A Two-parameter Weighted Inverse Lindley Distribution and Applications

Jirawat Kantalo[1*], and Jiraphan Suntornchost[2]

[1]*Mathematics and Computer Science, Chulalongkorn University, Bangkok, 10330, Thailand, jirawatkantalo@gmail.com*

[2]*Mathematics and Computer Science, Chulalongkorn University, Bangkok, 10330, Thailand, jiraphan.s@chula.ac.th*

Abstract

A two-parameter weighted inverse Lindley (TWIL) distribution, of which the inverse Lindley (IL) distribution is a particular case, has been introduced. It's properties such as survival function, hazard function, moments and other associated measures are obtained. Moreover, parameter estimations of the new distribution by the maximum likelihood estimation (MLE) and the bias-corrected maximum likelihood estimation (CMLE) are also provided. Simulation results and applications to two data sets show that the new distribution outperforms other extensions of the Lindley distribution.

***Keywords*:** Inverse Lindley distribution, Hazard function, Parameter estimation, Maximum Likelihood Estimators, Goodness of fit.

1. Introduction

The Lindley distribution, a mixture of an exponential distribution with a gamma distribution, was first proposed by Lindley (1958) to illustrate relations between fiducial distribution and posterior distribution. The distribution is based on a mixture of Exponential(θ) and Gamma($2,\theta$) with the mixing probability $\frac{\theta}{\theta+1}$, when θ is a positive real number. It's probability density function, for $x>0$, is given by

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}. \tag{1}$$

Later in 2008, Ghitany et al. studied it's properties and applied it to model waiting times and lifetime data. They showed that the Lindley distribution fitted their data better than the exponential distribution.

The distribution was found to be useful in a wide variety of fields such as lifetime modeling in medical sciences, engineering and biology. Therefore, the Lindley distribution has been extensively studied both in applications and distribution generalizations. Several new distributions extending the distribution have been proposed by either generalizing the mixed distributions or proposing adjustments to the mixing weight, for example, Ghitany et al.(2011) proposed a two-parameter weighted Lindley distribution which is an extension of the Lindley distribution with the probability density function, for $x>0$, is given by

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)}x^{\alpha-1}(1+x)e^{-\theta x}, \qquad \theta,\alpha>0. \tag{2}$$

Shanker et al. (2013) introduced a two-parameter Lindley distribution by modifying the mixing weight of the Lindley distribution for modeling waiting and survival times data and showed that the two-parameter distribution outperforms the original distribution. Elbatal et al. (2013) proposed a new class of Lindley distributions called new generalized Lindley distribution by mixing two gamma distributions. Nedjar and Zeghdoudi (2016) proposed the Gamma Lindley distribution, a mixture of gamma $(2, \theta)$ and one-parameter Lindley distributions. Shibu and Irshad (2016) obtained a new generalized Lindley distribution, which offers a more flexible distribution for modeling lifetime data. Ramos (2016) studied a new lifetime distribution, generalized weighted Lindley distribution which is a mixture of generalized gamma distributions.

Moreover, several inverse transformations of the family of Lindley distributions have been widely studied in statistics and other fields. For example, Sharma et al. (2015) used a mixture of inverse Exponential(θ) and inverse Gamma$(2, \theta)$ with the mixing probability $\dfrac{\theta}{\theta+1}$ to produce the inverse Lindley (IL) distribution with the probability density function, for $x > 0$, is given by

$$f(x;\theta) = \frac{\theta^2}{(1+\theta)}\frac{(1+x)}{x^3}e^{-\theta/x}, \qquad \theta > 0. \tag{3}$$

Recently,several generalizations of the inverse Lindley (IL) distributions have been studied. For example, Sharma et al. (2015) proposed a generalized inverse Lindley (GIL) distribution by considering a generalized inverse distribution transformation. Alkarni (2015) introduced a more flexible distribution with three parameters, extended inverse Lindley (EIL) distribution.

This paper offers a new generalization of the inverse Lindley (IL) distribution having two parameters called the *two-parameter weighted inverse Lindley (TWIL)* distribution. The distribution is a certain mixture of inverse Gamma(α, θ) and inverse Gamma$(\alpha+1, \theta)$ distributions with the mixing probability $\dfrac{\theta}{\theta+\alpha}$ when α and θ are parameters having positive real values. Various properties of the proposed distribution including the survival and hazard functions, moments, skewness, kurtosis, stochastic orderings and parameter estimations are provided. Moreover, the proposed distribution are compared to different variations of the Lindley distributions using two real data sets. The results show that the new distribution fits data better than other studied distributions.

This paper is organized as follows. In section 2, the two-parameter weighted inverse Lindley distribution is introduced and its properties are provided. Section 3 discusses parameter estimations by (1) Maximum likelihood estimation and (2) Bias-Corrected Maximum likelihood estimation. Sections 4 and 5 show simulation results and applications, respectively. Conclusion and discussion are provided in Section 6.

2. A two-parameter weighted inverse Lindley distribution and its properties

In this section, we present the definitions and some important properties of the two-parameter weighted inverse Lindley (TWIL) distribution.

Definition 1. *A continuous random variable X is said to follow the two-parameter weighted inverse Lindley (TWIL) distribution if its probability density function, for $x > 0$, can be expressed as a two-component mixture*

$$f(x) = pg_1(x) + (1-p)g_2(x), \tag{4}$$

where
$p = \dfrac{\theta}{\theta+\alpha}$ *for $\theta, \alpha > 0$ and for $i = 1, 2$, $g_i(x)$ is the probability density function of the inverse*

Gamma$(\alpha+i-1,\theta)$ *distribution defined as* $g_i(x)=\frac{\theta^{\alpha+i-1}}{\Gamma(\alpha+i-1)}x^{-\alpha-i}e^{-\theta/x}$.

Definition 2. *From Definition 1, we can say that a random variable X follows the two-parameter weighted inverse Lindley (TWIL) distribution if its probability density function (pdf), for $x>0$, is*

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)}(\frac{1+x}{x^{\alpha+2}})e^{-\theta/x}, \qquad \theta,\alpha>0. \tag{5}$$

From definitions 1 and 2, if $\alpha=1$, the distribution reduces to the inverse Lindley distribution with pdf (3).

The corresponding cumulative distribution function (cdf) of the TWIL distribution, for $x>0$, is given by

$$F(x:\theta,\alpha)=\frac{\Gamma(\alpha+1,\theta/x)+\theta\Gamma(\alpha,\theta/x)}{(\theta+\alpha)\Gamma(\alpha)}, \quad \theta,\alpha>0, \tag{6}$$

where $\Gamma(\alpha,\theta/x)=\int_{\theta/x}^{\infty}u^{\alpha-1}e^{-u}du$ is the upper incomplete gamma.

If we consider the inverse transformation $X=Y^{-1}$ where the random variable Y follows the two-parameter weighted Lindley distribution, we can obtain a new distribution as follows.

Theorem 3. *Let Y be a random variable following the two-parameter weighted Lindley distribution with the probability density function stated in (2), then its inverse $X=Y^{-1}$ is said to follow the two-parameter weighted inverse Lindley (TWIL) distribution with pdf stated in (5).*

Proof. We want to find the probability density function (pdf) of $X=Y^{-1}$, when Y follows the two-parameter weighted Lindley distribution. By density transformation and the definition of $f_Y(y)$ in (2), we obtain

$$\begin{aligned} f_X(x) &= f_Y(g^{-1}(x))\left|\frac{d}{dx}g^{-1}(x)\right| \\ &= \frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)}\frac{1}{x^{\alpha-1}}(\frac{1+x}{x})\frac{1}{x^2}e^{-\theta/x} \\ &= \frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)}(\frac{1+x}{x^{\alpha+2}})e^{-\theta/x}. \end{aligned}$$

Then, we have $f_X(x)$ is the probability density function (pdf) of TWIL distribution. □

Lemma 4. *The probability density function (pdf) of the two-parameter weighted inverse Lindley (TWIL) distribution is unimodal such that the x_{mode}, a value at which the probability density function (pdf) attains its maximum value, can be obtained as the solution of the first derivative of $f(x)$, is obtained from (7) as*

$$\frac{d}{dx}f(x)=-\frac{\theta^{\alpha+1}e^{-\theta/x}}{(\theta+\alpha)\Gamma(\alpha)x^{\alpha+4}}[(\alpha+1)x^2-(\theta-\alpha-2)x-\theta] \tag{7}$$

and $\frac{d}{dx}f(x)|_{x=x_{mode}}=0$. *Therefore,* x_{mode}, *the mode of the two-parameter weighted inverse Lindley random variable, is given by*

$$x_{mode}=\frac{(\theta-\alpha-2)+\sqrt{(\theta-\alpha-2)^2+4(\alpha+1)\theta}}{2(\alpha+1)}. \tag{8}$$

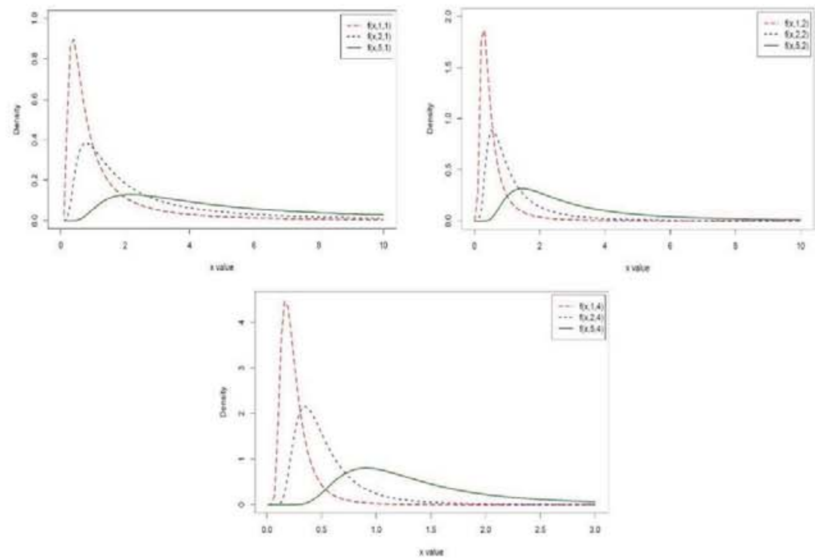

Figure 1: Plots of pdfs, $f(x;\theta,\alpha)$ of the TWIL distribution for some selected choices of α and θ.

In Figure 1, plots of pdf of the two-parameter weighted inverse Lindley distribution for some values of θ,α are presented. From the figure, we can see that those density functions are unimodal and x_{mode} increases when θ increases.

2.1 Survival and hazard functions

The survival and hazard functions of the two-parameter weighted inverse Lindley distribution are respectively given by

$$S(x) = 1 - F(x;\theta,\alpha) = 1 - \frac{\Gamma(\alpha+1,\theta/x) + \theta\Gamma(\alpha,\theta/x)}{(\theta+\alpha)\Gamma(\alpha)} \quad x>0,\theta,\alpha>0 \tag{9}$$

and

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta^{\alpha+1}(1+x)e^{-\theta/x}}{x^{\alpha+2}[(\theta+\alpha)\Gamma(\alpha) - (\theta\Gamma(\alpha,\theta/x) + \Gamma(\alpha+1,\theta/x))]} \quad x>0,\theta,\alpha>0. \tag{10}$$

The behaviors of hazard functions of the TWIL distribution, for different values of the parameter, are shown graphically in Figure 2. From the figure, we can see that those hazard functions first increase and then decrease and they are unimodal.

2.2 Moments and associated measures

Many important features and properties of a distribution such as mean, variance, kurtosis and skewness can be obtained through its moments. Therefore, in this section, we present some important moments, such as the moment and central moment among others.

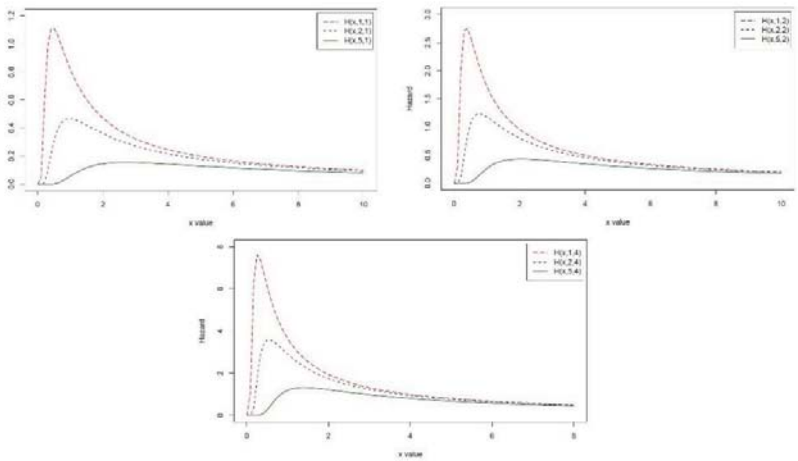

Figure 2: Plots of the hazard rate functions, $H(x;\theta,\alpha)$ of the TWIL distribution for different values of α and θ.

Theorem 5. *Let X be a random variable that follows the TWIL distribution with pdf as (5), then the r^{th} moment (about the origin) is given by*

$$\mu_r' = E(X^r) = \frac{\theta^r(\theta+\alpha-r)\Gamma(\alpha-r)}{(\theta+\alpha)\Gamma(\alpha)} \quad \alpha > r. \tag{11}$$

Proof: For $X \sim$TWIL(θ,α), we have

$$\begin{aligned}
\mu_r' = E(X^r) &= \int_{-\infty}^{\infty} x^r f(x)\,dx \\
&= \frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)} \int_0^{\infty} x^r \Big(\frac{1+x}{x^{\alpha+2}}\Big) e^{-\theta/x}\,dx \\
&= \frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)} \int_0^{\infty} \Big(\frac{1}{x^{\alpha-r+2}} + \frac{1}{x^{\alpha-r+1}}\Big) e^{-\theta/x}\,dx \\
&= \frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)} \Big(\frac{\Gamma(\alpha-r+1)}{\theta^{\alpha-r+1}} + \frac{\Gamma(\alpha-r)}{\theta^{\alpha-r}}\Big) \\
&= \frac{\theta^r(\theta+\alpha-r)\Gamma(\alpha-r)}{(\theta+\alpha)\Gamma(\alpha)}, \quad \alpha > r,
\end{aligned} \tag{12}$$

where we use the fact that $\int_0^{\infty} x^{-b-1} e^{-a/x}\,dx = \frac{\Gamma(b)}{a^b}$ in (12).

From theorem 5, we can obtain *skewness* and *kurtosis* of the two-parameter weighted inverse Lindley (TWIL) distribution as follows

Proposition 6. *Let X be a random variable following the two-parameter weighted inverse Lindley (TWIL) distribution. Then its skewness and kurtosis are*

$$\text{skewness} = \frac{\theta^3\Big((\theta+\alpha)^2\big(\theta\Gamma(\alpha-3)+\Gamma(\alpha-2)\big)\Gamma^2(\alpha) - 3(\theta+\alpha)\big(\theta\Gamma(\alpha-2)+\Gamma(\alpha-1)\big)\cdot\Gamma(\alpha)\big(\theta\Gamma(\alpha-1)+\Gamma(\alpha)\big) + 2\big(\theta\Gamma(\alpha-1)+\Gamma(\alpha)\big)^3\Big)}{(\theta+\alpha)^3\Gamma^3(\alpha)\left(\frac{\theta^2\left((\theta+\alpha)\Gamma(\alpha)(\theta\Gamma(\alpha-2)-\Gamma(\alpha-1))-(\theta\Gamma(\alpha-1)+\Gamma(\alpha))^2\right)}{(\theta+\alpha)^2\Gamma^2(\alpha)}\right)^{3/2}},$$

$$\text{kurtosis} = \frac{\Big((\theta+\alpha)^3\Gamma^3(\alpha)\big(\theta\Gamma(\alpha-4)+\Gamma(\alpha-3)\big) - 4(\theta+\alpha)^2\Gamma^2(\alpha)\big(\theta\Gamma(\alpha-3)+\Gamma(\alpha-2)\big)\cdot\big(\theta\Gamma(\alpha-1)+\Gamma(\alpha)\big) + 6(\theta+\alpha)\Gamma(\alpha)\big(\theta\Gamma(\alpha-2)+\Gamma(\alpha-1)\big)\big(\theta\Gamma(\alpha-1)+\Gamma(\alpha)\big)^2 - 3\big(\theta\Gamma(\alpha-1)+\Gamma(\alpha)\big)^4\Big)}{\Big(-(\theta+\alpha)\Gamma(\alpha)\big(\theta\Gamma(\alpha-2)+\Gamma(\alpha-1)\big) + \big(\theta\Gamma(\alpha-1)+\Gamma(\alpha)\big)^2\Big)^2}.$$

Remark 7. *The skewness and kurtosis of the two-parameter weighted inverse Lindley (TWIL) distribution has the following properties.*

1. *The skewness is an increasing function in $\theta > 0$ for fixed $\alpha > 3$.*
2. *The kurtosis is an increasing function in $\theta > 0$ for fixed $\alpha > 4$.*

2.3 Stochastic ordering

A random variable X is said to be smaller than a random variable Y in the following contexts, stochastic order $(X \le_{st} Y)$ if $F_X(x) \le F_Y(x)$, hazard rate order $(X \le_{hr} Y)$ if $h_Y(x) \le h_X(x)$, mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \le m_Y(x)$ and likelihood ratio order $(X \le_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreasing function of x for all x.

Shaked and Shanthikumar (1994) stated the results for stochastic ordering of distribution as follows:

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

The following theorem shows that the TWIL distribution is ordered with respect to "likelihood ratio" ordering.

Theorem 8. *Let X and Y be two independent random variables following the two-parameter weighted inverse Lindley (TWIL) distribution with parameters (θ_1, α_1) and (θ_2, α_2), respectively. If $\theta_1 \le \theta_2$ and $\alpha_1 = \alpha_2$ (or $\theta_1 = \theta_2$ and $\alpha_1 \ge \alpha_2$), then $X \le_{lr} Y$ for all x.*

Proof. Given that

$$X \sim TWIL(\theta_1, \alpha_1) \quad \text{and} \quad Y \sim TWIL(\theta_2, \alpha_2)$$

Then, the likelihood ratio is given by

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^{\alpha_1+1}}{\theta_2^{\alpha_2+1}}\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)}\left(\frac{\theta_2+\alpha_2}{\theta_1+\alpha_1}\right)\frac{x^{\alpha_2+1}}{x^{\alpha_1+2}}\exp\left\{\frac{\theta_2-\theta_1}{x}\right\}.$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^{\alpha_1+1}(\theta_2+\alpha_2)\Gamma(\alpha_2)}{\theta_2^{\alpha_2+1}(\theta_1+\alpha_1)\Gamma(\alpha_1)}\right] + \log\left(\frac{x^{\alpha_2+1}}{x^{\alpha_1+1}}\right) + \left(\frac{\theta_2-\theta_1}{x}\right).$$

Thus

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{(\alpha_2-\alpha_1)}{x} - \frac{(\theta_2-\theta_1)}{x^2}.$$

We have $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0$ if $\alpha_1 = \alpha_2$ and $\theta_1 \leq \theta_2$ or $\theta_1 = \theta_2$ and $\alpha_1 \geq \alpha_2$, this mean that $X \leq_{lr} Y$. □

3. Parameter estimations

In this section, we study parameter estimations of θ and α for the two-parameter weighted inverse Lindley (TWIL) distribution by two models: (1) Maximum likelihood estimation and (2) Bias-Corrected maximum likelihood estimation.

3.1 Maximum likelihood estimation

Let $x_1, \ldots, x_n$ be a random sample of size n from the two-parameter weighted inverse Lindley (TWIL) distribution with pdf in (5). The log-likelihood function is given by

$$\ln L(\alpha,\theta|x_1,\ldots,x_n) = n\ln\left(\frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)}\right) + \sum_{i=1}^{n}\ln(1+x_i) - (\alpha+2)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\frac{\theta}{x_i}. \quad (13)$$

Therefore the MLE of $\hat{\theta}_{MLE}$ and $\hat{\alpha}_{MLE}$ for the parameters θ and α can be obtained by solving the following equation as the simultaneous solution of the following nonlinear equations:

$$0 = \frac{\partial \ln L(\alpha,\theta|x_1,\ldots,x_n)}{\partial\alpha} = n\ln(\theta) - \frac{n}{\theta+\alpha} - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - \sum_{i=1}^{n}\ln(x_i) \quad (14)$$

$$0 = \frac{\partial \ln L(\alpha,\theta|x_1,\ldots,x_n)}{\partial\theta} = \frac{n(\alpha+1)}{\theta} - \frac{n}{\theta+\alpha} - \sum_{i=1}^{n}\frac{1}{x_i} \quad (15)$$

Then the $\hat{\theta}_{MLE}$ can be obtained in terms of α as

$$\hat{\theta} = \frac{\hat{\alpha}(n-S)+\sqrt{\hat{\alpha}}\sqrt{\hat{\alpha}n^2+nS(4+2\hat{\alpha})+\hat{\alpha}S^2}}{2S} \qquad \text{where} \quad S = \sum_{i=1}^{n}\frac{1}{x_i}.$$

To find $\hat{\alpha}$, we substitute $\hat{\theta}$ in

$$n\ln(\theta) - \frac{n}{\theta+\alpha} - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - \sum_{i=1}^{n}\ln(x_i) = 0. \quad (16)$$

Note that (16) cannot be solved analytically by so numerical iteration technique, the Newton-Raphson algorithm, is used to solve (16).

As an alternative to the maximum likelihood estimates, we can also consider another analytically bias-corrected MLE introduced by Firth (1993). The method involves modifying the score vector of the log-likelihood function *prior* to solving for the MLE as follows.

3.2 Bias-Corrected maximum likelihood estimation

For some arbitrary distribution, let $l(\tau)$ be the log-likelihood function with a p-dimensional vector of unknown parameters $\tau = (\tau_1, \ldots, \tau_p)'$ based on a sample of n observations.

The joint cumulants of the derivatives of $l(\tau)$ are

$$k_{ij} = E\left[\frac{\partial^2 l}{\partial\tau_i \partial\tau_j}\right], \qquad \text{for } i, j = 1, 2, \ldots, p, \tag{17}$$

$$k_{ijl} = E\left[\frac{\partial^3 l}{\partial\tau_i \partial\tau_j \partial\tau_l}\right], \qquad \text{for } i, j, l = 1, 2, \ldots, p, \tag{18}$$

$$k_{ij,l} = E\left[\left(\frac{\partial^2 l}{\partial\tau_i \partial\tau_j}\right)\left(\frac{\partial l}{\partial\tau_l}\right)\right], \qquad \text{for } i, j, l = 1, 2, \ldots, p, \tag{19}$$

$$k_{ij}^{l} = \frac{\partial k_{ij}}{\partial\tau_l}, \qquad \text{for } i, j, l = 1, 2, \ldots, p. \tag{20}$$

All of the k expressions are assumed to be of order $O(n)$. Cox and Snell (1968) showed that when the sample data are independent but not necessarily identically distributed, the bias of the s^{th} element of the maximum likelihood estimation can be written as

$$Bias(\hat{\tau}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} k^{si}k^{jl}\left[\frac{1}{2}k_{ijl} + k_{ij,l}\right] + O(n^{-2}), \qquad \text{for } s = 1, 2, \ldots, p, \tag{21}$$

where k^{ij} is the $(i,j)^{th}$ element of inverse of the Fisher's information matrix $K = \{-k_{ij}\}$.

Cordeiro and Klein (1994) showed that when all of the k terms are of order $O(n)$, the bias expression (21) still holds even if observations are not independent, it can be re-written as

$$Bias(\hat{\tau}_s) = \sum_{i=1}^{p} k^{si}\sum_{j=1}^{p}\sum_{l=1}^{p}\left[k_{ij}^{(l)} - \frac{1}{2}k_{ijl} + k_{ij,l}\right] + O(n^{-2}), \qquad \text{for } s = 1, 2, \ldots, p, \tag{22}$$

Define $a_{ij}^{(l)} = k_{ij}^{(l)} - \frac{1}{2}k_{ijl}$ for $i, j, l = 1, \ldots, p$. and define the following matrices

$$A = \left[\, A^{(1)} \;\mid\; A^{(2)} \;\mid\; \ldots \;\mid\; A^{(p)} \,\right] \text{ with } A^{(l)} = \left[a_{ij}^{(l)}\right].$$

A bias-corrected MLE for τ, denote $\hat{\tau}^{CMLE}$, can then be obtained as

$$\hat{\tau}^{CMLE} = \hat{\tau} - \hat{K}^{-1}\hat{A}\cdot \text{vec}(\hat{K}^{-1}),$$

where $\hat{\tau}$ is the MLE of the unknown parameter τ, $\hat{K} = K|_{\tau=\hat{\tau}}$ and $\hat{A} = A|_{\tau=\hat{\tau}}$.

For the two-parameter weighted inverse Lindley (TWIL) distribution, the Fisher information matrix for the TWIL distribution is given by

$$K = -n\begin{bmatrix} \dfrac{1}{(\theta+\alpha)^2} - \dfrac{(\alpha+1)}{\theta^2} & \dfrac{1}{\theta} + \dfrac{1}{(\theta+\alpha)^2} \\ \dfrac{1}{\theta} + \dfrac{1}{(\theta+\alpha)^2} & \dfrac{1}{(\theta+\alpha)^2} - \psi'(\alpha) \end{bmatrix}, \tag{23}$$

and the matrix of A can thus be written as

$$A = \left[A^{(1)} \quad | \quad A^{(2)}\right],$$

with

$$A^{(1)} = \begin{bmatrix} \frac{\alpha n\left(\theta^3 + 3\theta^2(\alpha+1) + 3\theta\alpha(\alpha+1) + \alpha^2(\alpha+1)\right)}{\theta^3(\theta+\alpha)^3} & -\frac{n\left(\theta^3 + 3\theta\alpha^2 + \alpha^3 + \theta^2(3\alpha+2)\right)}{2\theta^2(\theta+\alpha)^3} \\ -\frac{n\left(\theta^3 + 3\theta\alpha^2 + \alpha^3 + \theta^2(3\alpha+2)\right)}{2\theta^2(\theta+\alpha)^3} & -\frac{n}{(\theta+\alpha)^3} \end{bmatrix} \quad (24)$$

and

$$A^{(2)} = \begin{bmatrix} -\frac{n\left(\theta^3 + 3\theta\alpha^2 + \alpha^3 + \theta^2(3\alpha+2)\right)}{2\theta^2(\theta+\alpha)^3} & -\frac{n}{(\theta+\alpha)^3} \\ -\frac{n}{(\theta+\alpha)^3} & -\frac{n}{(\theta+\alpha)^3} - \frac{\psi''(\alpha)}{2} \end{bmatrix}, \quad (25)$$

where $\psi'(\alpha)$ and $\psi''(\alpha)$ are the first and second derivatives of $\psi(\alpha)$, respectively.

The bias-corrected estimators of the MLE of the TWIL distribution can be obtained as

$$\begin{pmatrix} \hat{\theta}^{CMLE} \\ \hat{\alpha}^{CMLE} \end{pmatrix} = \begin{pmatrix} \hat{\theta} \\ \hat{\alpha} \end{pmatrix} - \hat{K}^{-1}\hat{A}\cdot \text{vec}(\hat{K}^{-1}), \quad (26)$$

where $\hat{K} = K|_{\theta=\hat{\theta},\alpha=\hat{\alpha}}$ and $\hat{A} = A|_{\theta=\hat{\theta},\alpha=\hat{\alpha}}$.

4. Simulation study

In this section, we consider numerical simulations to study the behavior of the parameters θ and α with two different methods, the maximum likelihood estimation (MLE) and the Bias-Corrected maximum likelihood estimation (CMLE) in algorithm.

4.1 Algorithm 1

The algorithm has the following steps,

1. Generate $U_i \sim Uniform(0,1), \quad i = 1, 2, \ldots, n.$
2. If $U_i \leq \frac{\theta}{\theta+\alpha}$; $\quad X_i \sim InverseGamma(\alpha, \theta)$,
   otherwise; $\quad X_i \sim InverseGamma(\alpha+1, \theta)$
3. Estimate parameters $\hat{\theta}$ and $\hat{\alpha}$ by using MLE and CMLE methods.

Using the above algorithm to generate a random sample, our simulation study is carried out for $N = 1,000$ times for each triple (α, θ, n) with different values α and θ, for $n = 20, 40, 60, 80, 100$. In our study, we calculate the following measures.

1. Average bias (AB) of the estimators $\hat{\theta}$ and $\hat{\alpha}$.
2. Average root mean square error $(RMSE)$ of the estimators $\hat{\theta}$ and $\hat{\alpha}$.

α	n	θ					
		0.5		1.5		3	
		MLE	CMLE	MLE	CMLE	MLE	CMLE
2	20	0.0883	0.0804	0.2374	0.2001	0.5179	-11.2192
		(0.2277)	(0.2244)	(0.6445)	(0.9546)	(1.3542)	(235.3890)
	40	0.0412	0.0373	0.1190	0.1070	0.2374	-0.0535
		(0.1295)	(0.1282)	(0.3800)	(0.3773)	(0.7772)	(3.7912)
	60	0.0255	0.0229	0.0785	0.0706	0.1504	0.0449
		(0.1014)	(0.1007)	(0.2950)	(0.2934)	(0.5736)	(1.6619)
	80	0.0188	0.0169	0.0556	0.0498	0.1089	0.0542
		(0.0811)	(0.0806)	(0.2406)	(0.2395)	(0.4907)	(0.5623)
	100	0.0153	0.0138	0.0440	0.0394	0.0873	0.0520
		(0.0732)	(0.0729)	(0.2186)	(0.2179)	(0.4336)	(0.4443)
2.5	20	0.0878	0.0810	0.2610	0.2426	0.5149	0.2822
		(0.2297)	(0.2268)	(0.6682)	(0.6613)	(1.3390)	(7.0135)
	40	0.0448	0.0414	0.1128	0.1038	0.2262	0.2017
		(0.1352)	(0.1340)	(0.3877)	(0.3851)	(0.7533)	(0.7531)
	60	0.0274	0.0252	0.0737	0.0677	0.1333	0.1182
		(0.1025)	(0.1018)	(0.2921)	(0.2906)	(0.5746)	(0.5743)
	80	0.0204	0.0188	0.0526	0.0481	0.0937	0.0825
		(0.0852)	(0.0848)	(0.2422)	(0.2413)	(0.4784)	(0.4783)
	100	0.0204	0.0188	0.0442	0.0406	0.0717	0.0629
		(0.0738)	(0.0735)	(0.2146)	(0.2139)	(0.4274)	(0.4273)
3	20	0.0900	0.0840	0.2449	0.2292	0.4753	0.4448
		(0.2309)	(0.2285)	(0.6591)	(0.6530)	(1.2760)	(1.2689)
	40	0.0413	0.0383	0.1153	0.1075	0.2524	0.2380
		(0.1333)	(0.1324)	(0.3843)	(0.3818)	(0.7975)	(0.7941)
	60	0.0267	0.0247	0.0789	0.0737	0.1657	0.1562
		(0.1015)	(0.1009)	(0.2961)	(0.2947)	(0.6062)	(0.6042)
	80	0.0212	0.0197	0.0558	0.0519	0.1280	0.1209
		(0.0867)	(0.0864)	(0.2519)	(0.2510)	(0.5098)	(0.5084)
	100	0.0144	0.0132	0.0456	0.0425	0.0939	0.0883
		(0.0750)	(0.0748)	(0.2139)	(0.2132)	(0.4330)	(0.4321)

Table 1: Average bias (average RMSE) of the simulated estimates of θ.

Tables 1 and 2 represent average biases (AB) and the average root mean square errors $(RMSE)$ for fixed α and θ which are $2, 2.5, 3$ and $0.5, 1, 3$, respectively. From the two tables, we can see that the two different estimates, MLEs and CMLEs, give similar values of average bias (AB) and the average root mean square error $(RMSE)$ and those values decrease to zero as n increases.

α	n	θ					
		0.5		1.5		3	
		MLE	CMLE	MLE	CMLE	MLE	CMLE
2	20	0.4208	0.4256	0.3526	0.3430	0.3442	-7.4726
		(1.1108)	(1.1126)	(0.9672)	(1.3614)	(0.9382)	(154.5586)
	40	0.1986	0.2010	0.1719	0.1758	0.1655	-0.0156
		(0.6444)	(0.6451)	(0.5655)	(0.5693)	(0.5400)	(2.6240)
	60	0.1279	0.1295	0.1125	0.1154	0.1065	0.0445
		(0.5015)	(0.5019)	(0.4444)	(0.4465)	(0.4042)	(1.2620)
	80	0.0946	0.0958	0.0792	0.0815	0.0753	0.0463
		(0.4054)	(0.4057)	(0.3569)	(0.3582)	(0.3422)	(0.4060)
	100	0.0747	0.0757	0.0659	0.0677	0.0602	0.0431
		(0.3633)	(0.3635)	(0.3264)	(0.3273)	(0.3038)	(0.3182)
2.5	20	0.5221	0.5262	0.4815	0.4922	0.4522	0.3232
		(1.3602)	(1.3617)	(1.2479)	(1.2537)	(1.1929)	(4.8895)
	40	0.2716	0.2736	0.2103	0.2159	0.1987	0.1999
		(0.8204)	(0.8210)	(0.7247)	(0.7268)	(0.6710)	(0.6797)
	60	0.1604	0.1617	0.1398	0.1436	0.1240	0.1256
		(0.6207)	(0.6210)	(0.5480)	(0.5492)	(0.5135)	(0.5179)
	80	0.1228	0.1238	0.0958	0.0986	0.0786	0.0800
		(0.5211)	(0.5213)	(0.4580)	(0.4588)	(0.4260)	(0.4288)
	100	0.0873	0.0881	0.0838	0.0861	0.0603	0.0616
		(0.4449)	(0.4451)	(0.4008)	(0.4014)	(0.3725)	(0.3745)
3	20	0.6396	0.6431	0.5425	0.5528	0.4992	0.5131
		(1.6577)	(1.6590)	(1.4780)	(1.4821)	(1.3508)	(1.3620)
	40	0.2935	0.2952	0.2543	0.2595	0.2686	0.2763
		(0.9618)	(0.9623)	(0.8574)	(0.8591)	(0.8472)	(0.8515)
	60	0.1941	0.1953	0.1750	0.1784	0.1760	0.1812
		(0.7313)	(0.7316)	(0.6603)	(0.6613)	(0.6369)	(0.6394)
	80	0.1452	0.1460	0.1255	0.1281	0.1294	0.1333
		(0.6163)	(0.6165)	(0.5616)	(0.5622)	(0.5378)	(0.5394)
	100	0.1027	0.1034	0.0972	0.0993	0.0991	0.1023
		(0.5371)	(0.5372)	(0.4757)	(0.4762)	(0.4645)	(0.4656)

Table 2: Average bias (average RMSE) of the simulated estimates of α.

5. Applications

In this section, we discuss applications of our dustribution to two real data sets which are used extensively in comparing different extensions of the Lindley distribution and lifetime distributions. The first data set, called `data1`, discussed in Dumonceaux and Antle (1973) presenting the flood levels for the Susquehanna River at Harrisburg, Pennsylvania, over 20 four-year periods from 1890 to 1969. The data set was obtained in a civil engineering context and give the maximum flood level (in millions of cubic feet per second).

0.654	0.613	0.315	0.449	0.297	0.402	0.379	0.423
0.379	0.324	0.269	0.740	0.418	0.412	0.494	0.416
0.338	0.392	0.484	0.265				

Table 3: The data represent the flood level data for the Susquehanna river.

The second data set, called `data2` strength data, which was originally reported in Bader and Priest (1982) representing the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10 mm with sample size ($n = 63$).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445	2.454
2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.659
2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937	2.977	2.996	3.030
3.125	3.139	3.145	3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332
3.346	3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628	3.852
3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020			

Table 4: The data represent the strength data measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows.

In this section, we demonstrate the suitability of our proposed distribution comparing with other existing distributions in the family of inverse Lindley distributions which are available for modeling lifetime data. The distributions considered in this paper are the inverse Lindley model IL(θ), the generalized inverse Lindley model GIL(θ, α) and Extended inverse Lindley model EIL(θ, α, β).

In order to compare the models, we consider some goodness-of-fit measures including log-likelihood (log$L$), Kolmogorov-Smirnov (K-S) distances between the empirical distribution functions and the fitted distribution functions and corresponding p values. In general, the model with minimum values for Kolmogorov-Smirnov (K-S) could be chosen as the best model to fit the data, p value is large enough to accept the null hypothesis that the data sets are consistent with a specified distribution and also the model with maximum values for log$L$ could be chosen as the best model to fit the data.

Model	$\hat{\theta}_{MLE}$	$\hat{\alpha}_{MLE}$	$\hat{\beta}_{MLE}$	log$L$	K-S	p value
IL(θ)	0.6345	-	-	-0.5854	0.3556	0.0127
GIL(θ, α)	0.0898	3.0766	-	16.1475	0.1445	0.7977
EIL(θ, α, β)	0.1052	4.0439	2.9573	16.2317	0.1395	0.8311
TWIL(θ, α)	5.8241	14.0886	-	16.1457	0.1268	0.9047

Table 5: Parameter estimates, log-likelihood, K-S and p values of the four models using `data1`.

From table 5, we can see that the value of log$L$ of the extended inverse Lindley (EIL) distribution is slightly higher than that of other distributions. However, the EIL has three parameters in the models. Considering the K-S statistics, the two-parameter weighted inverse Lindley (TWIL) distribution gives the lowest value for K-S statistics. Moreover the p value of the two-parameter weighted inverse Lindley (TWIL) distribution is higher than that of other distributions. So for `data1`, the two-parameter weighted inverse Lindley (TWIL) distribution seems to give the best fit to the data.

From table 6, we can see that the value of log$L$ of the two- parameter weighted inverse Lindley (TWIL) distribution is higher than that of other distributions. It gives the lowest values for K-S statistics. Moreover the p value of the two-parameter weighted inverse Lindley (TWIL) distribution is higher than that of other distributions. So for `data2`, the two-parameter weighted inverse Lindley (TWIL) distribution fits `data2` significantly better than other distributions.

Model	$\hat{\theta}_{MLE}$	$\hat{\alpha}_{MLE}$	$\hat{\beta}_{MLE}$	$\log L$	K-S	p value
IL(θ)	3.5842	-	-	-131.8826	0.4630	3.712e-12
GIL(θ, α)	231.4422	5.4337	-	-58.9023	0.1001	0.5528
EIL(θ, α, β)	237.2141	1.7337	5.4558	-58.9037	0.1011	0.5396
TWIL(θ, α)	77.2653	26.0076	-	-56.2881	0.0795	0.8208

Table 6: Parameter estimates, log-likelihood, K-S and p values of the four models using data2.

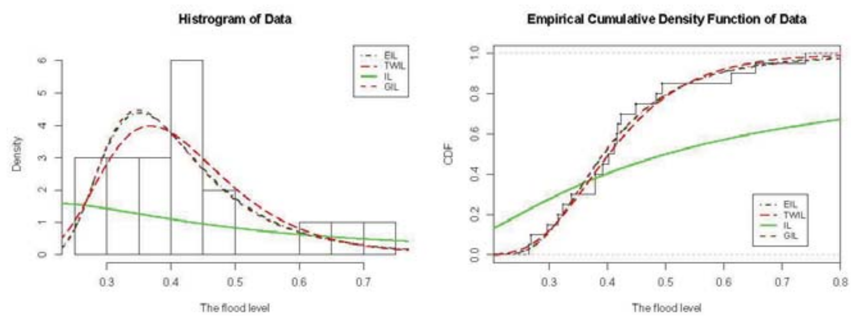

Figure 3: Plots showing the fitted densities and cdfs of the distributions listed in table 5.

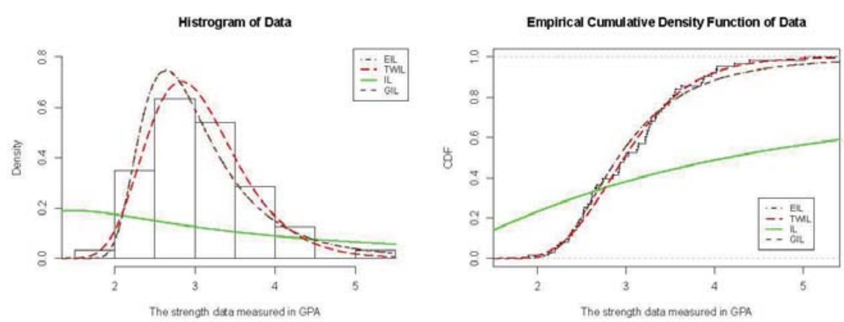

Figure 4: Plots showing the fitted densities and cdfs of the distributions listed in table 6.

6. Conclusion and Discussion

The structural properties of our new proposed two-parameter weighted inverse Lindley distribution are presented. The expressions for the survival and hazard function of the distribution have been obtained. The estimation of parameters by the maximum likelihood estimation (MLE) and the bias-corrected maximum likelihood estimation (CMLE) are also provided. The two benchmark data sets, the flood levels data and the strength data measured in GPA are considered here to illustrate that the proposed distribution provides consistently better fit than some other forms of inverse Lindley distribution. Therefore, the authors hope that our proposed distribution will provide users a more flexible distribution that could be applied to wider applications in many disciplines.

7. References

[1] Alkarni SH. Extended inverse Lindley distribution: Properties and application. SpringerPlus. 2015; 4(1): 690–703.
[2] Bader MG, Priest AM. Statistical aspects of fiber and bundle strength in hybrid composites. Progress in science and engineering of composites. 1982; 1129-1136.
[3] Cordeiro GM, Klein R. Bias correction in ARMA models. Statistics & Probability Letters. 1994; 19(3): 169-176.
[4] Cox DR, Snell EJ. A general definition of residuals. Journal of the Royal Statistical Society. 1968; Series B (Methodological): 248-275.
[5] Dumonceaux R, Antle CE, Discriminating between the log-normal and Weibull distribution. Technometrics. 1973; 15: 923-926.
[6] Elbatal I, Merovci F, Elgarhy M. A new generalized Lindley distribution. Mathematical Theory and Modeling. 2013; 3: 30-47.
[7] Firth D, Bias reduction of maximum likelihood estimates. Biometrika. 1993; 80(1): 27-38.
[8] Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its application. Mathematics and computers in simulation. 2008; 78(4): 493-506.
[9] Ghitany ME, Alqallaf F, Al-Mutairi DK, Husain HA. A two-parameter weighted Lindley distribution and its applications to survival data. Mathematics and Computers in Simulation. 2011; 81(6): 1190-1201.
[10] Lindley DV. Fiducial distributions and Bayes' theorem. Journal of the Royal Statistical Society. 1958; Series B (Methodological): 102-107.
[11] Nedjar S, Zeghdoudi H. On gamma Lindley distribution: properties and simulations. Journal of Computational and Applied Mathematics. 2016; 298: 167-174.
[12] Ramos PL, Louzada F. The generalized weighted Lindley distribution: Properties, estimation, and applications. Cogent Mathematics. 2016; 3(1): (Acticle 1256022).
[13] Shaked M, Shanthikumar J. Stochastic Orders and Their Applications. Boston: Academic Press; 1994.
[14] Shanker R, Mishra A. A two-parameter Lindley distribution. Statistics in Transition new series. 2013; 14(1): 45-56.
[15] Sharma V, Singh S, Singh U, Merovci F.The generalized inverse Lindley distribution: A new inverse statistical model for the study of upside-down bathtub survival data. Commun Stat Theory Methods. 2015 ; preprint.
[16] Sharma VK, Singh SK, Singh U, Agiwal V. The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data. Journal of Industrial and Production Engineering. 2015; 32(3): 162-173.
[17] Shibu DS, Irshad MR. Extended New Generalized Lindley Distribution. Statistica. 2016; 76(1): 41-56.