A Comparison of Parameter Estimation of Gamma Distribution by Maximum Likelihood, Bayes', and Markov Chain Monte Carlo Methods

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Abstract

The objective of this research is to estimate a parameter (λ) of gamma distribution by using Maximum Likelihood (ML), Bayes', Markov Chain Monte Carlo (MCMC), and applied Bayes' with Markov Chain Monte Carlo (Bayes'-MCMC) methods. The Bayes' method is depended on the prior distribution, so gamma distribution is considered the parameter for Bayes' estimator. The t-test is used for testing that the true parameter is equal the mean of estimated parameter, while it means the performant method for estimating parameter. Data is simulated from gamma distribution by setting the shape parameter (α) as 4,5,6,7, and 8 and the scale parameter or called true parameter (λ) as 2 based on the sample sizes (n) 30, 50 and 70. The results are found that ML, Bayes', MCMC is a good performance to estimate parameter in most all cases except moderate sample size and small shape parameter. The Bayes'-MCMC method outperforms all others when shape parameter and sample sizes are large.

Keywords: Bayes', Gamma Distribution, Markov Chain Monte Carlo, Maximum Likelihood

1. Introduction

The estimation is a field of statistical inference which is related to the process by which one makes inferences about a population, based on information obtained from a sample. Point estimation is a part of estimation which is contained a single value and estimated the population parameter or called estimator. The properties of good estimator are unbiasedness, sufficiency, consistency, efficiency and minimum variance [1]. There are several methods to approximate estimator such as moments method, maximum likelihood method, least squares method, and Bayes' method.

The maximum likelihood method is a well known method for estimating parameter of statistical model given observation. The estimator is estimated by maximize the likelihood function given the parameter. The properties of good estimator can be found in this method. The Bayes' method mentions the probability distribution given the parameter that related with prior distribution. The relation among probability and prior distributions imply to find the posterior distribution. So the Bayes' estimator is computed from mean of posterior distribution which is depended on the parameter of prior distribution. The Markov Chain Monte Carlo method (MCMC) [2] method is solved the problem of estimating parameter on prior distribution. The Gibbs sampling is a process of MCMC to generate the value from the posterior distribution that obtained MCMC estimator. From MCMC method, we obtain the estimated parameter of prior distribution which can be applied by Bayes and MCMC to find the estimator.

From the related research, Araveeporn A [3] studied the point parameter estimation of Poisson distribution by using Maximum Likelihood, Markov Chain Monte Carlo, and Bayes methods by estimated parameter of the prior distribution from MCMC method. Thetkham A and

Araveeporn A [4] studied a comparison of parameter estimation of exponential distribution using Maximum Likelihood, Bayes, Markov Chain Monte Carlo and applied Bayes with Markov Chain Monte Carlo methods, and the prior distribution of Bayes' method depended on gamma distribution.

In this research, we interested the data in term of a gamma distribution which is related the parameter of Poisson distribution. The number of successes that occurs in fixed interval of time and space is a random variable of Poisson distribution depended on average number of successes in fixed interval of time or called the parameter of Poisson distribution (λ). For gamma distribution, the random variable is a waiting times until α th success occurs, given that the average number of success per unit of time is λ as the meaning of parameter on Poisson distribution. In this respect, the gamma distribution is related to the exponential distribution in the average waiting time until successful for the first time.

The gamma distribution is often used to model waiting times, particularly in the case of lifespan testing in which the "waiting time" until death is modeled by a gamma distribution. It is also commonly used in applied fields such as finance, civil engineering, climatology (e.g. in estimating rainfall), and econometrics.

The research about the gamma distribution is studied by several literatures following : Pradhan B and Kundu D [5] developed Bayes estimation and prediction of two-parameter gamma distribution by assumed that the scale parameter has a gamma distribution prior and the shape parameter has any log-concave distribution prior. Kirimi E, Ouko A, and Kipkoech CW [6] studied Modified Moment Estimation (MME) for a two parameter gamma distribution which estimated the scale parameter using the MME and compared the performance with the Maximum Likelihood Estimates (MLE). Son YS and Oh M [7] focused Bayesian Estimation of the Two-Parameter Gamma Distribution was considered under the non informative prior.

For this reason, we interest in three methods with four estimating parameter such as maximum likelihood (ML), Bayes', Markov Chain Monte Carlo (MCMC), and applied Bayes with Markov Chain Monte Carlo (Bayes'-MCMC). We estimate the parameter of gamma distribution (λ) by using these methods that mentioned before.

2. Methods of Parameter Estimation

In this research, we have studied parameter estimation (λ) of gamma distribution with three methods.

2.1 Maximum Likelihood (ML) Method

Let $X_1,...,X_n$ are random samples via gamma distribution with parameter α and λ or called $Gamma(\alpha,\lambda)$. The shape parameter is α, and the scale parameter is λ.

The probability density function is written by

$$f(x;\alpha,\lambda)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x} & ,x>0 \\ 0 & ,\text{ otherwise}\end{cases},$$

where mean is $E(X) = \frac{\alpha}{\lambda}$, and variance is $Var(X) = \frac{\alpha}{\lambda^2}$.

The definition of likelihood function of gamma distribution $L(\alpha,\lambda)$ is

$$L(\alpha,\lambda) = \prod_{i=1}^{n} f(x_i;\alpha,\lambda)$$
$$= \left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{n} \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\lambda\sum_{i=1}^{n} x_i} .$$

The first step of maximum likelihood method is taken by natural logarithm (ln) on a likelihood function following

$$\ln L(\alpha,\lambda) = \ln\left(\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{n} \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\lambda\sum_{i=1}^{n} x_i}\right)$$

$$= \alpha\, n \ln(\lambda) - n\ln(\Gamma(\alpha)) + (\alpha-1)\sum_{i=1}^{n} \ln x_i - \lambda\sum_{i=1}^{n} x_i .$$

Next, the partial differential of λ is considered by

$$\frac{\partial \ln L(\alpha,\lambda)}{\partial\lambda} = \frac{\partial}{\partial\lambda}\left(\alpha n\ln(\lambda) - n\ln(\Gamma(\alpha)) + (\alpha-1)\sum_{i=1}^{n} \ln x_i - \lambda\sum_{i=1}^{n} x_i\right),$$

$$= \frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i = 0,$$

$$\frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i = 0,$$

$$\frac{n\alpha}{\lambda} = \sum_{i=1}^{n} x_i ,$$

$$\alpha = \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\lambda,$$

$$\alpha = \bar{x}\lambda,$$

$$\lambda = \frac{\alpha}{\bar{x}}.$$

The second derivative of $\ln L(\alpha,\lambda)$ is $\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\lambda^2} = -\frac{n\alpha}{\lambda^2} < 0$, so the estimators have the maximum value. Therefore, the ML estimator of λ is $\hat{\lambda}_{ML} = \frac{\alpha}{\bar{x}}$.

2.2 Bayes' Method

Let the prior distribution (λ) is a gamma distribution with parameter a and b which is the conjugate distribution of gamma distribution. The prior distribution is shown that the random variable is λ in the form of gamma distribution as the following

$$f(\lambda;a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda} & , \lambda > 0 \\ 0 & , \text{otherwise} \end{cases}.$$

From the Bayes' theorem [8], the posterior distribution function is approximated by

$$h(\lambda|x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_n|\lambda)f(\lambda;a,b)}{\int_0^\infty f(x_1,\ldots,x_n|\lambda)f(\lambda;a,b)d\lambda},$$

or it can be rewritten in term of conjugate distribution as

$$h(\lambda|x_1,\ldots,x_n) \propto L(\alpha,\lambda)f(a,b|\lambda),$$

then

$$h(\lambda|x_1,\ldots,x_n) \propto \left(\frac{\lambda^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1}e^{-\lambda\sum_{i=1}^n x_i}\left(\frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}\right),$$

$$\propto \frac{b^a}{(\Gamma(\alpha))^n(\Gamma(a))}\prod_{i=1}^n x_i^{\alpha-1}\lambda^{n\alpha+a-1}e^{-\lambda\sum_{i=1}^n x_i - b\lambda},$$

$$\propto \frac{b^a}{(\Gamma(\alpha))^n(\Gamma(a))}\prod_{i=1}^n x_i^{\alpha-1}\lambda^{n\alpha+a-1}e^{-\lambda\left(\sum_{i=1}^n x_i + b\right)},$$

$$\propto \frac{b^a}{(\Gamma(\alpha))^n(\Gamma(a))}\prod_{i=1}^n x_i^{\alpha-1}\lambda^{n\alpha+a-1}e^{-\lambda(n\bar{x}+b)}.$$

The posterior distribution is presented in a class of gamma distribution as

$$h(\lambda|x_1,\ldots,x_n) \sim Gamma(n\alpha+a, n\bar{x}+b).$$

Therefore, the Bayes' estimator of λ is $\hat{\lambda}_{Bayes'} = E(\lambda | x_1,...,x_n) = \frac{n\alpha + a}{n\bar{x} + b}$.

2.3 Markov Chain Monte Carlo (MCMC) Method

The MCMC method is a popular method for estimating parameter when the random variable can not define the prior distribution. This method consists of sampling random variable from Markov Chain method for estimating parameter on prior distribution, sampling random variable by Gibbs sampling algorithm [9]. The posterior distribution is used a Markov Chain and Gibbs sampling to approximate estimator from the MCMC method. We carry out the RJAGS package which provides an interface from R to the JAGS library for Bayesian data analysis [10]. JAGS uses Markov Chain Monte Carlo (MCMC) to generate a sequence of dependent samples from the posterior distribution of the parameters.

Sampling process from the MCMC method as follows :

1. Set initial value $a^{(t)}$ and $b^{(t)}$ from gamma distribution.

2. Generating $a^{(t)}$ and $b^{(t)}$ from 1., $t = 1,2,...,T$, where T is a number of Gibbs sampling algorithm. The number of T is approached to population parameter at large number replications. In this case, we use 5000 replications.

3. Generating $\lambda^{(t)}$ from posterior distribution of gamma distribution with parameter $a^{(t)}$ and $b^{(t)}$ from 2.

Therefore, the Markov Chain Monte Carlo estimator of λ is $\hat{\lambda}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T}\lambda^{(t)}$

Furthermore, we use this benefit of MCMC algorithm to approximate parameter (a and b) on prior distribution. From Bayes' estimator, we can apply this idea to estimate Bayes'-MCMC estimator which is written as

$$\lambda_{Bayes'-MCMC} = \frac{n\alpha + \hat{a}_{MCMC}}{n\bar{x} + \hat{b}_{MCMC}}, \text{ where } \hat{a}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} a^{(t)}, \text{ and } \hat{b}_{MCMC} = \frac{1}{T}\sum_{t=1}^{T} b^{(t)}.$$

3. Research Scope

3.1 The random variable X is simulated data from gamma distribution with R program by setting the scale parameter or called true parameter (λ) at 2 and shape parameter (α) at 4, 5, 6, 7, and 8 shown in Figure 1.

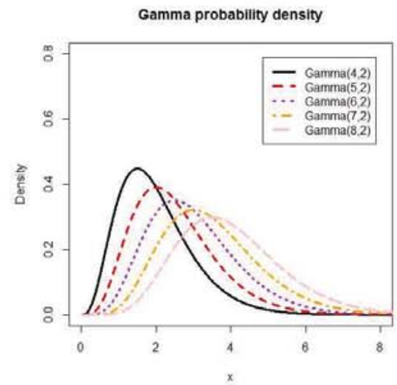

Figure 1: Gamma distribution with parameters (α, λ) as $(4,2),(5,2),(6,2),(7,2)$, and $(8,2)$.

3.2 The prior distribution of λ is defined as the gamma distribution with $\alpha = 2$ and $\lambda = 1$ shown in Figure 2.

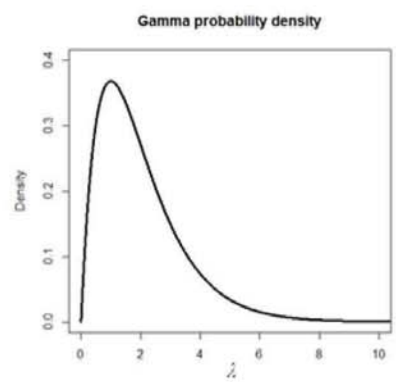

Figure 2: Gamma distribution with parameter $\alpha = 2$ and $\lambda = 1$.

3.3 The sample sizes (n) are studied at 30, 50, and 70.

3.4 The significant confidence level (α) is defined at 2 levels as 0.01 and 0.05.

3.5 In this research, we use R program versions 3.3.1 that generated 1,000 replications in each situation.

4. Statistical Hypothesis Testing

A t statistic is applied to test the different between the mean of sample with the mean of true parameter. The hypothesis are

$$H_0 : \mu_{\hat{\lambda}} = \lambda \text{ and } H_1 : \mu_{\hat{\lambda}} \neq \lambda$$

The t statistic was computed as follows:

$$t = \frac{\bar{\hat{\lambda}} - \lambda}{S_{\hat{\lambda}} / \sqrt{m}}$$

where $\bar{\hat{\lambda}}$ is the mean of estimated parameter.

$S_{\hat{\lambda}} = \sqrt{\frac{\sum_{j=1}^{m}\left(\hat{\lambda}_j - \bar{\hat{\lambda}}_j\right)^2}{m-1}}$ is a standard deviation (SD) of sample .

$\hat{\lambda}_j$ is the estimated parameter in repeat j times.

$\bar{\hat{\lambda}}_j$ is the mean of estimated parameter in repeat j times.

m is the number of replications, here given $m = 1{,}000$ and $df = m-1$.

In this case, we will reject H_0, when $|t| > t_{\alpha/2,m-1}$ and set the level of significance at $\alpha = 0.01$ and 0.05. The lower confident interval (LCI), upper confident interval (UCI) of λ on confident interval $(1-\alpha)100\%$ can compute from $\lambda = \bar{\hat{\lambda}} \pm t_{\alpha/2,m-1}\frac{S_{\hat{\lambda}}}{\sqrt{m}}$.

5. The Results

In this section, the parameter estimation of gamma distribution by Maximum Likelihood (ML), Bayes', Markov Chain Monte Carlo (MCMC), and applied Bayes' with MCMC (Bayes'-MCMC) methods are presented in Tables 1-4.

From Table 1-3, by observing the p-values, the ML, Bayes', and MCMC estimators shown that the means of the estimated parameters do not different from the true parameters ($\lambda = 2$) as follows: $\alpha = 4$ when $n = 50$, $\alpha = 5,8$ when $n = 50,70$, $\alpha = 6,7$ in all sample sizes. The estimated parameter is a good performance in 11 situations from 15 situations.

From Table 4, the p-value of the Bayes'- MCMC estimators is larger than the significance level (0.01 and 0.05) when the mean of the estimated parameter do not different from the true parameter ($\lambda = 2$) at $\alpha = 4$ when $n = 70$, $\alpha = 5$ when $n = 50,70$ and $\alpha = 6,7,8$ in all sample sizes. The estimated parameter is a good performance in 12 situations from 15 situations.

In the part of the histograms, the estimated parameters of all the true parameters are shown in Figure 3-22. The histograms are found that the estimated value tends to the normal distribution on true parameter ($\lambda = 2$) when the sample sizes increased.

Table 1: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α), mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by the ML method.

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01534	0.18505	2.00386	2.02682	2.62140	0.00889**
	50	2.00872	0.14230	1.99989	2.01756	1.93879	0.05281
	70	2.00825	0.11849	2.00090	2.01560	2.20196	0.02790*
5	30	2.01236	0.16623	2.00205	2.02268	2.35139	0.01890*
	50	2.00628	0.12549	1.99849	2.01406	1.58188	0.11399
	70	2.00552	0.10697	1.99888	2.01216	1.63185	0.10303
6	30	2.00858	0.15144	1.99918	2.01797	1.79090	0.07361
	50	2.00298	0.11359	1.99593	2.01003	0.82965	0.40694
	70	2.00244	0.09837	1.99633	2.00854	0.78295	0.43384
7	30	2.00815	0.13688	1.99966	2.01665	1.88332	0.05995
	50	2.00301	0.10922	0.49906	0.50245	0.87191	0.38347
	70	2.00218	0.09113	1.99653	2.00784	0.75790	0.44869
8	30	2.00832	0.13132	2.00017	2.01647	2.00392	0.04535*
	50	2.00363	0.10399	1.99717	2.01008	1.10306	0.27027
	70	2.00261	0.08427	1.99738	2.00784	0.97808	0.32827

**indicates significance at 1% level and *indicates significance at 5% level

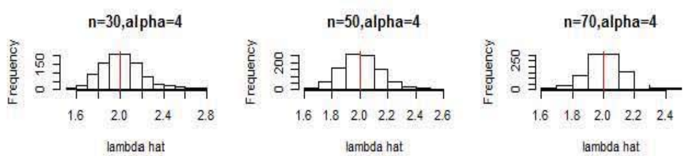

Figure 3: Histograms of estimated parameters λ by ML method when $\alpha = 4, \lambda = 2$

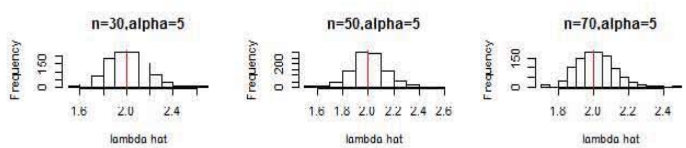

Figure 4: Histograms of estimated parameters λ by ML method when $\alpha = 5, \lambda = 2$

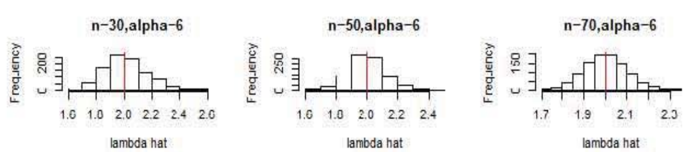

Figure 5: Histograms of estimated parameters λ by ML method when $\alpha = 6, \lambda = 2$

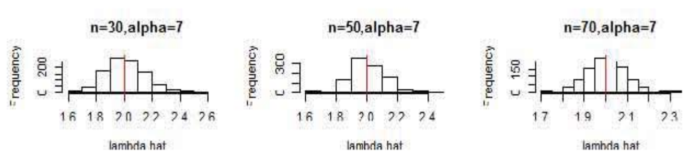

Figure 6: Histograms of estimated parameters λ by ML method when $\alpha = 7, \lambda = 2$

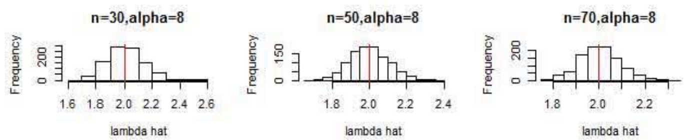

Figure 7: Histograms of estimated parameters λ by ML method when $\alpha = 8, \lambda = 2$

Table 2: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α), mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by the Bayes' method with the prior distribution is gamma distribution.

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01481	0.18183	2.00353	2.02610	2.57585	0.01014*
	50	2.00854	0.14085	1.99980	2.01728	1.91704	0.05552
	70	2.00814	0.11764	2.00084	2.01544	2.18889	0.02884*
5	30	2.01202	0.16394	2.00184	2.02219	2.31807	0.02065*
	50	2.00617	0.12447	1.99844	2.01389	1.56638	0.11758
	70	2.00546	0.10635	1.99886	2.01206	1.62236	0.10504
6	30	2.00836	0.14972	1.99907	2.01765	1.76519	0.07784
	50	2.00292	0.11283	1.99592	2.00992	0.81784	0.41365
	70	2.00240	0.09790	1.99633	2.00848	0.77561	0.43816
7	30	2.00799	0.13555	1.99958	2.01640	1.86336	0.06270
	50	2.00296	0.10858	0.50119	0.50459	0.86222	0.38877
	70	2.00216	0.09075	1.99653	2.00779	0.75208	0.45218
8	30	2.00818	0.13019	2.00010	2.01626	1.98734	0.04716*
	50	2.00358	0.10346	1.99716	2.01000	1.09495	0.27380
	70	2.00258	0.08397	1.99737	2.00780	0.97338	0.33060

**indicates significance at 1% level and *indicates significance at 5% level

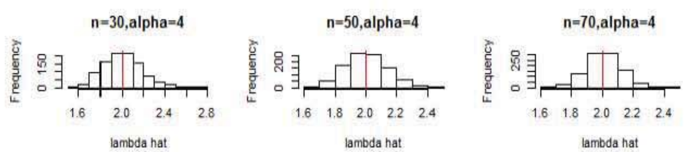

Figure 8: Histograms of estimated parameters λ by Bayes' method when $\alpha = 4, \lambda = 2$

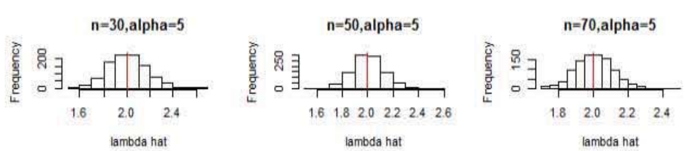

Figure 9: Histograms of estimated parameters λ by Bayes' method when $\alpha = 5, \lambda = 2$

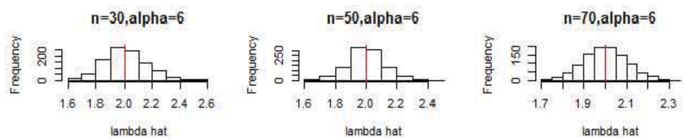

Figure 10: Histograms of estimated parameters λ by Bayes' method when $\alpha = 6, \lambda = 2$

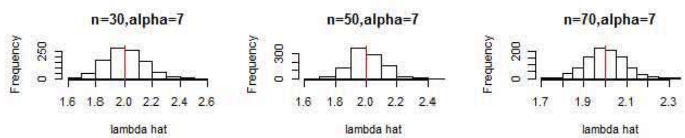

Figure 11: Histograms of estimated parameters λ by Bayes' method when $\alpha = 7, \lambda = 2$

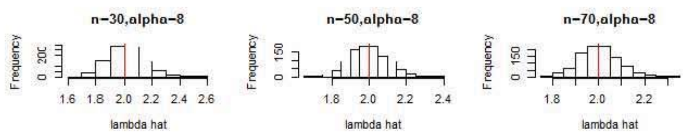

Figure 12: Histograms of estimated parameters λ by Bayes' method when $\alpha = 8, \lambda = 2$

Table 3: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α), mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by the MCMC method

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01548	0.18527	2.00398	2.02698	2.64203	0.00837**
	50	2.00863	0.14272	1.99978	2.01749	1.91279	0.05606
	70	2.00821	0.11864	2.00085	2.01557	2.18763	0.02893*
5	30	2.01263	0.16678	2.00228	2.02298	2.39519	0.01679*
	50	2.00619	0.12570	1.99839	2.01399	1.55833	0.11947
	70	2.00539	0.10700	1.99875	2.01203	1.59412	0.11123
6	30	2.00901	0.15182	1.99959	2.01843	1.87699	0.06081
	50	2.00307	0.11355	1.99602	2.01012	0.85485	0.39284
	70	2.00240	0.09845	1.99629	2.00851	0.76954	0.44175
7	30	2.00812	0.13732	1.99960	2.01664	1.87001	0.06177
	50	2.00303	0.10919	0.49905	0.50244	0.87680	0.38081
	70	2.00234	0.09129	1.99667	2.00800	0.80901	0.41870
8	30	2.00819	0.13142	1.99983	2.01614	1.97182	0.04891*
	50	2.00379	0.10439	1.99732	2.01027	1.14932	0.25070
	70	2.00250	0.08444	1.99726	2.00774	0.93714	0.34891

**indicates significance at 1% level and *indicates significance at 5% level

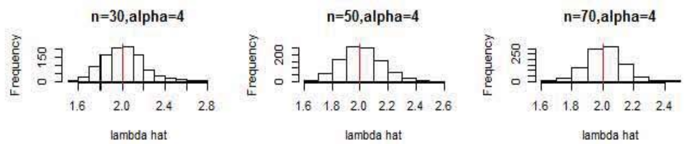

Figure 13: Histograms of estimated parameters λ by MCMC method when $\alpha = 4, \lambda = 2$

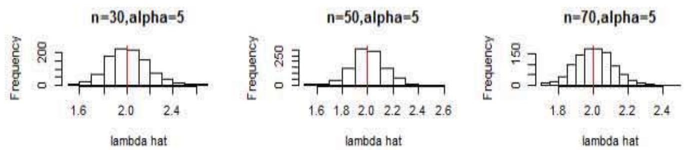

Figure 14: Histograms of estimated parameters λ by MCMC method when $\alpha = 5, \lambda = 2$

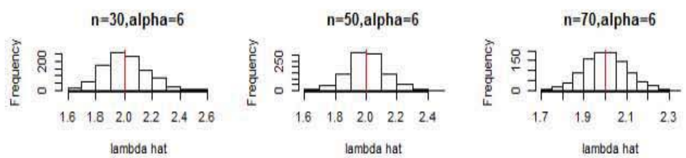

Figure 15: Histograms of estimated parameters λ by MCMC method when $\alpha = 6, \lambda = 2$

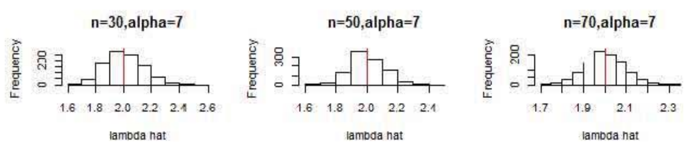

Figure 16: Histograms of estimated parameters λ by MCMC method when $\alpha = 7, \lambda = 2$

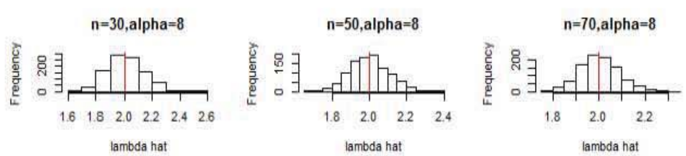

Figure 17: Histograms of estimated parameters λ by MCMC method when $\alpha = 8, \lambda = 2$

Table 4: The true parameter ($\lambda = 2$), sample sizes (n), parameter (α), mean, standard deviation (SD), lower confident interval (LCI), upper confident interval (UCI), t statistic (t-test) and p-value by applied Bayes' with MCMC method

α	n	Mean	SD	LCI	UCI	t-test	p-value
4	30	2.01502	0.18340	2.00364	2.02640	2.58934	0.00976**
	50	2.00957	0.14168	2.00078	2.01837	2.13696	0.03284*
	70	2.00723	0.11793	1.99991	2.01455	1.93772	0.05294
5	30	2.01171	0.16491	2.00148	2.02195	2.24579	0.02494*
	50	2.00659	0.12503	1.99884	2.01435	1.66796	0.09564
	70	2.00482	0.10657	1.99820	2.01143	1.42925	0.15324
6	30	2.00814	0.15058	1.99879	2.01748	1.70878	0.08780
	50	2.00298	0.11322	1.99595	2.01001	0.83206	0.40557
	70	2.00168	0.09803	1.99560	2.00776	0.54175	0.58811
7	30	2.00712	0.13605	1.99868	2.01556	1.65427	0.09839
	50	2.00306	0.10891	0.49889	0.50227	0.88747	0.37504
	70	2.00230	0.09096	1.99666	2.00794	0.79944	0.42423
8	30	2.00804	0.13068	1.99963	2.01585	1.94636	0.05189
	50	2.00351	0.10372	1.99708	2.00995	1.07095	0.28445
	70	2.00220	0.08408	1.99698	2.00742	0.82779	0.40799

**indicates significance at 1% level and *indicates significance at 5% level

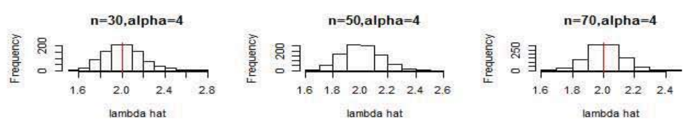

Figure 18: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 4, \lambda = 2$

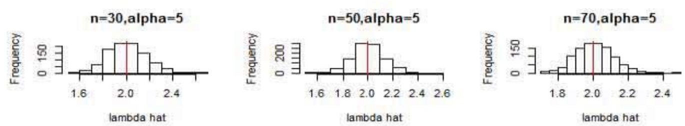

Figure 19: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 5, \lambda = 2$

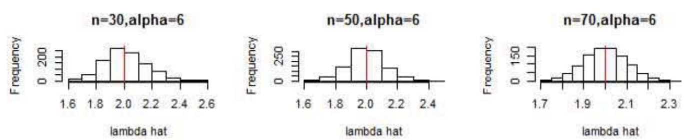

Figure 20: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 6, \lambda = 2$

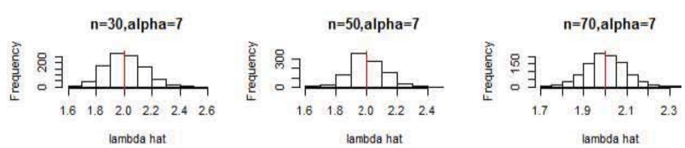

Figure 21: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 7, \lambda = 2$

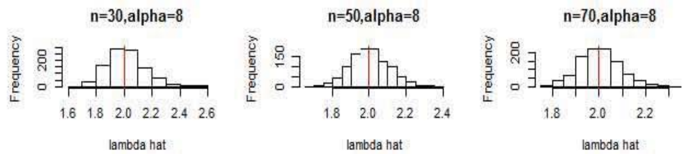

Figure 22: Histograms of estimated parameters λ by Bayes'- MCMC when $\alpha = 8, \lambda = 2$

From Tables 1-4, we can be summarized the estimation method that the estimated parameters are equal the true parameter shown in Table 5 and 6 based on α.

Table 5: An efficiency of parameter estimation in each situation with $\alpha = 0.01$

n	α (Shape Parameter)				
	4	5	6	7	8
30	Bayes'	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC
50	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC
70	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC

Table 6: An efficiency of parameter estimation in each situation with $\alpha = 0.05$

n	α (Shape Parameter)				
	4	5	6	7	8
30	-	-	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	Bayes'- MCMC
50	ML Bayes' MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC
70	Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC	ML Bayes' MCMC Bayes'- MCMC

6. Conclusion

In this research, we have tested the hypothesis of mean among estimated parameter (λ) from gamma distribution and mean of estimator from ML, Bayes', MCMC, and Bayes'- MCMC method. Through a simulation study, ML method performs a good method in most all cases except of moderate sample size and small shape parameter. The Bayes' method is used the parameter from gamma distribution to estimate parameter of prior distribution, so it makes the posterior estimator is outperforms than the other methods in all cases at significance level 0.01. The MCMC method is a reasonable working method as well as Bayes' and ML when the sample sizes and shape parameter are presented on moderate values. However, we apply Bayes'- MCMC that shown the good estimator more than the other method especially large shape parameter and large sample sizes. For this case, the user can use Bayes'- MCMC method without to set the parameter of prior distribution on Bayes' method.

References

[1] Harville DA, Maximum Likelihood Approaches to Variance Component Estimation and to Related Problems. Journal of the American Statistical Association. 1977; 72(358):320-338.

[2] Gilks W, Richardson S, Speigelhalter D, Markov Chain Monte Carlo in Practice, Interdisciplinary Statistics, Chapman & Hall, Suffolk; 1996.

[3] Araveeporn A, Parameter Estimation of Poisson Distribution by Using Maximum Likelihood, Markov Chain Monte Carlo, and Bayes method. Thammasat International Journal of Science and Technology. 2014; 19(3):1-14.

[4] Thetkham A, Araveeporn A, A Comparison of Parameter Estimation of Exponential Distribution using Maximum Likelihood, Bayes, and Markov Chain Monte Carlo methods. International Conference on Applied Statistics 2016, July 13-15, Phuket, THAILAND.

[5] Pradhan B, Kundu D, Bayes estimation and prediction of the two-parameter gamma distribution. Journal of Statistical Computation and Simulation. 2011; 81(9):1187-1198.

[6] Kirimi E, Ouko A, Kipkoech CW, Modified Moment Estimation for a Two Parameter Gamma Distribution. IOSR Journal of Mathematics (IOSR-JM). 2014; 10(6):42-50.

[7] Son YS, Oh M, Bayesian Estimation of the Two-Parameter Gamma Distribution. Communications in Statistics-Simulation and Computation. 2006; 35(2):285-293.

[8] Bradley PC, and Thomas AL, Bayesian Methods for Data Analysis, 3th Ed., Chapman & Hall/CRC, London, 2008.

[9] Gelfand A, Hills S, Racine-Poon A, Smith A. Illustration of Bayesian inference in normal data models using Gibbs sampling. Journal of the American Statistical Association, 1990; 85:972-985.

[10] Lunn D, Spiegelhalter D, Thomas A, Best N. (2009) The BUGS project: Evolution, critique and future directions. Statistics in Medicine, 2009; 28:3049-67.