Stress Analysis on Crack Tip Using Q8 and Adaptive Meshes

Wiroj Limtrakarn
Department of Mechanical Engineering, Faculty of Engineering,
Thammasat University (Rangsit Campus), Khlong Luang, Pathum Thani, 12120, Thailand.
Tel: (02) 564–3001–9 Ext. 3144, Fax: (02) 564–3010
E–Mail: limwiroj@engr.tu.ac.th

Abstract

Evaluation of a adaptive finite element method for analysis of two–dimensional crack problems is presented. A combined procedure of the finite element method and an adaptive remeshing technique is used. Eight–node quadrilateral elements (Q8) are placed around a crack tip, while constant strain triangular elements are used in other regions. The finite element formulation and computational procedure are described. The performance of the adaptive finite element method is evaluated using a single edge notched (SEN) problem, and the circular cut out hole with centrally cracked plate problem.

Keywords: crack tip, adaptive finite element method, stress intensity factor, linear fracture mechanics

1. Introduction

In elastic-plastic fracture mechanics, stresses inside the plastic zone are represented by a path independent integral J which is directly related to the stress intensity factor in linear elastic problems. In linear elastic fracture mechanics, the in–plane stresses and strains around the crack tip can be described as symmetric and antisymmetric fields, called mode I and mode II, respectively. Classical fracture mechanics theory assumes that in mode I the near–crack–tip stresses and strains can be characterized by a single parameter such as stress intensity factor K_I or J integral. The stress intensity factor normally depends on the geometry and loading conditions. Many researchers have attempted to study the crack tip stresses and K_I using experimental methods such as photoelasticity [1] or by computational methods such as finite element methods [2, 3].

This paper focuses on an adaptive finite element method to predict accurate crack tip stress intensity factors. Stresses near a crack tip are larger than those in the far away fields. The adaptive remeshing technique is then proposed to automatically construct smaller triangular elements near the crack tip to produce an accurate solution and larger triangular elements in the far fields having small stress gradients. In addition, an accurate stress intensity factor near the crack tip is obtained using eight–node quadrilateral elements. An evaluation of the adaptive finite element method for linear elastic fracture problems is demonstrated by a single edge notched (SEN) problem and a circular cut out hole and centrally cracked plate problem.

2. Theory

2.1 Stress and Stress intensity factor

In linear elastic fracture mechanics, the stress intensity factor near a crack tip in mode I is related to stresses by the series expansion:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{1}{2}\theta\left(1 - \sin\frac{1}{2}\theta\sin\frac{3}{2}\theta\right) \quad (1a)$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{1}{2}\theta\left(1 + \sin\frac{1}{2}\theta\sin\frac{3}{2}\theta\right) \quad (1b)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta\cos\frac{3}{2}\theta\right) \quad (1c)$$

where K_I is the mode I stress intensity factor and r-θ is the in–plane polar coordinate system with the crack tip located at the origin as shown in

Fig. 1. In the Cartesian coordinate system σ_{xx}, σ_{yy} and τ_{xy} are the normal stresses in the x, y directions and shear stress in the x–y plane, respectively.

The stress intensity factor may be determined by several methods such as the displacement extrapolation near a crack tip [4], the J–integral [5], and the energy domain integral [6]. In this paper the displacement extrapolation is used. The stress intensity factor is calculated from:

$$K_I = \frac{2G}{3(1+\kappa)}\sqrt{\frac{2\pi}{L}}\left(4(v_b - v_d) - \frac{(v_c - v_e)}{2}\right) \tag{2}$$

where G is the shear modulus, κ is the elastic parameter defined by (3–4ν) for plane strain and (3–ν)/(1+ν) for plane stress, ν is the Poisson's ratio, and L is the element length. The subscripts for v, which is the y-displacement component, indicate their positions as shown in Fig. 2.

2.2 Finite Element Equations

The governing differential equations of the problem are:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{3a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \tag{3b}$$

By applying Bubnov–Galerkin finite element method [7] to Eq. (3a)–(3b), the finite element equations for determining nodal displacements are obtained. These equations can be written in matrix form as:

$$[K]\{\delta\} = \{F\} \tag{4}$$

where $[K] = \int_A [B]^T [C][B]\, t\, dA$ (5a)

$$\{\delta\}^T = \lfloor u_1 \quad v_1 \quad \cdots \quad u_n \quad v_n \rfloor \tag{5b}$$

$$\{F\} = \int_\Gamma \lfloor N \rfloor \{T\} d\Gamma \tag{5c}$$

In the above Eq. (5a)–(5c), $[B]$ is the strain–displacement matrix; $[C]$ is the material stiffness matrix that depends on plane stress or plane strain case; t is the element thickness; A is the element area, u and v are the nodal displacement in the x and y direction, respectively; $\lfloor N \rfloor$ is the element interpolation function matrix and $\{T\}$ is the surface traction matrix.

To obtain high resolution accuracy near the crack tip, the eight–node quadrilateral elements (Q8) that form up a circular zone surrounding the crack tip are used. These elements have their mid–side nodes displaced from their nominal position to quarter points of the tip as shown in Fig. 3. The radius of the circular zone is specified to be no longer than one–eight of the initial crack length, with roughly one element every 30° in the circumferential direction [4].

2.3 Adaptive remeshing technique

The adaptive remeshing technique generates an entirely new mesh based on the solution obtained from the previous mesh [8]. To determine appropriate element sizes at different locations in the domain, the Von Mises stress σ is used as an indicator for computing proper element sizes. As small elements are required in the region where changes in the Von Mises stress gradients are large, the second derivatives of the Von Mises stress at a point with respect to global coordinates x and y are needed. Using the concept of principal directions from a given state of stresses at a point, the principal quantities in the principal directions X and Y where the cross–derivatives vanish are determined:

$$\begin{bmatrix} \frac{\partial^2 \sigma}{\partial x^2} & \frac{\partial^2 \sigma}{\partial x \partial y} \\ \frac{\partial^2 \sigma}{\partial x \partial y} & \frac{\partial^2 \sigma}{\partial y^2} \end{bmatrix} \Longrightarrow \begin{bmatrix} \frac{\partial^2 \sigma}{\partial X^2} & 0 \\ 0 & \frac{\partial^2 \sigma}{\partial Y^2} \end{bmatrix} \tag{6}$$

The maximum principal quantities are then used to compute the proper element size, h_i, by requiring the error to be uniform for all elements [8]:

$$h_i^2 \lambda_i = h_{min}^2 \lambda_{max} = \text{constant} \tag{7}$$

where λ_i is the higher principal quantity of an element considered:

$$\lambda_i = \max\left(\left|\frac{\partial^2\sigma}{\partial X^2}\right|, \left|\frac{\partial^2\sigma}{\partial Y^2}\right|\right) \tag{8}$$

In Eq. (7), λ_{max} is the maximum principal quantity of all elements and h_{min} is the minimum element size specified by the user.

3. Application

Performance of the adaptive finite element method using the eight–node quadrilateral elements at the crack tip is evaluated by a single edge notched (SEN) plate under tension loading, and the circular cut out with centrally cracked plate under tension loading.

3.1 Single edge notched (SEN) problem

The problem statement in this case is presented in Fig. 4 [8]. A square flat plate, with dimension of the width b = 72 mm and the thickness t = 32 mm, has a crack length a = 30 mm in the middle of the left side. Plate's material properties are Young's modulus = 2.5E+09 Pa and Poisson's ratio = 0.38. The plate is under a tension stress, σ_{far} = 8.51 kPa. A finite element model can be generated on the upper half plate due to its symmetry condition. Figure 5 shows the first finite element model consisting of 3,036 nodes and 5,828 elements. At the crack tip, 6 eight–node quadrilateral elements (Q8) are constructed as shown in Fig. 5. After applying the tension loading and symmetrical boundary conditions, the displacements and stresses are computed. Figure 6 shows the Von Mises stress distribution. The computed Von Mises stress from the first finite element model is used to compute the proper mesh size based on the adaptive remeshing technique. Its final adaptive mesh as shown in Fig. 7 consists of 165 triangles, 6 quads, 6 eight–node quads, and 116 nodes for the upper half model. The computed stress intensity factor from the final adaptive mesh is 0.1788 comparing to that of the boundary collocation method [9], 0.1817, calculated by Eq. (9). The difference is less than 1.62 %.

$$K_I = F\sigma_{far}\sqrt{\pi a} \tag{9}$$

where $F = 1.12 - 0.231\alpha + 10.55\alpha^2 - 21.72\alpha^3 + 30.39\alpha^4$

$$\alpha = \frac{a}{b}$$

3.2 Circular cut out hole with centrally cracked plate problem

A rectangular plate with the width b = 80 mm and the thickness t = 32 mm has a circular hole at the center as shown in Fig. 8. There are cracks with length a = 10 mm at both sides of the hole. The plate is under tension loading, σ_{far} = 7.66 kPa. Based on symmetry, a finite element model is then created on one quarter of the plate. Fig. 9 displays the first finite element model consisting of 952 nodes and 1,756 elements. The stress solutions are computed as depicted in Fig. 10. After applying the adaptive remeshing technique, the final adaptive mesh consists of 168 triangles, 6 quads, 6 eight–node quads, and 124 nodes as shown in Fig. 11. The computed stress intensity factor from the solution of the final adaptive mesh is 0.07667. The difference is less than 1.73.%, as compared to the boundary collocation method [10] of K_I = 0.07802.

4. Conclusions

The applications of Q8 and adaptive meshes for crack problems were presented. The concepts of the combined method, the finite element equations and an adaptive remeshing technique, were then described. A single edge notched plate under tension loading and the circular cut out with centrally cracked plate under tension loading were selected to evaluate the performance of the adaptive finite element method using eight–node quadrilateral elements (Q8) at the crack tip. These examples demonstrated capability of the proposed method for solving linear fracture mechanics problem accurately.

5. Acknowledgements

The author is pleased to acknowledge the Thailand Research Fund (TRF) for supporting this research work.

6. References

[1] Smith DG and Smith CW Photoelastic Determination of Mixed Mode Stress Intensity Factors. *Exp. Mech.* Vol. 18, pp. 357–366, 1972.

[2] Ayatollahi MR, Pavier MJ, Smith DJ Determination of T–stress from Finite Element Analysis for Mode I and Mixed Mode I/II Loading. *Int. J. Fract.* Vol. 91, pp. 283–298, 1998.

[3] Phongthanapanich S., Dechaumphai P. Adaptive Delaunay Triangulation with Object–Oriented Programming for Crack Propagation Analysis. *Finite Elements in Analysis and Design* (in press.) 2004.

[4] Guinea G.V., Planas J., Elices M. K Evaluation by the Displacement Extrapolation Technique. *Eng. Fract. Mech.* Vol. 66, No. 3, pp. 243–255, 2000.

[5] Parks D.M. A Stiffness Derivative Finite Element Technique for Determination of Crack Tip Stress Intensity Factors. *Int. J. Fract.* Vol. 10, No. 4, pp. 487–502, 1974.

[6] Moran B., Shih C.F. A General Treatment of Crack Tip Contour Integrals. *Eng. Fract. Mech.* Vol. 35, pp. 295–310, 1987.

[7] Zienkiewicz O.C. and Taylor R.L. *Finite Element Method.*, 5th Edition, Butterworth–Heinemann, Woburn, 2000.

[8] Limtrakarn W. Crack Evaluation Using and Adaptive Finite Element Method. *The 8th Annual National Symposium on Computational Science and Engineering*, 2004.

[9] Benthem J. P. and Koiter W. T. Asymtotic Approximations to Crack Problems, Methods of Analysis and Solutions of Crack Problems, Mechanics of Fracture 1, Noordhoff Int. Pub., 1972.

[10]Newman J. C. Jr. An Improved Method of Collocation for the Stress Analysis of Cracked Plates with Various Shaped Boundaries. *NASA Report TN D-6376*, 1971.

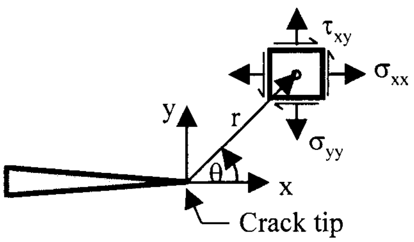

Fig. 1 In–plane polar and Cartesian coordinates.

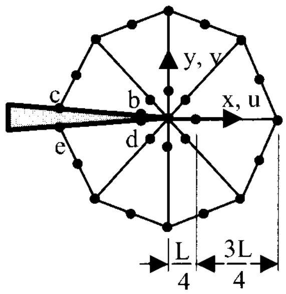

Fig. 2 Quarter–point quadrilateral elements around the crack tip.

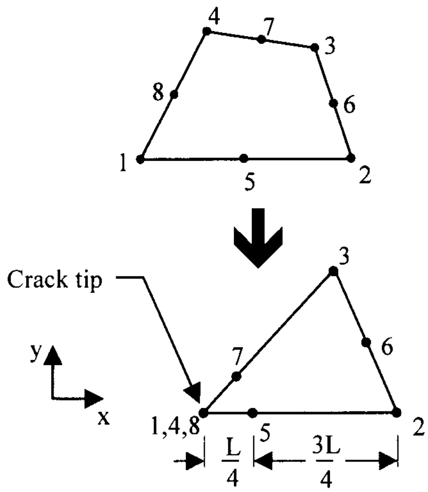

Fig. 3 A quarter–point eight–node quadrilateral elements (Q8).

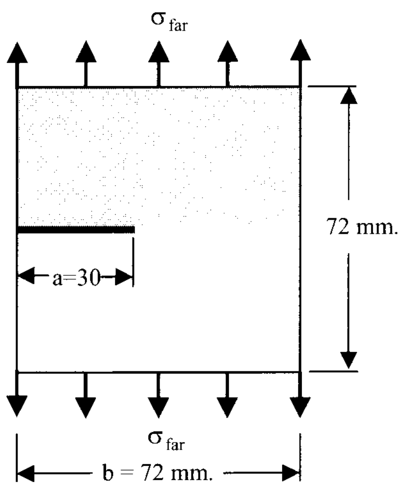

Fig. 4 Problem statement of single edge notched (SEN) plate under tension loading.

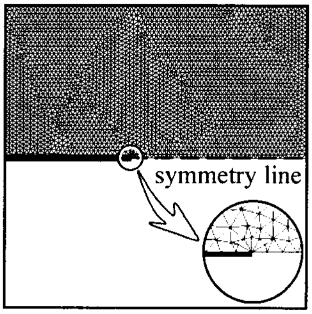

Fig. 5 First finite element model of single edge notched plate under tension loading.

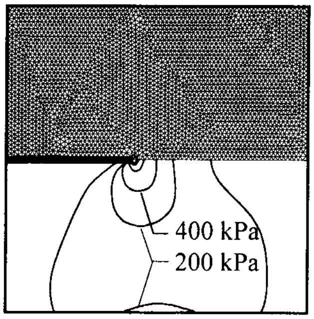

Fig. 6 Von Mises stress distribution of SEN.

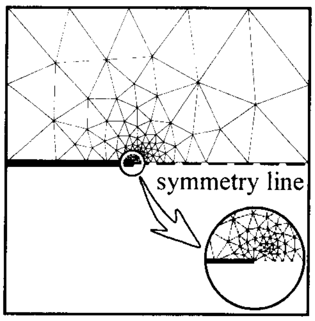

Fig. 7 Final adaptive mesh of SEN.

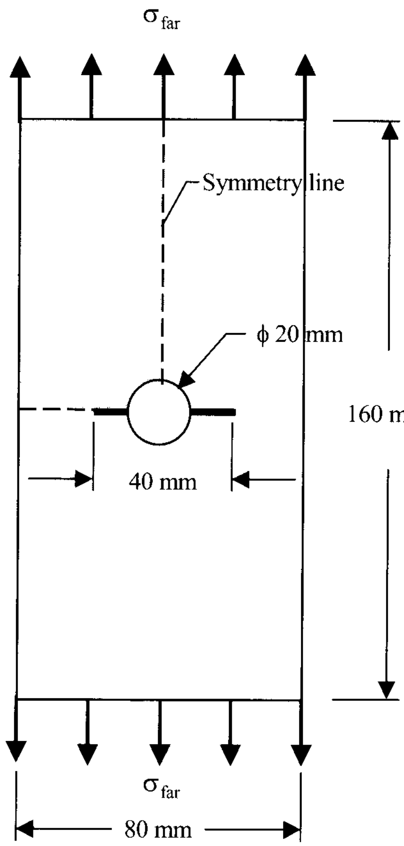

Fig. 8 Problem statement of circular hole with centrally cracked plate under tension loading.

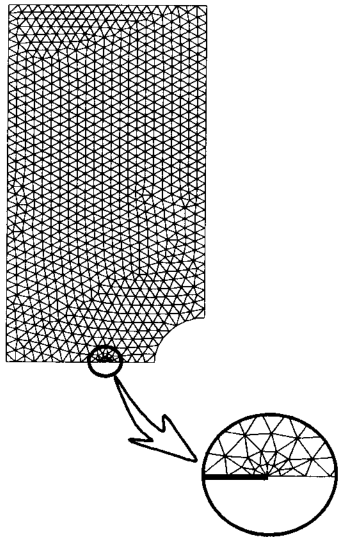

Fig. 9 First finite element model of circular hole and centrally cracked plate under tension loading.

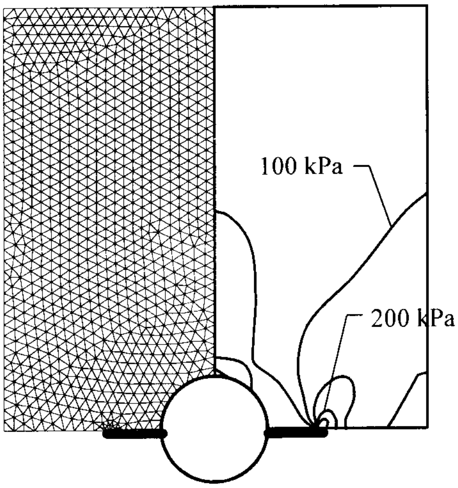

Fig. 10 Von Mises stress distribution of circular cut out hole and centrally cracked problem.

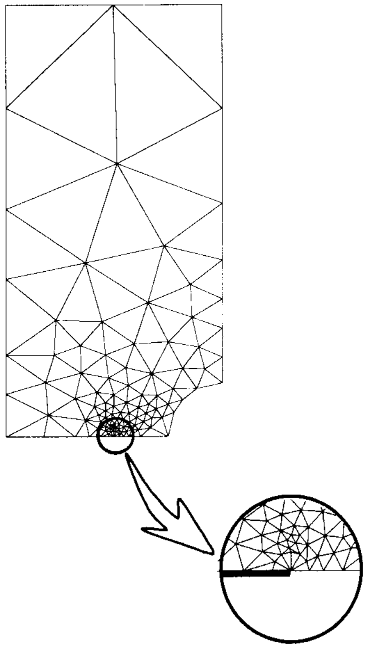

Fig. 11 Final adaptive mesh of circular hole and centrally cracked problem.