# Bending Analysis of Symmetrically Laminated Rectangular Plates with Arbitrary Edge Supports by the Extended Kantorovich Method

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## Abstract

The extended Kantorovich method is employed to solve the bending problem of symmetrically laminated composite rectangular plates with various edge supports. The variational principle with a separable displacement function is utilized to derive a set of governing ordinary differential equations whose exact general solutions are obtained. The solution of the displacement function is determined from iterative calculations of these governing equations using an initial trial function that can be selected arbitrarily. The convergence rate for cross-ply laminated plates is very fast, within only a few iterations. The accuracy of this method is validated numerically with the available Levy-type solutions. The results demonstrate that the proposed semi-analytical approach can be used to solve efficiently the bending problem of isotropic and symmetric cross-ply laminated plates with any combinations of simple, clamped and free supports. New numerical examples of symmetric cross-ply plates with complicated boundary conditions are also presented.

**Keywords:** bending analysis, Kantorovich method, laminated plate, separation of variables

## 1. Introduction

Due to their high stiffness-to-weight and strength-to-weight ratios along with other desirable properties, laminated composite materials are increasingly considered in mechanical, civil, and aerospace engineering applications. Furthermore, composite materials can be designed to have the desired properties in the specified directions without over-designing in other directions. In designing plate components to withstand a particular type of loading, a solution for stress analysis must be obtained first. There have been studies on stress analysis of laminated composite rectangular plates, but closed-form solutions are possible only for the case for which all or opposite edges are simply supported [1, 2]. For other types of boundary conditions, either approximate methods such as the Ritz method [3, 4] or numerical methods such as the FEM and BEM are usually employed. The following proposed method of calculation can be used with any combinations of simple, clamped and free supports

In this paper, the Kantorovich method [5] is extended to obtain the bending solution of symmetrically laminated rectangular plates with various combinations of the boundary conditions. Earlier, Kerr [6] used the extended Kantorovich method for the problem of bending and buckling of an isotropic rectangular plate successfully. The efficiency and accuracy of the method have also been demonstrated in the stress analysis of clamped rectangular isotropic plate by Kerr and Alexander [7] and clamped rectangular orthotropic plate by Dalaei and Kerr [8] also. Aghdam and Falahatgar [9] employed the extended Kantorovich method to the bending problem of moderately thick clamped laminated plates. Although the extended Kantorovich method is based on the variational principle, the works of several researchers, e.g., [7, 8], have shown that initial trial functions are neither required to satisfy the geometric, nor the force boundary conditions, because the iterative procedure will force the solution to satisfy all boundary conditions eventually. Furthermore, the proposed method reduces the problem of solving the partial differential equations to a set of ordinary differential equations in the x and y directions. These two outstanding features make the extended Kantorovich method more attractive than the Galerkin or Ritz methods for certain circumstances.

## 2. Derivation of the iterative differential equations

The extended Kantorovich method is a semi-analytical method which requires iterative calculation by reducing the governing partial differential equations to two sets of governing ordinary differential equations (ODE). These iterative equations are derived by the variational principle because equations for all boundary conditions are established automatically. The total potential energy for bending of symmetrically laminated composite plates subjected to a uniform lateral load $q$ is given by Reddy [2] as:

$$\Pi = U + V$$

$$\Pi = \frac{1}{2}\int_0^a\int_0^b [D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}^2 + 4D_{66}w_{,xy}^2 + 4(D_{16}w_{,xx} + D_{26}w_{,yy})w_{,xy}]dxdy - \int_0^a\int_0^b q w dxdy \quad (1)$$

where comma denotes the differentiation with respect to the subscripted variable and $D_{ij}$ is the bending stiffness of the laminated composite material.

For the classical Kantorovich method, the solution $w(x,y)$ is assumed to be separable as:

$$w(x,y) = X(x)Y(y) \quad (2)$$

Substituting Equation (2) into (1), the total potential energy becomes:

$$\Pi = \frac{1}{2}\int_0^a\int_0^b [D_{11}X_{,xx}^2Y^2 + 2D_{12}X_{,xx}Y_{,yy}XY + D_{22}X^2Y_{,yy}^2 + 4D_{66}X_{,x}^2Y_{,y}^2 + 4(D_{16}X_{,xx}Y + D_{26}XY_{,yy})X_{,x}Y_{,y}]dxdy - \int_0^a\int_0^b qXYdxdy \quad (3)$$

If $X(x)$ is priorly specified, eq. (3) can be written as:

$$\Pi = \frac{1}{2}\int_0^b [S_{1x}D_{11}Y^2 + 2S_{2x}D_{12}YY_{,yy} + S_{3x}D_{22}Y_{,yy}^2 + 4S_{4x}D_{66}Y_{,y}^2 + 4(S_{5x}D_{16}YY_{,y} + S_{6x}D_{26}Y_{,y}Y_{,yy})]dy - \int_0^b S_{7x}qYdy \quad (4)$$

If $Y(y)$ is priorly specified, eq. (3) can be written as:

$$\Pi = \frac{1}{2}\int_0^a [S_{3y}D_{11}X_{,xx}^2 + 2S_{2y}D_{12}XX_{,xx} + S_{1y}D_{22}X^2 + 4S_{4y}D_{66}X_{,x}^2 + 4(S_{6y}D_{16}X_{,x}X_{,xx} + S_{5y}D_{26}XX_{,x})]dx - \int_0^a S_{7y}qXdx \quad (5)$$

where:

$$S_{1x} = \int_0^a X_{,xx}^2dx,\quad S_{2x} = \int_0^a XX_{,xx}dx,\quad S_{3x} = \int_0^a X^2dx$$

$$S_{4x} = \int_0^a X_{,x}^2dx,\quad S_{5x} = \int_0^a X_{,x}X_{,xx}dx, S_{6x} = \int_0^a XX_{,x}dx$$

$$S_{7x} = \int_0^a Xdx$$

$$S_{1y} = \int_0^b Y_{,yy}^2dy,\quad S_{2y} = \int_0^b YY_{,yy}dy,\quad S_{3y} = \int_0^b Y^2dy$$

$$S_{4y} = \int_0^b Y_{,y}^2dy,\quad S_{5y} = \int_0^b Y_{,y}Y_{,yy}dy,\quad S_{6y} = \int_0^b YY_{,y}dy$$

$$S_{7y} = \int_0^b Ydy \quad (6)$$

When $X(x)$ is priorly specified, to obtain the equilibrium equation, the variational principle requires the stationary condition for the functional equation (4), $\delta\Pi = 0$. The procedure yields the following governing ODE and the associated boundary conditions (BCs):

$$S_{3x}D_{22}\frac{d^4Y}{dy^4} + 2(S_{2x}D_{12} - 2S_{4x}D_{66})\frac{d^2Y}{dy^2} + S_{1x}D_{11}Y = S_{7x}q \quad (7)$$

BCs along $y=0$ and $y=b$ are :
Either:

$$S_{3x}D_{22}\frac{d^3Y}{dy^3} + (S_{2x}D_{12} - 4S_{4x}D_{66})\frac{dY}{dy} - 2S_{5x}D_{16}Y = 0 \quad (8)$$

or $\quad \delta Y = 0 \quad (9)$

And either:

$$S_{3x}D_{22}\frac{d^2Y}{dy^2} + 2S_{6x}D_{26}\frac{dY}{dy} + S_{2x}D_{12}Y = 0 \quad (10)$$

or $\quad \delta Y_{,y} = 0 \quad (11)$

These conditions, eqs. (8)-(11), correspond to the following edge supports:

Simply supported edge: eqs. (9) and (10)
Clamped edge: eqs. (9) and (11)
Free edge: eqs. (8) and (10)

When $Y(y)$ is priorly specified, the variational principle requires that the first variation of functional equation (5) to equal zero. The procedure yields the following governing ODE and the associated BCs:

$$S_{3y}D_{11}\frac{d^4X}{dx^4} + 2(S_{2y}D_{12} - 2S_{4y}D_{66})\frac{d^2X}{dx^2} + S_{1y}D_{22}X = S_{7y}q \quad (12)$$

BCs along $x=0$ and $x=a$ are:
Either:

$$S_{3y}D_{11}\frac{d^3X}{dx^3} + (S_{2y}D_{12} - 4S_{4y}D_{66})\frac{dX}{dx} - 2S_{5y}D_{26}X = 0 \quad (13)$$

or $\quad \delta X = 0 \quad (14)$

And either:

$$S_{3y}D_{11}\frac{d^2X}{dx^2} + 2S_{6y}D_{16}\frac{dX}{dx} + S_{2y}D_{12}X = 0 \quad (15)$$

or $\quad \delta X_{,x} = 0 \quad (16)$

Hence, we have completed the derivation of two sets of ODE for iterative calculation. One set is for $X(x)$ specified beforehand, and the other set is for $Y(y)$ specified beforehand. Next the exact solutions for both sets of differential equations will be obtained.

## 3. Solutions of the iterative ODE

From eq. (7), one has the following fourth order ODE.

$$\frac{d^4Y}{dy^4} + 2k_1\frac{d^2Y}{dy^2} + k_2Y = q_y \quad (17)$$

where :

$$k_1 = \frac{(S_{2x}D_{12} - 2S_{4x}D_{66})}{S_{3x}D_{22}} \quad (18)$$

$$k_2 = \frac{S_{1x}D_{11}}{S_{3x}D_{22}} \quad (19)$$

$$q_y = \frac{S_{7x}q}{S_{3x}D_{22}} \quad (20)$$

The characteristic equation of eq. (17) is :

$$s^4 + 2k_1s^2 + k_2 = 0 \quad (21)$$

whose four roots are :

$$s_{1,2,3,4} = \pm\sqrt{-k_1 \pm \sqrt{k_1^2 - k_2}} \quad (22)$$

Depending on the values of $k_1$ and $k_2$, and since $k_2$ is always positive, there are three kinds of roots in eq. (22). Therefore, the general solution of eq. (17) can be written in one of the following three forms:

$$Y(y) = A_y \sin(p_1 y) + B_y \cos(p_1 y) + C_y \sin(p_2 y) + D_y \cos(p_2 y) + q_y / k_2 \quad (23)$$

$$Y(y) = [A_y \cos(p_2 y) + B_y \sin(p_2 y)]\cosh(p_1 y) + [C_y \cos(p_2 y) + D_y \sin(p_2 y)]\sinh(p_1 y) + q_y / k_2 \quad (24)$$

$$Y(y) = A_y \sinh(p_1 y) + B_y \cosh(p_1 y) + C_y \sinh(p_2 y) + D_y \cosh(p_2 y) + q_y / k_2 \quad (25)$$

The conditions for each type of solution are as follows.

1. If $k_1 > 0$ and $(k_1^2 - k_2) > 0$, all four roots are imaginary, eq. (23) is the solution with :

$s_{1,2} = \pm ip_1$, $s_{3,4} = \pm ip_2$, where

$$p_1 = \sqrt{k_1 + \sqrt{k_1^2 - k_2}}\,, \quad p_2 = \sqrt{k_1 - \sqrt{k_1^2 - k_2}} \quad (26)$$

2. If $k_1 < 0$ or $k_1 > 0$ and $(k_1^2 - k_2) < 0$, roots are in complex conjugate pairs, eq. (24) is the solution with :

$s_{1,2} = p_1 \pm ip_2$, $s_{3,4} = -p_1 \pm ip_2$,

where

$$p_1 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{k_2} - k_1}\,, \quad p_2 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{k_2} + k_1} \quad (27)$$

3. If $k_1 < 0$ and $0 < k_2 < k_1^2$, all four roots are real, eq. (25) is the solution with :

$s_{1,2} = \pm p_1$, $s_{3,4} = \pm p_2$, where

$$p_1 = \sqrt{-k_1 + \sqrt{k_1^2 - k_2}}\,, \quad p_2 = \sqrt{-k_1 - \sqrt{k_1^2 - k_2}} \quad (28)$$

Similarly, if $Y(y)$ is known, eq. (12) can be arranged as :

$$\frac{d^4 X}{dx^4} + 2k_3 \frac{d^2 X}{dx^2} + k_4 X = q_x \quad (29)$$

where :

$$k_3 = \frac{(S_{2y} D_{12} - 2S_{4y} D_{66})}{S_{3y} D_{11}} \quad (30)$$

$$k_4 = \frac{S_{1y} D_{22}}{S_{3y} D_{11}} \quad (31)$$

$$q_x = \frac{S_{7y} q}{S_{3y} D_{11}} \quad (32)$$

whose four roots are :

$$s_{1,2,3,4} = \pm\sqrt{-k_3 \pm \sqrt{k_3^2 - k_4}} \quad (33)$$

Similarly, three possible general solutions for this case are:

$$X(x) = A_x \sin(p_3 x) + B_x \cos(p_3 x) + C_x \sin(p_4 x) + D_x \cos(p_4 x) + q_x / k_4 \quad (34)$$

$$X(x) = [A_x \cos(p_4 x) + B_x \sin(p_4 x)]\cosh(p_3 x) + [C_x \cos(p_4 x) + D_x \sin(p_4 x)]\sinh(p_3 x) + q_x / k_4 \quad (35)$$

$$X(x) = A_x \sinh(p_3 x) + B_x \cosh(p_3 x) + C_x \sinh(p_4 x) + D_x \cosh(p_4 x) + q_x / k_4 \quad (36)$$

The conditions for each type of solution are as discussed previously. The roles of $k_1, k_2, p_1, p_2$ are replaced by $k_3, k_4, p_3, p_4$, respectively.

## 4. Iterative procedure

To obtain the solution usually requires only a few iterations. The first calculation is to use the assumed function $X(x)$ to obtain the exact solution of $Y(y)$ and then use this $Y(y)$ to obtain the exact solution of $X(x)$. Hence, the following algorithm is devised.

1. Assume an initial solution $X(x)$ in the x direction which may or may not satisfy any boundary conditions, and then evaluate $k_1, k_2$ according to eqs. (18)-(19).
2. Calculate the four roots in eqs. (26)-(28) and select the form of solution $Y(y)$ from eqs. (23)-(25) corresponding to the roots.
3. Apply the BCs to the solution $Y(y)$ and determine all their constants. Hence, the solution $Y(y)$ is obtained. This

completes the simple Kantorovich method.

4. For the extended Kantorovich method, use the $Y(y)$ obtained in step 3 to evaluate $k_3, k_4$ according to eqs. (30)-(31).
5. Calculate the four roots and select the form of solution $X(x)$ from eqs. (34)-(36) corresponding to the roots.
6. Apply the BCs to the solution $X(x)$ and determine all their constants. Hence, the solution $X(x)$ is obtained.
7. The complete solution is $w(x,y) = X(x)Y(y)$. With an assumed geometry and materials, calculate the deflection $w$, say at its center, and compare its value with the previous one. If the difference satisfies the specified tolerance, the last solution is taken as the final solution. This completes the extended Kantorovich procedure. Otherwise continue the iterative calculation by repeating steps 1 to 3 using the most recent $X(x)$ as the trial function.

Observe that if the assumed initial function in step 1 is identical to the solution, the first iteration will give good results.

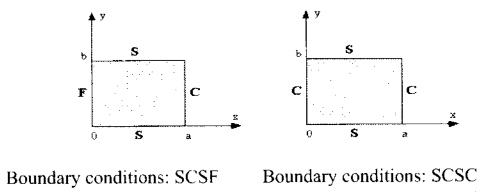


**Fig. 1** Boundary conditions of plate denoted by S, C, and F

## 5. Numerical verification and accuracy

The iterative procedure outlined in the previous section may be applied to rectangular plates with any combinations of simple support (S), clamped support (C), and free edge (F). A simple-clamped-simple-free (SCSF) plate as shown in Fig. 1 is a specimen with simple supports on $y = 0$ and $y = b$, and free and clamped on $x = 0$ and $x = a$, respectively. The other example in Fig. 1 is a SCSC specimen which is simply supported on $y = 0$ and $y = b$, and clamped on $x = 0$ and $x = a$. The iterative example for a symmetric cross-ply, $[0/90]_s$, CCCF square plate, subjected to a uniformly distributed load is illustrated in Table1. Mechanical properties of this plate are $\frac{E_1}{E_2} = 25$, $G_{12} = 0.5E_2$, $\nu_{12} = 0.25$. The first iteration begins with assuming the function $X(x)$ is $\sin\left(\frac{\pi x}{a}\right)$ and solving for $Y(y)$ according to eq. (17). This selected function is not required to satisfy the F-C boundary conditions on $x = 0$ and $x = a$. The function $Y(y)$ obtained from the first iteration denoted as **A** is forced to satisfy the boundary conditions in the y-direction automatically. The non-dimentional out-of-plane displacement $\bar{w}(x,y) = \left(\frac{E_2 h^3}{qa^4}\right) w(x,y)$ is plotted in the last column of Table 1. Its value evaluated at $x = 0, y = b/2$ is used as the convergent criterion. Notice that $\bar{w}(0,b/2) = 0$ in the first iteration because the assumed function $X(x)$ does not correspond to the free edge. The second iteration employs function "**A**" obtained from the first iteration as an assumed function $Y(y)$. The solution $X(x)$ obtained from the second iteration is denoted as "**B**" with $\bar{w}(0,b/2) = 0.00900$. The third iteration is performed using the function "**B**" from the second iteration, which yields the solution "**C**" with $\bar{w}(0,b/2) = 0.009236$. The fourth iteration gives function "**D**" with $\bar{w}(0,b/2) = 0.009258$. The next iteration yields the same value of $\bar{w}(0,b/2)$. Hence, the iteration processes conclude with $\bar{w}(0,b/2)$ converging to 0.009258 for the CCCF, $[0/90]_s$ square plate.

The present method is numerically verified by comparing the solution obtained from this method with those of known isotropic and laminated plate solutions of various combinations of support. In Tables 2-6, deflections and bending moments of uniformly loaded rectangular isotropic plates of CCCC, CSCS, CSFS, CSSS and SSFS are compared with the exact solutions in the book of Timoshenko and Woinowsky-Krieger [10]. The convergence of the solution requires only three iterations. Here the non-dimensional variables of isotropic plates are defined as follows.

$$\bar{w} = w\left(\frac{D}{qa^4}\right), \quad \bar{M}_x = \frac{M_x}{qa^2}, \text{ and}$$

$$\bar{M}_y = \frac{M_y}{qa^2}. \tag{37}$$

The present method applies to symmetric cross-ply plates of SSSS, SCSS, SCSC and SSSF. SCSF is also verified with the available Levy-type solutions [2] as shown in Table 7. The non-dimensional deflection function is $\bar{w}(x,y) = \left(\frac{E_2 h^3}{qa^4}\right) w(x,y)$. Convergence to the exact Levy-type solutions is very fast within four iterations, even although only a one term solution is used and the initial assumed function does not satisfy the boundary conditions. Hence, it is verified that the extended Kantorovich method can be used for solving the bending problem of isotropic and symmetric cross-ply laminated plates with high accuracy and efficiency.

## 6. Other numerical results

Numerical results for symmetric cross-ply laminated plates with complicated BCs, which are not available elsewhere, are shown in Table 8. They are CCCC, CCCS and CCCF plates with stacking sequence of $[0/90]_s$, $\frac{E_1}{E_2} = 25$, $G_{12} = 0.5E_2$, $\nu_{12} = 0.25$. The required number of iterations for the results is only four. The deformed configurations of square plates in Tables 7-8 are shown in Fig. 2 also.

## 7. Conclusion and discussion

Bending problems of isotropic and symmetrically cross-ply laminated plates are solved efficiently by the extended Kantorovich method. The boundary conditions of the rectangular plates can be any combinations of simple support, clamped and free edges. Although the solution by Kantorovich method is obtained from solving a set of ODE, this method requires a series of iterative calculations, so it is considered to be a semi-analytical method. By assuming that the out-of-plane displacement function of plates is separable and either one of $X(x)$ or $Y(y)$ is known beforehand, applying the variational principle to the total potential energy, yields a set of ODE in the x or y direction. These ODE and the associated boundary conditions are used in the iterative calculations to obtain the deflection function. The procedure is repeated until the deflection at a specified point converges to the specified tolerance. The final product of $X(x)$ and $Y(y)$ indicates the solution of the deflection function.

The extended Kantorovich method is verified numerically by comparing the deflections and bending moments with the known solutions in references [10] and [2]. The results for isotropic and cross-ply laminated plates agreed very well. Therefore, the extended Kantorovich method is validated for isotropic rectangular plates and composite plates with unidirectional or cross-ply symmetric stacking sequence, i.e. specimens with $D_{16} = D_{26} = 0$. For specimens with the presence of $D_{16}$ and $D_{26}$, i.e. angle-ply laminates; the approximation of the displacement function is required to include additional terms in order to simulate the actual deflection patterns. The approximate function could be in a form of $w(x,y) = X_1(x)Y_1(y) + X_2(x)Y_2(y) + X_3(x)Y_3(y)$ which will lead to iterative calculations involving three sets of simultaneous ODE. The method is prohibitive without efficient software for ODE solving [11]. Solving these ODE are tedious and not in the scope of this study. Webber [12] also suggested simplification of multi-term solutions by assuming that $X_1(x)$ and $Y_1(y)$,obtained from the first iterative procedure, were held constant while $X_2(x)$ and $Y_2(y)$ were determined through the iterative procedure. New numerical examples of specimens with combinations of simple, clamped, and free boundary conditions are also included in our work.

An advantage of this proposed method is that there is no need to solve the governing partial differential equations. They are transformed to a set of ODE with an assumed displacement function. Another advantage is that the initial assumed displacement function in the first iteration could be arbitrarily selected regardless of the type of boundary conditions. The displacement functions are automatically forced to satisfy the boundary conditions in the next iterations.

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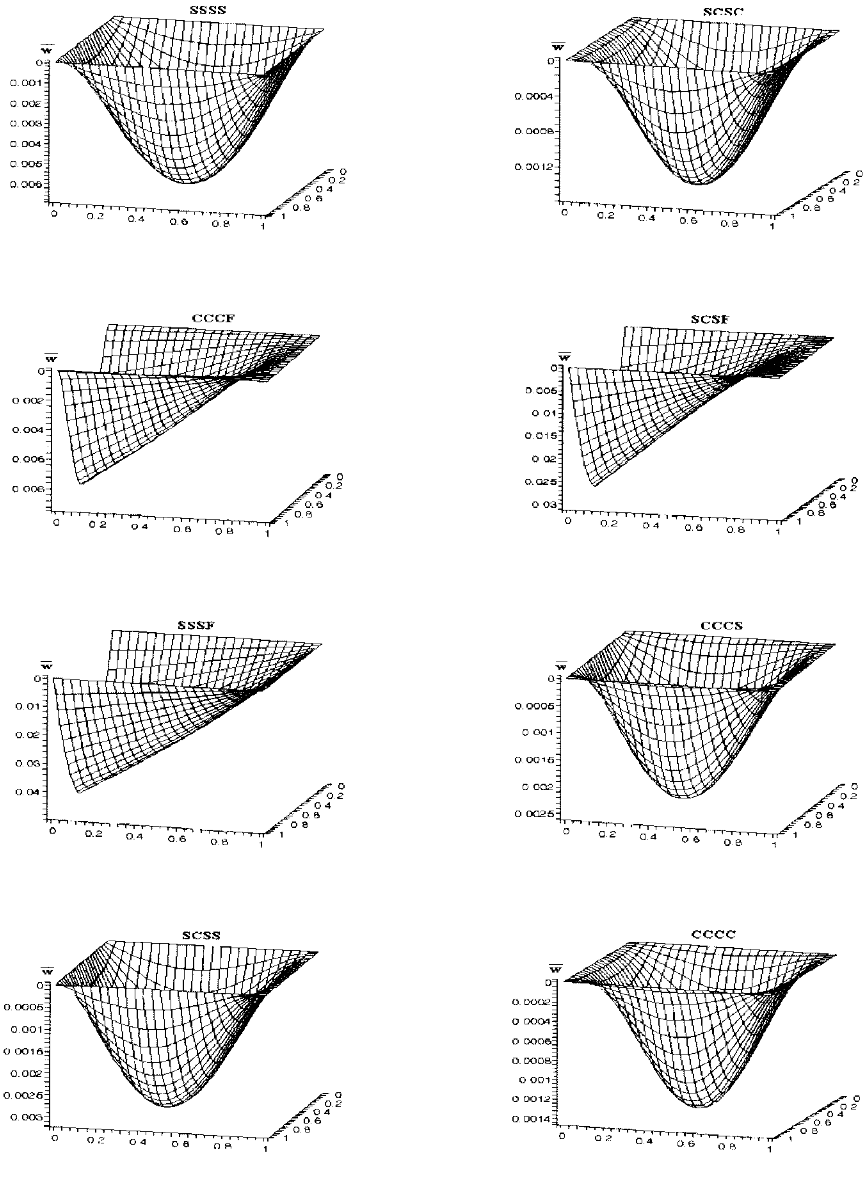
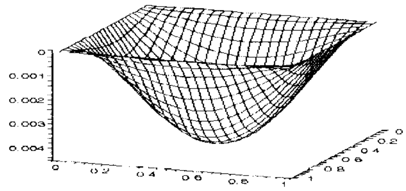
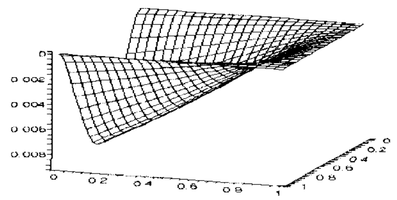
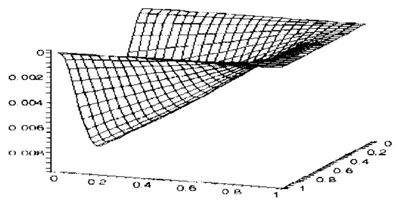
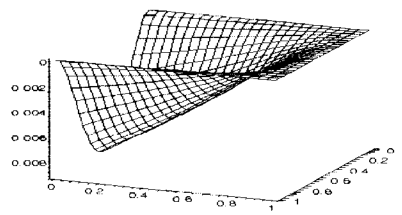



**Fig. 2.** Deflection of plates with various edge supports

**Table 1**. Iteration example for $[0/90]_s$ plate with CCCF boundary condition

| Iteration **No.** | **X(x)** | **Y(y)** | **Non-dimensional Deflection** $\overline{w}(x,y)$ |
|---|---|---|---|
| 1 | $\sin\left(\frac{\pi x}{a}\right)$ | **A** |  |
| 2 | **B** | **A** |  |
| 3 | **B** | **C** |  |
| 4 | **D** | **C** |  |

Note: $E_1/E_2 = 25, G_{12} = 0.5E_2, \nu_{12} = 0.25$

$$\mathbf{A} = (-0.0071\cos(3.17y)\cosh(3.62y) - 0.0086\sin(3.17y)\cosh(3.62y) + 0.0075\cos(3.17y)\sinh(3.62y) + 0.0086\sin(3.17y)\sinh(3.62y) + 0.0071)\frac{qa^4}{E_2h^3}$$

$$\mathbf{B} = 0.3390\cos(2.13x)\cosh(2.28x) - 0.4836\sin(2.13x)\cosh(2.28x) - 0.4642\cos(2.13x)\sinh(2.28x) + 0.0048\sin(2.13x)\sinh(2.28x) + 1.7028$$

$$\mathbf{C} = (-0.0214\cos(2.15y)\cosh(2.34y) - 0.0222\sin(2.15y)\cosh(2.34y) + 0.0204\cos(2.15y)\sinh(2.34y) + 0.0168\sin(2.15y)\sinh(2.34y) + 0.0214)\frac{qa^4}{E_2h^3}$$

$$\mathbf{D} = 0.3345\cos(2.11x)\cosh(2.27x) - 0.4908\sin(2.11x)\cosh(2.27x) - 0.4704\cos(2.11x)\sinh(2.27x) + 0.0053\sin(2.11x)\sinh(2.27x) + 1.7123$$

**Table 2**. Deflections and bending moments of uniformly loaded rectangular isotropic plate (CCCC) with $\nu = 0.3$

| b/a | Exact Solution | Present Solution | Exact Solution | Present Solution | Exact Solution | Present Solution |
|---|---|---|---|---|---|---|
| | $\overline{w}$ x=a/2,y=b/2 | $\overline{w}$ x=a/2,y=b/2 | $\overline{M}_x$ x=a, y=b/2 | $\overline{M}_x$ x=a, y=b/2 | $\overline{M}_y$ x=a/2, y=b | $\overline{M}_y$ x=a/2, y=b |
| 0.5 | - | 0.00016 | - | -0.0145 | - | -0.0214 |
| 1.0 | 0.00126 | 0.00126 | -0.0513 | -0.0522 | -0.0513 | -0.0522 |
| 1.5 | 0.00220 | 0.00220 | -0.0757 | -0.0776 | -0.0570 | -0.0581 |
| 2.0 | 0.00254 | 0.00253 | -0.0829 | -0.0855 | -0.0571 | -0.0582 |
| 2.5 | - | 0.00260 | - | -0.0861 | - | -0.0583 |
| 3.0 | - | 0.00261 | - | -0.0854 | - | -0.0583 |

**Table 3**. Deflections and bending moments of uniformly loaded rectangular isotropic plate (CSCS) with $\nu = 0.3$

| b/a | Exact Solution | Present Solution | Exact Solution | Present Solution | Exact Solution | Present Solution |
|---|---|---|---|---|---|---|
| | $\overline{w}$ x=a/2,y=b/2 | $\overline{w}$ x=a/2,y=b/2 | $\overline{M}_x$ x=a/2,y=b/2 | $\overline{M}_x$ x=a/2,y=b/2 | $\overline{M}_y$ x=a/2, y=b | $\overline{M}_y$ x=a/2, y=b |
| 0.5 | - | 0.00016 | - | 0.0035 | - | -0.0217 |
| 1.0 | 0.00192 | 0.00191 | 0.0244 | 0.0240 | -0.0697 | -0.0720 |
| 1.5 | 0.00531 | 0.00532 | 0.0585 | 0.0578 | -0.1049 | -0.1077 |
| 2.0 | 0.00844 | 0.00843 | 0.0869 | 0.0860 | -0.1191 | -0.1221 |
| 2.5 | - | 0.01049 | - | 0.1037 | - | -0.1266 |
| 3.0 | 0.01168 | 0.01167 | 0.1144 | 0.1136 | -0.1246 | -0.1278 |

**Table 4**. Deflections and bending moments of uniformly loaded rectangular isotropic plate (CSFS) with $\nu = 0.3$

| b/a | Exact Solution | Present Solution | Exact Solution | Present Solution | Exact Solution | Present Solution |
|---|---|---|---|---|---|---|
| | $\overline{w}$ x=a/2, y=b | $\overline{w}$ x=a/2, y=b | $\overline{M}_x$ x=a/2, y=b | $\overline{M}_x$ x=a/2, y=b | $\overline{M}_y$ x=a/2, y=0 | $\overline{M}_y$ x=a/2, y=0 |
| 0.5 | 0.0036 | 0.0036 | 0.0293 | 0.0285 | -0.0797 | -0.0824 |
| 1.0 | 0.0113 | 0.0112 | 0.0972 | 0.0959 | -0.1190 | -0.1220 |
| 1.5 | 0.0141 | 0.0141 | 0.1230 | 0.1220 | -0.1240 | -0.1270 |
| 2.0 | 0.0150 | 0.0149 | 0.1310 | 0.1295 | -0.1250 | -0.1280 |
| 2.5 | - | 0.01515 | - | 0.1315 | - | -0.1282 |
| 3.0 | 0.0152 | 0.0152 | 0.1330 | 0.1321 | -0.1250 | -0.1283 |

**Table 5**. Deflections and bending moments of uniformly loaded rectangular isotropic plate (CSSS) with $\nu = 0.3$

| b/a | Exact Solution | Present Solution | Exact Solution | Present Solution | Exact Solution | Present Solution |
|---|---|---|---|---|---|---|
| | $\bar{w}$ x=a/2,y=b/2 | $\bar{w}$ x=a/2,y=b/2 | $\bar{M}_x$ x=a/2,y=b/2 | $\bar{M}_x$ x=a/2,y=b/2 | $\bar{M}_y$ x=a/2, y=0 | $\bar{M}_y$ x=a/2, y=0 |
| 0.5 | 0.0003 | 0.0003 | 0.0058 | 0.0058 | -0.0305 | -0.0314 |
| 1.0 | 0.0028 | 0.0028 | 0.0340 | 0.0336 | -0.0840 | -0.0865 |
| 1.5 | 0.0064 | 0.0064 | 0.0690 | 0.0685 | -0.1120 | -0.1151 |
| 2.0 | 0.0093 | 0.0093 | 0.0940 | 0.0934 | -0.1220 | -0.1246 |
| 2.5 | - | 0.0110 | - | 0.1080 | - | -0.1275 |
| 3.0 | - | 0.0119 | - | 0.1159 | - | -0.1282 |

**Table 6**. Deflections and bending moments of uniformly loaded rectangular isotropic plate (SSFS) with $\nu = 0.3$

| b/a | Exact Solution | Present Solution | Exact Solution | Present Solution | Exact Solution | Present Solution |
|---|---|---|---|---|---|---|
| | $\bar{w}$ x=a/2, y=b | $\bar{w}$ x=a/2, y=b | $\bar{M}_x$ x=a/2, y=b | $\bar{M}_x$ x=a/2, y=b | $\bar{M}_y$ x=a/2,y=b/2 | $\bar{M}_y$ x=a/2,y=b/2 |
| 0.5 | 0.00710 | 0.00709 | 0.0600 | 0.0594 | 0.0220 | 0.0227 |
| 1.0 | 0.01286 | 0.01284 | 0.1120 | 0.1106 | 0.0390 | 0.0392 |
| 1.5 | 0.01462 | 0.01460 | 0.1280 | 0.1265 | 0.0420 | 0.0422 |
| 2.0 | 0.01507 | 0.01506 | 0.1320 | 0.1310 | 0.0410 | 0.0414 |
| 2.5 | - | 0.01518 | - | 0.1319 | - | 0.0401 |
| 3.0 | 0.01520 | 0.01520 | 0.1330 | 0.1322 | 0.0390 | 0.0390 |

**Table 7**. Non-dimensional deflections of uniformly loaded rectangular $[0/90]_s$ plates determined from the Levy solution and Kantorovich method.

| b/a | | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| SSSS x=a/2,y=b/2 | Levy Solution | 0.00204 | 0.00680 | 0.00768 | 0.00746 | 0.00721 | 0.00720 |
| | Present Solution | 0.00203 | 0.00679 | 0.00767 | 0.00746 | 0.00720 | 0.00710 |
| SCSS x=a/2,y=b/2 | Levy Solution | 0.00149 | 0.00305 | 0.00303 | 0.00288 | 0.00283 | 0.00283 |
| | Present Solution | 0.00148 | 0.00303 | 0.00301 | 0.00287 | 0.00281 | 0.00281 |
| SCSC x=a/2,y=b/2 | Levy Solution | 0.00106 | 0.00157 | 0.00146 | 0.00141 | 0.00142 | 0.00142 |
| | Present Solution | 0.00107 | 0.00156 | 0.00146 | 0.00141 | 0.00141 | 0.00142 |
| SSSF x=0,y=b/2 | Levy Solution | 0.00277 | 0.04698 | 0.21880 | 0.57601 | 1.16870 | 1.99502 |
| | Present Solution | 0.00276 | 0.04879 | 0.21836 | 0.57838 | 1.16589 | 1.99063 |
| SCSF x=0,y=b/2 | Levy Solution | 0.00285 | 0.03063 | 0.05950 | 0.07107 | 0.07371 | 0.07290 |
| | Present Solution | 0.00284 | 0.03050 | 0.05926 | 0.07078 | 0.07341 | 0.07261 |

Note: $E_1/E_2 = 25, G_{12} = G_{13} = 0.5E_2, \nu_{12} = 0.25$

**Table 8.** Non-dimensional deflections of uniformly loaded rectangular $[0/90]_s$ plates determined from the Kantorovich method.

| b/a | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|
| CCCC x=a/2,y=b/2 | 0.00044 | 0.00143 | 0.00149 | 0.00143 | 0.00141 | 0.00141 |
| CCCS x=a/2,y=b/2 | 0.00049 | 0.00247 | 0.00300 | 0.00293 | 0.00285 | 0.00283 |
| CCCF x=0,y=b/2 | 0.00050 | 0.00926 | 0.03264 | 0.05490 | 0.06674 | 0.07094 |

Note: $E_1/E_2 = 25, G_{12} = 0.5E_2, \nu_{12} = 0.25$