Stochastic Modeling for SET50 Index in Thailand Using the Black-Scholes Model with A Time-Dependent Drift Parameter and Its Application to SET50 Futures Pricing

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Abstract

In this paper we adopt the Black-Scholes model with a time-dependent drift parameter to describe the SET50 index which is a market-cap weighted index composed of the biggest 50 stocks traded in the Stock Exchange of Thailand (SET). An analytical formula for pricing futures contracts written on the SET50 index is derived by solving a partial differential equation (PDE) formulated for the futures contract pricing. Furthermore, we derive some interesting properties of the futures prices using our formula. We develop a method to estimate the model parameters by using the market data of SET50 and conduct the Monte-Carlo method to demonstrate the accuracy and efficiency of our formula. Finally, we display an evolution of futures prices over the lifetime of the futures contract with several maturity times.

Keywords: SET50 Index, Black-Scholes Model with Time-Dependent Parameters, PDE, Futures Prices.

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1. Introduction

The SET50 index is a stock market index calculated from the first fifty stock prices of companies in the Stock Exchange of Thailand (SET) ranking based on high market capitalization and high liquidity. The index can be used as a tool for financial managers or investors to describe the market movement. The method for calculating SET50 index is the same method that used for the SET index, called Market Capitalization Weighted [1]. The base date is 16 August 1995 with the base value of 1000 points. The list of companies in the SET50 index is modified every six months to take account of any changes that happened in the stock market. Therefore, the Black-Scholes model with time-dependent parameters (BS-T model) is suitable for describing the dynamics of the SET50 index in our investigation.

SET50 futures have been traded on Thailand Futures Exchange (TFEX) since April 28th, 2006. One of the advantages of SET50 futures is that they can be used to manage and reduce the investment risks. Because SET50 futures prices are related to the SET50 index, investors should consider whether those futures prices are reasonable or not. If those prices are not reasonable, there may be an arbitrage opportunity. However, we should not allow the arbitragers to get a profit without risk in the futures market.

From theorical point of view, a no-arbitrage futures price (or a fair futures price) at time t of a futures contract, having maturity at $T \geq t$, of a selected risky asset, denoted by F_t, is the expected value of the spot price of the risky asset at time T conditioned on the information at time t, i.e., $F_t = E_{\mathbb{Q}}\left[S_T \mid S_t\right]$ (see pages 61-62 in [2]), where S_t is the spot price of the risky asset at time $t \leq T$ in which we assume that S_t is a stochastic process (a collection of random variables indexed by time t) with respect to a risk-neutral probability measure $\mathbb{Q}$ (see pages 244-246 in [2]).

In terms of computing the no-arbitrage futures price F_t, the Monte-Carlo (MC) method [3] is a method of choice that can be adopted to obtain an approximate of F_t based on the law of large number such that the sample mean of spot prices $\bar{S}_T(n)$ converges to F_t as n approaches infinity. However, the major drawback of the MC method is time-consuming. In other words, it requires that n must be a sufficiently large number to ensure the convergence of $\bar{S}_T(n)$ to F_t.

Consequently, most of researchers avoid using the MC method and prefer to derive an analytical formula for futures prices based on the partial differential equation (PDE) method (see pages 242-244 in [2]). Therefore, the main contribution of the paper is to derive an analytical formula based on the PDE method for computing no-arbitrage futures prices for market participants trading SET50 futures contracts in TFEX.

The remaining of this paper is organized as follows. In Section 2, we adopt the BS-T model to describe the SET50 index and present a solution of the stochastic differential equation (SDE) for the SET50 index. In Section 3, by utilizing the Feynman-Kac theorem [4], we analytically solve a PDE formulated for the financial derivative pricing to obtain an analytical formula for pricing futures contracts based on the BS-T model and prove some interesting properties of our solution. We use the historical data of SET50 index from October 2nd, 2017 to March 29th, 2019 as showed in Figure 1, to estimate the model parameters and use the estimated parameters to demonstrate the accuracy and efficiency of our formula by comparing with the MC method. Furthermore, we display an evolution of futures prices over the lifetime of the futures contract with several maturities. We conclude the paper in Section 4.

2. Mathematical Model

Stochastic models play an important role in pricing derivatives such as futures or options under the no-arbitrage assumption. Due to our preliminary investigation concerning the daily return of SET50 index data from October 2017 to March 2019, we have found that the increasing (or decreasing) rates of SET50 index returns for short period of time exhibit linear trends with respect to time. On the other hand, the volatility of the index returns remains almost the same for short period of time. Therefore, we adopt the Black-Scholes model with a time-dependent drift parameter to describe the dynamics of SET50 index as follows.

Let S_t be the SET50 index value at time t and we assume that

$$dS_t = \mu(t)S_t dt + \sigma S_t dW_t \qquad (1)$$

for $t \geq 0$ and $S_0 > 0$ where $\mu(t) := \mu_0 + \mu_1 t$ represents an increasing (or decreasing) rate of the SET50 index at t depending on the parameters $\mu_0, \mu_1 \in \mathbb{R}$ and $\sigma > 0$ is the volatility of the SET50 index under a one-dimension Brownian motion W_t [5].

The next proposition can be used to show that the SET50 index as described by the SDE (1) is always positive that is $S_t > 0$ for all $t > 0$ if $S_0 > 0$ by considering a solution of the SDE (1).

Proposition 2.1

Let $t \in [0,\infty)$ and X_t be the SDE satisfying:

$$dX_t = a(t)X_t dt + b(t)X_t dW_t \ . \qquad (2)$$

Then,

$$X_t = X_0 e^{\left(\int_o^t \left(a(s) - \frac{1}{2}b^2(s)\right)ds + \int_0^t b(s)dW_s\right)} . \qquad (3)$$

Proof. See [6]. □

Theorem 2.2

Let $t \in [0,\infty)$ then

$$S_t = S_0 e^{\left((\mu_0 - \frac{\sigma^2}{2})t + (\frac{\mu_1}{2})t^2 + \sigma W_t\right)} > 0 \qquad (4)$$

providing that $S_0 > 0.$

Proof. By applying Proposition 2.1 with $a(t) = \mu(t)$ and $b(t) = \sigma$, we obtain

$$S_t = S_0 e^{\left(\int_0^t \left(\mu_0 + \mu_1 s - \frac{\sigma^2}{2}\right)ds + \int_0^t \sigma dW_s\right)}$$

$$= S_0 e^{\left((\mu_0 - \frac{\sigma^2}{2})t + (\frac{\mu_1}{2})t^2 + \sigma(W_t - W_0)\right)} .$$

Since $W_0 = 0$, we immediately obtain the solution of the SED (1) as in (4). Moreover, because the exponential term is greater than zero and $S_0 > 0$, S_t is positive for all $t \in (0,\infty)$ leading to the inequality (4) as desired. □

In the next section we shall apply the Feynman-Kac theorem [4] to derive an analytical formula for pricing contract.

3. Results and Discussion

3.1 No-arbitrage Futures Prices

Proposition 3.1 (The no-arbitrage assumption)

Under the no-arbitrage assumption in a futures market, the no-arbitrage futures price (fair-price) at time t with maturity date T, denoted by F, must equal to the expected value of its underlying commodity spot price at the maturity date T under the equivalent martingale measure $\mathbb{Q}$, i.e.,

$$F(S,\tau) = E_{\mathbb{Q}}\left[S_T \mid S_t = S\right] \qquad (5)$$

for $S>0$ and $\tau = T-t$ where the conditional expectation is taken under a risk-neutral probability measure $\mathbb{Q}$.

Proof. See [2].

Next, we shall apply the Feynman-Kac theorem [4] to derive an analytical formula for the conditional expectation (5) by solving the following PDE:

$$\frac{\partial F}{\partial \tau} = A^S(F) \tag{6}$$

where $A^s(F)$ is an infinitesimal generator of the SDE (2) as

$$A^s(F) = a(t)x\frac{\partial F}{\partial X} + \frac{1}{2}b^2(t)x^2\frac{\partial^2 F}{\partial X^2} \tag{7}$$

subject to the initial condition

$$F(S,0) = S. \tag{8}$$

In the following theorem, we show that the PDE (6) subject to the initial condition (8) has an explicit solution and thus we can obtain an analytical formula for the no-arbitrage futures prices.

Theorem 3.2 (No-arbitrage futures price formula) For given and fixed maturity date T, the solution to the PDE (6) subject to the initial condition (8) can be written as

$$F(S,\tau) = Se^{A(\tau)} \tag{9}$$

for all $S>0$ and $\tau \in [0,T]$ where

$$A(\tau) = (\mu_0 + \mu_1 T)\tau - \frac{\mu_1}{2}\tau^2. \tag{10}$$

Proof. Applying the Feynman-Kac theorem [5] to the fair futures price defined in (5), we have

$$\frac{\partial F}{\partial \tau} = A^s(F) \tag{11}$$

where $A^s(F)$ is an infinitesimal generator of the SDE (1) as

$$A^s(F) = (\mu_0 + \mu_1(T-\tau))S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}. \tag{12}$$

Replacing (12) in (11), we obtain a PDE:

$$\frac{\partial F}{\partial \tau} = (\mu_0 + \mu_1(T-\tau))S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2} \tag{13}$$

with an initial condition

$$F(S,0) = S. \tag{14}$$

Suppose that the solution of (13) with condition (14) can be written as

$$F(S,\tau) = Se^{A(\tau)} \tag{15}$$

where $A(\tau)$ is a time dependent function that will be determined later. So, we get

$$\frac{\partial F}{\partial \tau} = Se^{A(\tau)} \cdot A'(\tau), \tag{16}$$

$$\frac{\partial F}{\partial S} = e^{A(\tau)}, \tag{17}$$

$$\frac{\partial^2 F}{\partial S^2} = 0. \tag{18}$$

Substituting (16)-(18) into the PDE (13), we have that $A(\tau)$ must satisfy the ordinary differential equation:

$$A'(\tau) = \mu_0 + \mu_1(T - \tau) \tag{19}$$

for all $\tau \in (0,T]$. The initial condition (14) implies that

$$A(0) = 0. \tag{20}$$

Solve (19) with (20), we obtain

$$A(\tau) = (\mu_0 + \mu_1 T)\tau - \frac{\mu_1}{2}\tau^2.$$

Finally, we obtain the analytical solution for the no-arbitrage futures prices represented by (9). □

By utilizing our solution (9), we can deduce the following properties of the futures prices.

Corollary 3.3 According to Theorem 3.2, we have the following properties for the futures prices.

(P1) If $\mu_1 = 0$ and $\mu_0 = 0$ then $F(S,\tau) = S$ for all $\tau \ge 0$.

(P2) If $\mu_1 = 0$ and $\mu_0 > 0$ then $F(S,\tau)$ is a strictly increasing function with respect to τ on $[0,\infty)$.

(P3) If $\mu_1 = 0$ and $\mu_0 < 0$ then $F(S,\tau)$ is a strictly decreasing function with respect to τ on $[0,\infty)$.

(P4) If $\mu_1 > 0$ then $F(S,\tau)$ has a maximum value at $\tau = \tau_{\max} := \dfrac{\mu_0 + \mu_1 T}{\mu_1}$.

(P5) If $\mu_1 < 0$ then $F(S,\tau)$ has a minimum value at $\tau = \tau_{\min} := \dfrac{\mu_0 + \mu_1 T}{\mu_1}$.

Proof. (P1) - (P3) are obvious from the formulas (9)-(10). To show (P4) - (P5), we shall apply the first and second derivative tests for finding maximum and minimum values of $F(S,\tau)$. From (16) and (19), we have

$$\frac{\partial F}{\partial \tau}(S,\tau) = Se^{A(\tau)}A'(\tau) = Se^{A(\tau)}\left(\mu_0 + \mu_1(T - \tau)\right)$$

and this implies that $\dfrac{\partial F}{\partial \tau}(S,\tau) = 0$ if and only if $\tau = \tau^* := \dfrac{\mu_0 + \mu_1 T}{\mu_1}$. Next, we compute the second derivative of $F(S,\tau)$ with respect to τ at $\tau = \tau^*$. Then, we obtain

$$\frac{\partial^2 F}{\partial \tau^2}(S,\tau^*) = -\mu_1 Se^{\frac{(\mu_0+\mu_1 T)^2}{2\mu_1}}.$$

By considering the sign of μ_1, we immediately obtain (P4) and (P5). □

Corollary 3.4 According to Theorem 3.2, if we fix $S > 0$ and $t \geq 0$, we have the following properties for the futures prices.

(P6) If $\mu_0 = 0$ and $\mu_1 = 0$ then $F(S,T-t) = S$ for all $T \geq t$.

(P7) If $\mu_0 = 0$ and $\mu_1 > 0$ then $F(S,T-t)$ is a strictly increasing function with respect to T on $(0,\infty)$.

(P8) If $\mu_0 = 0$ and $\mu_1 < 0$ then $F(S,T-t)$ is a strictly decreasing function with respect to T on $(0,\infty)$.

(P9) If $\mu_1 = 0$ and $\mu_0 > 0$ then $F(S,T-t)$ is a strictly increasing function with respect to T on $(0,\infty)$.

(P10) If $\mu_1 = 0$ and $\mu_0 < 0$ then $F(S,T-t)$ is a strictly decreasing function with respect to T on $(0,\infty)$.

(P11) If $\mu_1 \neq 0$ and $\mu_0 \neq 0$ then $F(S,T-t)$ is a strictly decreasing with respect to T on $(-\infty,\bar{\mu})$ and strictly increasing function on $(\bar{\mu},\infty)$ where $\bar{\mu} = -\mu_0 / \mu_1$.

Proof. (P6) is obvious from the formulas (9) - (10). From formulas (9) - (10), we have

$$\frac{\partial F(S,T-t)}{\partial T} = Se^{A(T-t)}\frac{dA(T-t)}{dT}$$

$$= Se^{A(T-t)}(\mu_0 + \mu_1 T).$$

Since $Se^{A(T-t)} > 0$ for all T, the sign of $\dfrac{\partial F}{\partial T}$ depends on the sign of $\mu_0 + \mu_1 T$ thus we obtain (P7)-(P10). To show (P11), we consider $\dfrac{dA(T-t)}{dT}$ when $\mu_0 + \mu_1 T < 0$. We get that $F(S,T-t)$ is a strictly decreasing function with respect to T on $(-\infty,\bar{\mu})$. Similarly, when $\mu_0 + \mu_1 T > 0$, we have that $F(S,T-t)$ is a strictly increasing function with respect to T on $(\bar{\mu},\infty)$. We now obtain (P11). □

3.2 Parameter Estimation

It should be noted that our solutions (9) and (10) cannot be used to compute the futures prices without providing values of the model parameters μ_0 and μ_1. By using the historical data of the SET50 index as shown in Figure 1, we provide a method to estimate the model parameters as follows.

Since S_t follows the model (1) and the expectation of $S_t dW_t$ is zero [4]. After we take the expectation to the both sides of (1) then we have that the actual return of the SET50 index on date t can be approximated by

$$r_t := \frac{dS_t}{S_t} \approx (\mu_0 + \mu_1 t)dt \,. \tag{21}$$

Let S_n be the SET50 index on date t_n, r_n be the return (known) of the SET50 index on date t_n that is $r_n = \frac{S_{n+1} - S_n}{S_n}$ and $\mu_{0,n}, \mu_{1,n}$ be unknown parameters which can be determined from the n^{th} pair daily return from 366 data point over 18 months so we have 365 time steps. The unknown parameters can be estimated using the following procedure. For example, $n = 1$, we have

$$r_1 = (\mu_{0,1} + \frac{1}{365}\mu_{1,1})dt$$

and

$$r_2 = (\mu_{0,1} + \frac{2}{365}\mu_{1,1})dt.$$

By solving these two linear equations, we obtain the parameters as follow:

$$\mu_{0,1} = \frac{(2r_1 - r_2)}{dt} \text{ and } \mu_{1,1} = \frac{365(r_2 - r_1)}{dt}.$$

Since we have 183 pairs of the daily return of the SET50 index from October 2017 to March 2019. Similarly, we can estimate $\mu_{0,n}$ and $\mu_{1,n}$ for $n = 2, 3, \ldots, 183.$ By using the sample means as follows:

$$\mu_0 \approx \bar{\mu}_0 = \frac{1}{183}\sum_{n=1}^{183}\mu_{0,n} \approx -73.358 \tag{22}$$

$$\mu_1 \approx \bar{\mu}_1 = \frac{1}{183}\sum_{n=1}^{183}\mu_{1,n} \approx 92.182\,. \tag{23}$$

Substituting $\bar{\mu}_0$ and $\bar{\mu}_1$ into equation (9), we obtain the approximation of future prices $(\bar{F})$ as

$$F(S,\tau) \approx \bar{F}(S,\tau) := Se^{(\bar{\mu}_0 + \bar{\mu}_1 T)\tau - \frac{\bar{\mu}_1}{2}\tau^2} \tag{24}$$

when $S > 0$ and $\tau \in [0,T].$

3.3 Comparison with the MC method

We set a maturity time $T = 1$ and an initial spot price $S = 1000$. Then, the approximates the futures prices $\bar{F}(S,\tau)$ computed from (24) and $\bar{S}_T(n)$ obtained from the MC method for a number of samples $n = 10,000$ in which we apply the Euler-Maruyama approximation [6] to the SDE (1) to obtain $S_{t_i}(n)$ for all $t_i \in \{0.01, 0.02, \ldots, T\}$, are plotted on the time interval $[0,T]$ as shown in Figure 2.

One can clearly observe that the result from our analytical formula (24) perfectly match the result from the MC method. However, in terms of computational time, the MC method consumes 5.68 seconds while our analytical solution just consumes 0.05 seconds for computing one futures price. This is not surprising at all since time-consuming is a major drawback of the MC method.

3.3 Properties of futures prices

From (P4) in Corollary 3.3, one can determine

$$\tau_{\max} \approx \frac{-73.358 + 92.182(1)}{92.182} \approx 0.204 \in [0,1].$$

Therefore, the maximum value of $F(S,\tau)$ on the time interval $[0,T]$ can be approximated by $\bar{F}(S,\tau_{\max}) = 6834.44.$ The approximation of the futures prices $\bar{F}(S, T-t)$ at time t is shown in Figure 3. It shows that the maximum value of futures price occurs when $t = t_{\max} = T - \tau_{\max} \approx 0.796.$ Figure 4 shows the approximations of the futures prices $\bar{F}(S, T-t)$ when $t = 0.49, 0.50, 0.51$ and $S = 1000.$ Using (22)-(23) and (P11) in Corollary 3.4, we have that

$$\bar{\mu} = \frac{-\mu_0}{\mu_1} \approx \frac{73.358}{92.182} \approx 0.7958.$$

Therefore, $\bar{F}(S,T-t)$ is a strictly decreasing function with respect to T on $(0, 0.7958)$. On the other hand, $\bar{F}(S,T-t)$ is a strictly increasing function with respect to T on $(0.7958, \infty)$. We further illustrate a surface of $\bar{F}(S,T-t)$ over the lifetime of a futures contract in Figure 5 showing an evolution of futures prices for all $t \in [0, T]$.

4. Conclusions

The paper has adopted the Black-Scholes model with a time-dependent parameter to describe the SET50 index traded in the Stock Exchange of Thailand (SET). An analytical formula for pricing futures contracts written on the SET50 index has been derived by solving the PDE using the Feynman-Kac theorem. We also quantify the dependency of the futures prices on the model parameters. We have proposed a method to estimate the model parameters from the historical data of the SET50 index from 2017 to 2019 and conducted the MC method to demonstrate the accuracy and efficiency of our formula. Finally, we have investigated some interesting properties of futures prices based on the estimated values of parameters with several maturity times.

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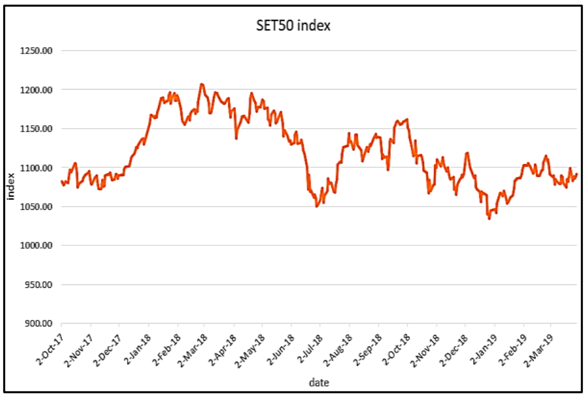

Figure 1 Historical data of SET50 index from October 2nd, 2017 to March 29th, 2019, used to estimate the model parameters

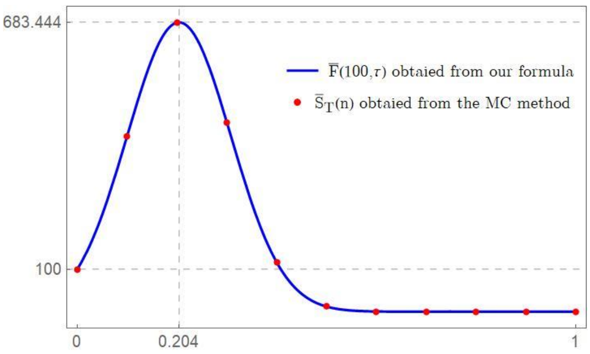

Figure 2 The approximates of the futures prices $\bar{F}(S,\tau)$ computed from (24) and $\bar{S}_T(n)$ obtained from the MC method for a number of samples $n = 10,000$

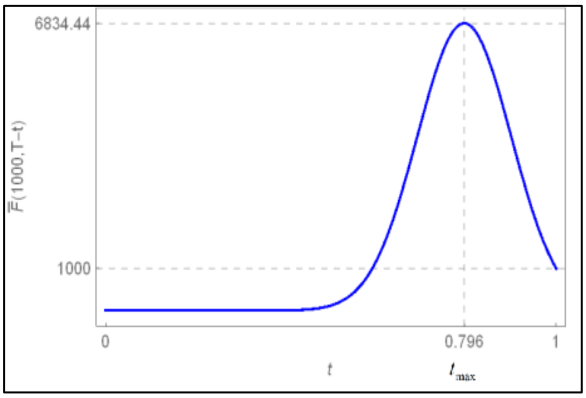

Figure 3 The approximate of the futures price $\bar{F}(S,\tau)$ as a function of t obtained with $S = 1000$ and $T = 1$

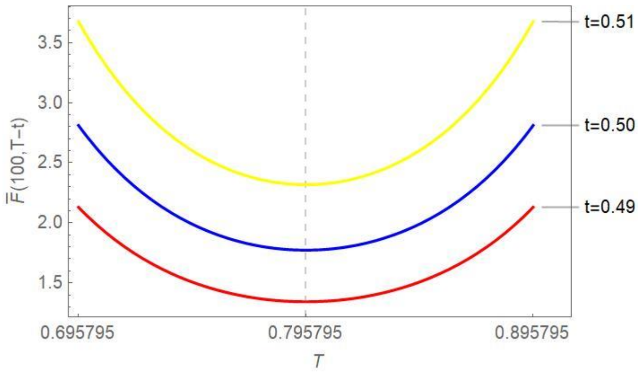

Figure 4 The approximates of the futures prices $\bar{F}(S,T-t)$ as a function of T obtained with $S = 1000$ on the interval $(\bar{\mu}-0.1, \bar{\mu}+0.1)$

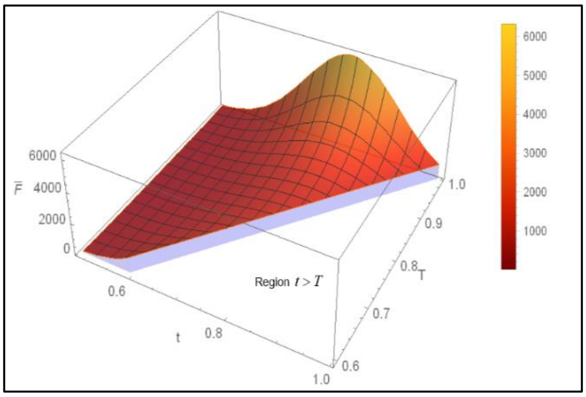

Figure 5 The evolution of the futures price $\bar{F}(S,T-t)$ as a function of t and T obtained with $S=1000$ for $t \in [0,T]$