 and T_2 both must exceed t_k for the holder to be paid. Otherwise, the loss in the portfolio will exceed the limit a, and dissipate the principal of the first tranche.

$$\begin{aligned} E[V_i] &= V\,\mathrm{P}(T_1 \ge t_i \wedge T_2 \ge t_i) \\ &= V\,\mathrm{P}(T_1 \ge t_i \wedge T_2 \ge t_i) \\ &= V\,\mathrm{P}(T_1 > t_i \mid T_2 > t_i)\,\mathrm{P}(T_2 > t_i) \end{aligned} \tag{8}$$

The second equality follows from the assumption that T_is are continuous. Observe that the first term of (8) is the tail dependence at quantile $G(t_i)$. Therefore, let q_i be $G(t_i)$, and we have:

$$E[V_i] = V\alpha_{q_i}\overline{G}(t_i) \tag{9}$$

where $\overline{G}(t) = 1 - G(t)$. From (7), we then obtain a formula for the rate of tranche one in the case $a \le 0.5$ as:

$$y = k\frac{1 - \gamma_k \alpha_{qk}\overline{G}(t_k)}{\sum_{i=1}^{k}\gamma_i \alpha_{qi}\overline{G}(t_i)} \tag{10}$$

On the other hand, consider when $a > 0.5$. In this case, when there are no defaults, the holder receives full payment. When only one name defaults, the holder receives partial payment. If both names default, the holder receives no payment. Hence, at each t_i,

$$\begin{aligned} E[V_i] &= (a-0.5)V\,\mathrm{P}(T_1 < t_i \wedge T_2 \ge t_i) \\ &\quad + (a-0.5)V\,\mathrm{P}(T_1 < t_i \wedge T_2 \ge t_i) \\ &\quad + V\,\mathrm{P}(T_1 \ge t_i \wedge T_2 \ge t_i) \\ &= (a-0.5)V\,\mathrm{P}(T_1 \le t_i \wedge T_2 > t_i) \\ &\quad + (a-0.5)V\,\mathrm{P}(T_1 \le t_i \wedge T_2 > t_i) \\ &\quad + V\,\mathrm{P}(T_1 > t_i \wedge T_2 > t_i) \\ &= (a-0.5)V\,\mathrm{P}(T_2 > t_i) - \mathrm{P}(T_1 > t_i \wedge T_2 > t_i) \\ &\quad + (a-0.5)V\,\mathrm{P}(T_1 > t_i) - \mathrm{P}(T_1 > t_i \wedge T_2 > t_i) \\ &\quad + V\,\mathrm{P}(T_1 > t_i \wedge T_2 > t_i) \\ &= 2(a-0.5)V\left(\overline{G}(t_i) - \alpha_{q_i}\overline{G}(t_i)\right) + V\alpha_{q_i}\overline{G}(t_i) \\ &= 2(a-0.5)V\overline{G}(t_i) + 2(1-a)\,V\alpha_{q_i}\overline{G}(t_i) \end{aligned} \tag{11}$$

Substitute (11) into (7), we obtain a formula for the rate of tranche one in the case $a > 0.5$ as:

$$y = k\,\frac{1 - 2(a-0.5)\gamma_k\overline{G}(t_k) - 2(1-a)\,\gamma_k\alpha_{q_k}\overline{G}(t_k)}{\sum_{i=1}^{k}2(a-0.5)\gamma_i\overline{G}(t_i) - 2(1-a)\,\gamma_i\alpha_{q_i}\overline{G}(t_i)} \tag{12}$$

Note that $(1 - a)$ in (12) is always nonnegative.

In both (10) and (12), with other things fixed, the more α_{qk}, the lower y. In other words, in this extreme case, the higher the tail dependence at q quantile, the lower the price of a first tranche CDO. Although not realistic, this model provides a valuable insight into the effect of dependence at the extreme on the first tranche

CDO prices. For bivariate times to default, high tail dependence results in both high co-default rate and high co-survive rate. In this case, the effect of high co-survive rate outweighs the effect of high co-default rate, resulting in low CDO price. In addition, it also implies that, at $\rho = 0.1$ or 0.5, in this extreme case, the Gaussian copula model gives a higher CDO price than that from the Student's t copula model, and, in turn, the Student's t copula model gives a higher CDO price that that from the double t copula model. In case $\rho = 0.9$, the Gaussian copula model gives a higher CDO price from the other two models, but the order of the prices between the other two models depends on G.

We proceed to do a numerical experiment with a more realistic case. Now suppose there exist 50 names, each has equal contribution to the portfolio value. Assume that the first tranche absorbs the first 10% losses, and the discount rate is 8%. The coupon is paid semiannually, i.e. $k = 2$. The CDO matures in one year. The time to default of every name is identically distributed by exponential distribution with expectation E[T]. We assume identical correlation, ρ, for all pairs of names in a copula model. The pricing is computed via Monte Carlo Simulation technique. Table 2, 3 and 4 show the CDO price when $\rho = 0.1$, 0.5 and 0.9, respectively. In each row, the table shows the expected time to default of each name, the default intensity, the average times to default in one year and the prices from the three copula models with respect to the intensity at the head of the row. Note that the expected time to default is equal to one over the default intensity. The result when $\rho = 0.5$ is selected to draw a graph. The graph of the other two ρ looks the same, but at different scale.

The CDO price from the Gaussian copula is the highest among the three copula models. This is consistent with the previous result that the Gaussian copula lacks dependence at the extreme, and, hence, it has low probability to co-survive, resulting in high price. However, the prices between the Student's t and the double t copula models are comparable. This cannot be explained by the tail dependence alone since the tail dependence of the Student's t and the double t copula models are not that close. We suspect that there must be more effects of co-default in a higher dimensions that we are unable to explain by the tail dependence since the tail dependence is defined only for two dimensions.

Table 2: Comparisons of CDO prices among the three copula models at $\rho = 0.1$

$E[T]$	λ	$\rho = 0.1$					
		Avg. defaults at 1 year			Rate (CDO price)		
		Gaussian	Student	Double t	Gaussian	Student	Double t
10	0.1000	7.3470	4.7497	4.7415	127.97%	89.12%	92.49%
15	0.0667	5.4659	3.2187	3.2070	79.07%	53.49%	54.56%
20	0.0500	4.4068	2.4359	2.4281	58.54%	38.45%	38.88%
25	0.0400	3.7113	1.9641	1.9548	46.98%	30.33%	30.45%
30	0.0333	3.2130	1.6423	1.6337	39.56%	25.24%	25.17%
35	0.0286	2.8418	1.4072	1.4058	34.51%	21.71%	21.69%
40	0.0250	2.5512	1.2328	1.2307	30.75%	19.19%	19.13%
45	0.0222	2.3191	1.0947	1.0941	27.86%	17.26%	17.20%
50	0.0200	2.1285	0.9848	0.9853	25.58%	15.77%	15.71%
55	0.0182	1.9699	0.8968	0.8956	23.71%	14.59%	14.51%
60	0.0167	1.8328	0.8223	0.8188	22.15%	13.62%	13.51%
65	0.0154	1.7155	0.7600	0.7576	20.84%	12.82%	12.72%
70	0.0143	1.6156	0.7077	0.7043	19.73%	12.15%	12.05%
75	0.0133	1.5260	0.6598	0.6581	18.76%	11.55%	11.47%
80	0.0125	1.4442	0.6206	0.6160	17.87%	11.07%	10.95%
85	0.0118	1.3749	0.5840	0.5795	17.14%	10.61%	10.51%
90	0.0111	1.3107	0.5512	0.5480	16.46%	10.21%	10.13%
95	0.0105	1.2515	0.5236	0.5191	15.84%	9.88%	9.78%
100	0.0100	1.1977	0.4979	0.4935	15.29%	9.57%	9.48%

Table 3: Comparisons of CDO prices among the three copula models at $\rho = 0.5$

$E[T]$	λ	$\rho = 0.5$					
		Avg. defaults at 1 year			Rate (CDO price)		
		Gaussian	Student	Double t	Gaussian	Student	Double t
10	0.1000	7.3375	4.7433	4.7338	55.87%	39.79%	43.12%
15	0.0667	5.4603	3.2106	3.2124	40.15%	28.17%	29.58%
20	0.0500	4.4093	2.4288	2.4246	32.27%	22.47%	22.93%
25	0.0400	3.7150	1.9532	1.9469	27.33%	18.98%	19.05%
30	0.0333	3.2294	1.6347	1.6262	24.04%	16.69%	16.51%
35	0.0286	2.8602	1.4026	1.4007	21.64%	15.00%	14.70%
40	0.0250	2.5722	1.2332	1.2275	19.78%	13.74%	13.34%
45	0.0222	2.3382	1.0965	1.0911	18.27%	12.74%	12.27%
50	0.0200	2.1474	0.9894	0.9846	17.07%	11.94%	11.44%
55	0.0182	1.9847	0.9020	0.8959	16.04%	11.29%	10.76%
60	0.0167	1.8462	0.8257	0.8218	15.19%	10.72%	10.17%
65	0.0154	1.7303	0.7637	0.7598	14.48%	10.25%	9.69%
70	0.0143	1.6265	0.7079	0.7067	13.83%	9.81%	9.29%
75	0.0133	1.5355	0.6602	0.6618	13.27%	9.45%	8.94%
80	0.0125	1.4550	0.6187	0.6204	12.77%	9.13%	8.63%
85	0.0118	1.3841	0.5827	0.5841	12.34%	8.85%	8.35%
90	0.0111	1.3167	0.5502	0.5513	11.94%	8.59%	8.09%
95	0.0105	1.2590	0.5215	0.5237	11.60%	8.37%	7.88%
100	0.0100	1.2054	0.4944	0.4981	11.27%	8.15%	7.69%

Table 4: Comparisons of CDO prices among the three copula models at $\rho = 0.9$

$E[T]$	λ	$\rho = 0.9$					
		Avg. defaults at 1 year			Rate (CDO price)		
		Gaussian	Student	Double t	Gaussian	Student	Double t
10	0.1000	0.5173	0.3698	0.3770	24.08%	17.45%	17.68%
15	0.0667	0.3935	0.2766	0.2718	18.66%	13.35%	13.10%
20	0.0500	0.3278	0.2295	0.2220	15.72%	11.14%	10.78%
25	0.0400	0.2866	0.2011	0.1930	13.84%	9.84%	9.45%
30	0.0333	0.2588	0.1823	0.1748	12.56%	8.97%	8.62%
35	0.0286	0.2377	0.1691	0.1614	11.59%	8.36%	8.00%
40	0.0250	0.2215	0.1591	0.1513	10.83%	7.89%	7.52%
45	0.0222	0.2085	0.1511	0.1437	10.22%	7.52%	7.17%
50	0.0200	0.1976	0.1446	0.1376	9.71%	7.21%	6.88%
55	0.0182	0.1891	0.1390	0.1324	9.31%	6.95%	6.54%
60	0.0167	0.1815	0.1343	0.1284	8.95%	6.73%	6.35%
65	0.0154	0.1750	0.1304	0.1249	8.64%	6.45%	6.19%
70	0.0143	0.1696	0.1270	0.1218	8.38%	6.29%	6.04%
75	0.0133	0.1649	0.1243	0.1190	8.16%	6.16%	5.91%
80	0.0125	0.1605	0.1217	0.1166	7.95%	6.04%	5.79%
85	0.0118	0.1564	0.1193	0.1144	7.75%	5.93%	5.69%
90	0.0111	0.1528	0.1173	0.1125	7.58%	5.83%	5.60%
95	0.0105	0.1494	0.1154	0.1109	7.42%	5.74%	5.52%
100	0.0100	0.1465	0.1139	0.1093	7.28%	5.67%	5.45%

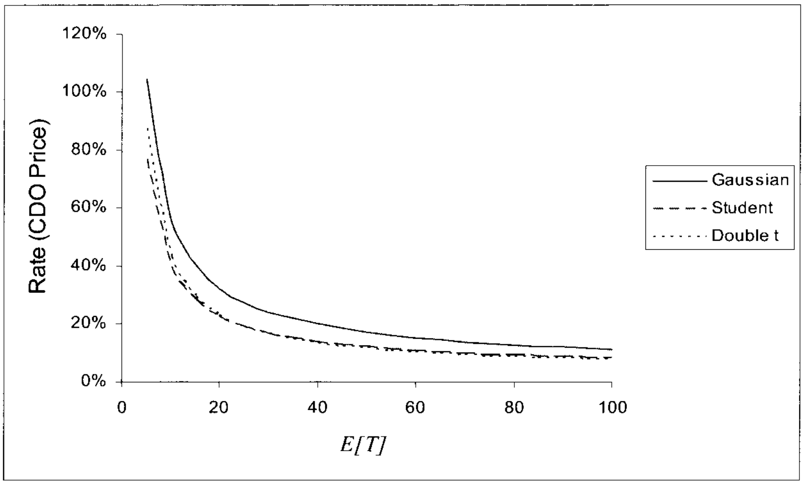

Figure 5: CDO prices among different copula models at $\rho = 0.5$

4. Conclusion

In this article, we have done two things. First, we study the differences of the tail dependence among three widely used copula models, namely the Gaussian copula, the Student's t copula, and the double t copula. Second, we try to explain some phenomenon found in CDO pricing through the light of the tail dependence. For the first task, we define the tail dependence at quantile q and observe that, at low and medium correlation values, the Gaussian model has the lowest tail dependence at each q, followed by the double t and the Student's t model, respectively. At high value of correlation, the Gaussian model is still the lowest of the three, but the other two models are similar. For the second task, in an extreme case of the CDO having two names in the portfolio of bonds, the tail dependence helps explain that high dependence at the extreme can be regarded as high probability of co-default and simultaneously high probability to co-survive. In this extreme case, the effect of co-survival dominates that of the co-default, and can be used to determine the direction of the CDO price. However, in a higher dimensional case, the reasoning is beyond the reach of tail dependence.

5. References

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