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Accurate Domination Number of Butterfly Graphs

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Abstract: Butterfly graphs and domination are very important ideas in computer architecture and communication techniques. We present results about one important domination parameter Accurate Domination Number for Butterfly Graph. We find the relation between domination number and accurate domination number for $BF(n)$.

In this paper we present results about Accurate Domination Number of Butterfly Graphs $BF(n)$. We show that Domination number and accurate domination number of butterfly graphs $BF(n)$ are related to each other as

$$\gamma_a(BF(n)) = \gamma(BF(n)) + 1 \qquad \text{if } n = 2, 3, 4k$$
$$\gamma_a(BF(n)) = \gamma(BF(n)) \qquad \text{if } n \neq 4k.$$

Keywords: Butterfly graph, domination number, accurate domination number

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1 Introduction

Domination in a graph along with its many variations provide an extremely rich area of study. Berge[2] and Ore[6] were the first to define dominating sets. Among

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various applications of domination, one of the most often discussed is communication networks. A fixed connection communication network is a graph $G(V, E)$ whose nodes represent processors and whose edges represent communication links between the processors. Examples of fixed connection machines are mesh, hypercube, butterfly etc.

Butterfly graph of order n, $BF(n)$ has a very good symmetry in structure and is regular with degree 4. This is an advantage in networks as it lowers hardware costs and allows unbounded size architectures using the same processors. One of current interests of researchers is butterfly graphs, because they are studied as a topology of parallel machine architectures.

Various domination parameters like total domination, perfect domination, inverse domination, accurate domination etc. are defined for graphs. In this paper we present a result about the relation between accurate domination number and domination number of butterfly graphs $BF(n)$.

2 Butterfly Graph $BF(n)$

2.1 Definition : Vertex set V of $BF(n)$ is the set of ordered pairs $(\alpha; v)$ where $\alpha = 0, 1, 2, \ldots, n-1$ and $v = x_{n-1} \ldots x_1 x_0$, a binary word of length n. There is an edge from a vertex $(\alpha; v)$ to a vertex $(\alpha'; v')$ in V where $\alpha' \equiv \alpha + 1 (\mathrm{mod}\, n)$ and $x_j = x'_j \forall j \neq \alpha'$. Thus there are three types of edges (i) straight edges joining $(\alpha; v)$ to $(\alpha+1; v)$, (ii) slanting edges joining $(\alpha; v)$ to $(\alpha+1; v')$, and (iii) winged edges joining $(0, v)$ to $(n-1, v)$. These edges are shown for $BF(3)$ in figure 1.

A Butterfly graph $BF(n)$ is an n partite graph with n levels. For $k = 0, 1, \ldots, n-1$ each level

$$L_k = \{(k; v)/v = x_{n-1} \ldots x_1 x_0, x_i = 0 \text{ or } 1\}$$

has 2^n vertices. Using decimal representation of the binary word we can rewrite

$$L_k = \left\{ (k; m)/ \text{ where } m = \sum_{0}^{n-1} x_j 2^j \right\}.$$

2.2 Recursive construction of $BF(n)$: Barth and Raspaud [1] have given recursive construction of $BF(n)$ from $BF(n-1)$. Consider two copies of $BF(n-1)$

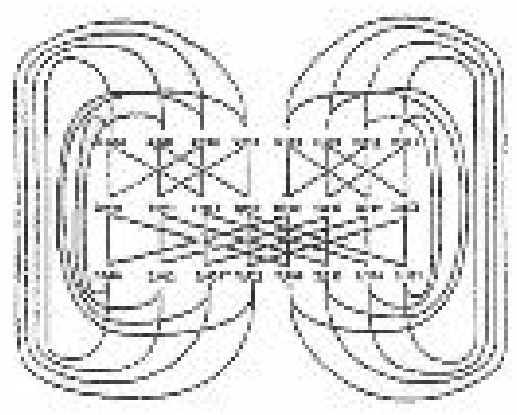

Figure 1: (straight, slanting and winged edges in $BF(3)$)

and place them adjacent to each other. Add a new level L_{n-1} and define straight and slanting edges between the vertices of L_{n-2} and L_{n-1}. Winged edges between L_0 and L_{n-2} are extended from the left and right side to vertices of newly added level L_{n-1}. A general construction is illustrated in the figure below.

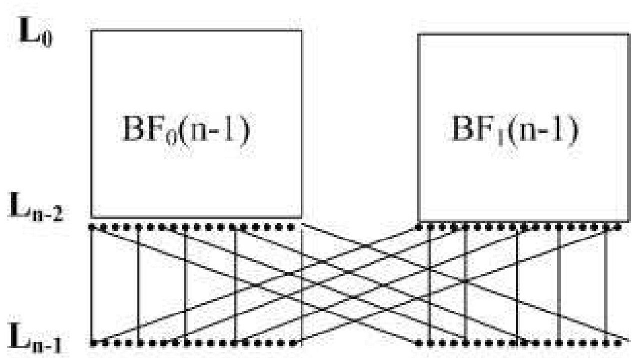

Figure 2: Recursive construction of $BF(n)$

2.3 Special Recursive Construction of $BF(4k)$ **:** Here we present a special recursive construction of $BF(4k)$ using $BF(4)$ pattern. Consider 2^4 copies of the graph $BF(4(k-1))$, and place them next to each other. Next call a set of 2^{n-4} vertices as a vertex group and add four new levels with 2^4 such vertex groups. So each level has $2^{4k-1} \times 2^4 = 2^{4k}$ vertices. Define a set of straight and slanting edges between these vertex groups similar to edges in $BF(4)$. Now the last 4 levels consist of 2^4 vertex groups in each level with edges as that in $BF(4)$ without winged edges. We shall call this a pattern of $BF(4)$. Now extend the winged edges joining vertices from level L_0 to level L_{4k-5} to last level L_{4k-1}.

Thus $BF(4k)$ is obtained recursively from $BF(4(k-1))$ using the idea of pattern of $BF(4)$. Figure 3 illustrates the recurssive construction of $BF(8)$ from $BF(4)$ where each reactangle represents a copy of $BF(4)$ and each dark circle in the next 4 levels represents a vertex group with 2^4 vertices.

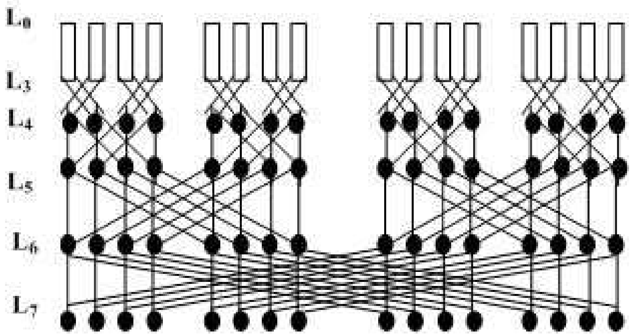

Figure 3: Special Recursive Construction of BF(8)

3 Results on Domination Number of Butterfly Graph $BF(n)$

From the definition of edges in $BF(n)$ we present following simple results without proof.

Lemma 3.1 : Two vertices $(k;m)$ and $(k;m')$ in the level L_k of $BF(n)$, dominate a pair of vertices $(k-1;m)$ and $(k-1;m\prime)$ in the preceding level L_{k-1} if $\mid m - m\prime \mid = 2^k$.

Lemma 3.2 : Two vertices $(k;m)$ and $(k;m\prime)$ in the level L_k of $BF(n)$ dominate two vertices $(k+1;m)$ and $(k+1;m\prime)$ in the succeeding level L_{k+1} if $\mid (m - m\prime) \mid = 2^{k+1}$.

Combining Lemmas 1 and 2 we get following result

Lemma 3.3 : The number of vertices of level L_k required to dominate all the vertices of levels L_{k-1} and L_{k+1} is 2^{n-1}. Theorem 4.3 of Chapter 4 from Ph D Thesis, Indrani Kelkar, [9] states that.

Theorem 3.4 : The domination number of $BF(n)$ is

$$\begin{aligned}
\gamma(BF(n)) &= 2 && \text{if } n = 2 \\
&= 6 && \text{if } n = 3 \\
&= k2^n && \text{if } n = 4k \\
&= (2k+1)2^{(n-1)} && \text{if } n = 4k+1 \\
&= (k+1)2^n && \text{otherwise.}
\end{aligned}$$

Consider graph $BF(4)$. $BF(4)$ has $4 \times 2^4 = 64$ vertices. Every vertex in $BF(4)$ has degree 4, so it can dominate at most 4 vertices. Clearly cardinality of any dominating set must be > 12. As $12 \times 4 = 48 + 12 = 60$ so 4 vertices still remain to be dominated. Symmetric nature of Butterfly graph gives that cardinality of any of its minimal dominating set must be even, so cardinality of any dominating set must be ≥ 14. Suppose there is a dominating set D of cardinality 14. Then as $BF(4)$ is a symmetric graph we must have 7 vertices in D are from the left part and 7 from right part of $BF(4)$. Each of the two parts of $BF(4)$, which contain a $BF(3)$ without winged edges, is also symmetric. But 7 vertices can not be symmetrically distributed in any of this part. So D must contain 8 vertices from each part giving $\mid D \mid \geq 16$. Now we show the existence of a dominating set of cardinality 16. From the recursive construction 1, we know that $BF(4)$ has 2 copies of $BF(3)$ and a level L_3 with 2^4 vertices. Making use of Lemma 3.3, from level L_1 we choose 4 vertices from left copy of $BF(3)$ and a mirror image of this selection in the right copy of $BF(3)$, Let us call this set as S_1 and its mirror image as S_2

$$S_1 = \{(1; m_1), (1; m_2), (1; m_3), (1; m_4) \,/\, 0 \le m_i \le 2^1 - 1 \text{ and } \mid m_i - m_j \mid \ne 2 \text{ or } 4\}$$

$$S_2 = \{(1; m_i') \text{ where } (1; m_i) \in S_1\}$$

Set $D_1 = S_1 \cup S_2$ Then $\mid D_1 \mid = 8$

These 8 vertices of L_1 included into D_1 dominate all the vertices of levels L_0 and L_2. Similarly to dominate 8 vertices in $L_1 \setminus D_1$ and the vertices of level L_3 we choose 8 vertices into D_2 from levels L_2 (or L_0) as follows :

$$S_3 = \{(2; m_1), (2; m_2), (2; m_3), (2; m_4) \,/\, 0 \le m_i \le 2^1 - 1 \text{ and } \mid m_i - m_j \mid \ne 4\}$$

and its mirror image

$$S_4 = \{(2; m_i') \text{ where } (2; m_i) \in S_3\}$$

Set $D_2 = S_3 \cup S_4$ Then $\mid D_2 \mid = 8$

As vertices in D_1 are from level L_1 and vertices in D_2 are from L_2, it is clear that D_1 and D_2 are disjoint sets. Let $D = D_1 \bigcup D_2$. So $\mid D \mid = \mid D_1 \mid + \mid D_2 \mid = 8 + 8 = 16$. Hence D is a minimum dominating set of $BF(4)$ with cardinality 16 giving $\gamma(BF(4)) = 2^4$.

One such dominating set of cardinality 16, shown by using dark circles, is illustrated in figure 4.

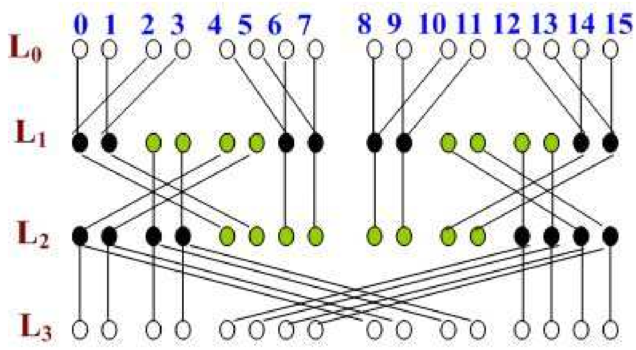

Figure 4: Dominating set of $BF(4)$

Remark 3.5 : From the adjacency of vertices in $BF(4)$ we observe that a set of vertices $D' = L_1 \setminus D_1 \bigcup L_2 \setminus D_2$, with same cardinality as that of D, also forms a minimum dominating set of $BF(4)$.

4 Main Results

4.1 Accurate domination number : Accurate domination number was introduced by Kulli and Kattimani [4]. They have studied this concept for various standard graphs. Here we discuss accurate dominating sets and find the accurate domination number of Butterfly graph $BF(n)$ from $BF(4k)$.

4.2 Definition : A dominating set D of G is called an accurate dominating set if the induced sub graph $< V \setminus D >$ contains a dominating set D_1 of cardinality not equal to $\mid D \mid$. Minimum cardinality of an accurate dominating set is called an accurate domination number of G is denoted by $\gamma_a(G)$.

Theorem 4.1 : Accurate domination number of $BF(4k)$ is

$$\gamma_a(BF(4k)) = \gamma(BF(4k)) + 1.$$

Proof : Consider the graph $BF(4k)$. From Remark 3.5 and the special recurssive construction of $BF(4k)$ we can extend the result for $BF(4)$ to $BF(4k)$ and state that complement of the set of vertices included into a dominating set of $BF(4k)$ in the corresponding levels is also a minimum dominating set of $BF(4k)$. That is for any minimal dominating set D the induced sub graph $< V \setminus D >$ contains a dominating set D' of cardinality equal to $\mid D \mid$. Thus any minimal dominating set of $BF(4k)$ is not an accurate dominating set of $BF(4k)$. Let D be a minimum dominating set of $BF(4k)$. Let L_0 be a level from which no vertex is included into D.

Consider a set $D_1 = D \bigcup (0;0)$.

So $\mid D_1 \mid = \mid D \mid +1$. We know that any superset of a dominating set is a dominating set so D_1 is a dominating set of $BF(4k)$. Now by the same argument as above $BF(4k) \setminus D_1$ has a dominating set of cardinality $\mid D \mid \neq \mid D_1 \mid$. Thus D_1 is an accurate dominating set of $BF(4k)$. Hence $\gamma_a(BF(4k)) = \gamma(BF(4k)) + 1$.

Corrollary 4.2 : Using similar idea we state that $\gamma_a(BF(n)) = \gamma(BF(n)) + 1$ for $n = 2, 3$.

Theorem 4.3 : Accurate domination number of $BF(4k+r)$ $r = 1, 2, 3$ is

$$\gamma_a(BF(4k+r)) = \gamma(BF(4k+r)).$$

Proof : Consider graph $BF(4k+r)$. By Recursive Construction 1, we know that $BF(4k+r)$ has 2^r copies of $BF(4k)$ without winged edges and r levels with 2^n vertices. Any minimal dominating set of $BF(4k+r)$ is union of vertices of dominating set of $BF(4k)$ and to dominate vertices of last r levels 2^{n-1} vertices are added to D for $r = 1$ and 2^n vertices are added to D for $r = 2, 3$.

Now consider induced graph $< BF(4k+r) \setminus D >$. From Remark 3.5 complement set of vertices from minimal dominating set for first $4(k-1)$ levels also form a dominating set. For domination of vertices in the induced graph in last $4+r$ levels let us consider the case $n = 5$.

For $n = 5$, $\gamma(BF(5)) = 3 \times 2^4 = 48$. Let D be a minimum dominating set of $BF(5)$ of cardinality 48. Now consider graph $< BF(5) \setminus D >$. Here we observe that a set D_1, of cardinality 32, as shown in figure 5 forms a minimum dominating set. Thus $< BF(5) \setminus D >$ has a dominating set of cardinality not equal to $\mid D \mid$. Hence D is an accurate dominating set of $BF(5)$.

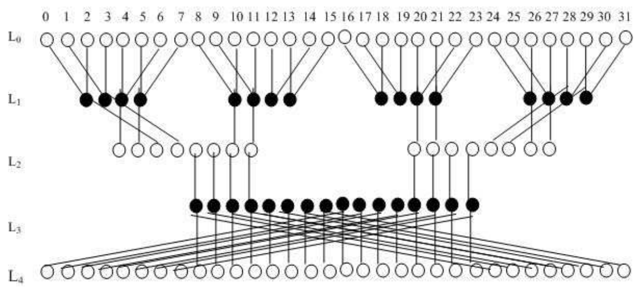

Figure 5: Dominating set of induced graph

Similarly for $r = 2, 3$ $< BF(4k+r) \setminus D >$ has a dominating set of lesser cardinality than D. Hence D is an accurate dominating set of $BF(4k+r)$. As D is the minimum dominating set of $BF(4k+r)$ we get that D is a minimum accurate dominating set of $BF(4k+r)$. Thus accurate domination number of $BF(4k+r)$ is $\gamma_a(BF(4k+r)) = \gamma(BF(4k+r))$.

5 Conclusion

Domination number and accurate domination number of butterfly graphs $BF(n)$ are related to each other as

$$\gamma_a(BF(n)) = \gamma(BF(n)) + 1 \qquad \text{if } n = 2, 3, 4k$$
$$\gamma_a(BF(n)) = \gamma(BF(n)) \qquad \text{if } n \neq 4k.$$

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