

Spectral Graph Theory and the Inverse Eigenvalue Problem of a Graph

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Abstract: The Inverse Eigenvalue Problem of a Graph is to determine the possible spectra among real symmetric matrices whose pattern of nonzero off-diagonal entries is described by a graph. In the last fifteen years a number of papers on this problem have appeared. Spectral Graph Theory is the study of the spectra of certain matrices defined from a given graph, including the adjacency matrix, the Laplacian matrix and other related matrices. Graph spectra have been studied extensively for more than fifty years. In 1990 Colin de Verdière introduced the first of several graph parameters defined as the maximum multiplicity of eigenvalue 0 among real symmetric matrices described by a graph and satisfying additional conditions. Recent work on Colin de Verdière-type parameters is bringing the two areas closer together. This paper surveys results on the Inverse Eigenvalue Problem of a Graph, Spectral Graph Theory, and Colin de Verdière-type parameters, and examines the connections between these fields.

Keywords: Spectral Graph Theory, Inverse Eigenvalue Problem, Colin de Verdière-type parameter, maximum eigenvalue multiplicity, maximum nullity, minimum rank

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1 Introduction

Spectral Graph Theory originally focused on specific matrices, such as the adjacency matrix or the Laplacian matrix, whose entries are determined by the graph, with the goal of obtaining information about the graph from the spectra of the matrices; a very brief introduction is given in Section 3. By contrast, The Inverse Eigenvalue Problem of a Graph seeks to determine information about the possible spectra of the family of real symmetric matrices whose pattern of nonzero off-diagonal entries is described by a given graph (see Section 2). In recent years spectral graph theorists have considered Colin de Verdière-type parameters based on families of matrices described by a graph, and there are now many connections between the two fields; some of these are discussed in Section 4.

Throughout this discussion, all matrices are real and symmetric. The *ordered spectrum* (the list of eigenvalues, repeated according to multiplicity in nondecreasing order) of an $n \times n$ matrix B will be denoted by $\sigma(B) = (\beta_1, \dots, \beta_n)$ with $\beta_1 \leq \dots \leq \beta_n$.

A *graph* G means a simple undirected graph (no loops, no multiple edges) with a nonempty set of vertices. The *order* of G is the number of vertices and is denoted by $|G|$. The *degree* of vertex v , $\deg_G v$, is the number of edges incident with v . The graph G is *regular* or *r -regular* if every vertex has degree r .

A *vertex cut-set* of G is a subset of vertices of G whose deletion increases the number of connected components of G ; a *cut-vertex* is a vertex cut-set of order one. The *vertex connectivity* of G , $\kappa_0(G)$, is the minimum number of vertices in a vertex cut-set (for a graph that is not the complete graph; by convention or $\kappa_0(K_n) = n - 1$). A graph G is *k -connected* if $\kappa_0(G) \geq k$. We usually restrict our attention to connected graphs, because each connected component can be analyzed separately. A *tree* is a connected graph with no cycles.

The subgraph $G[R]$ of G *induced* by a subset R of vertices is the subgraph with vertex set R and all edges of G having both endpoints in R . The subdigraph induced by the complement \overline{R} is also denoted by $G - R$, or if R is a single vertex v , by $G - v$.

If B is an $n \times n$ matrix and $R \subseteq \{1, \dots, n\}$, the *principal submatrix* $B[R]$ is the matrix consisting of the entries in the rows and columns indexed by R , and $B(R)$ is the complementary principal submatrix obtained from B by deleting the rows and columns indexed by R . In the special case when $R = \{k\}$, we use $B(k)$ to denote $B(R)$. If $\mathcal{G}(B) = G$, then by a slight abuse of notation $\mathcal{G}(B[R])$ can be

identified with $G[R]$, where $\mathcal{G}(B)$ denotes the graph of B (cf. Section 2).

Notation:

- P_n is a path on n vertices.
- C_n is a cycle on n vertices.
- K_n is the complete graph on n vertices.
- $K_{1,n}$ is a star on $n+1$ vertices, i.e., a complete bipartite graph on sets of 1 and n vertices.
- W_{n+1} is a wheel on $n+1$ vertices, i.e., a graph obtained by joining one additional vertex to every vertex of C_n .

2 The Inverse Eigenvalue Problem of a Graph

For a symmetric real $n \times n$ matrix B , the graph of B , $\mathcal{G}(B)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} \mid b_{ij} \neq 0 \text{ and } i \neq j\}$. Note that the diagonal of B is ignored in determining $\mathcal{G}(B)$.

Example 2.1. For the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3.1 & -1.5 & 2 \\ 0 & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$, $\mathcal{G}(B)$ is shown in

Figure 1.

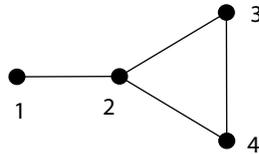


Figure 1: The graph $\mathcal{G}(B)$ for B in Example 2.1

Let S_n be the set of real symmetric $n \times n$ matrices. For G a graph with vertices $\{1, \dots, n\}$, define the *symmetric matrices described by G* ,

$$\mathcal{S}(G) = \{B \in S_n \mid \mathcal{G}(B) = G\}.$$

The *Inverse Eigenvalue Problem of a Graph (IEPG)* is to characterize the possible spectra of matrices in $\mathcal{S}(G)$.

This is a difficult problem and very little progress has been made. A first step is to determine the maximum possible multiplicity of an eigenvalue of a matrix in $\mathcal{S}(G)$. The *multiplicity* of β as an eigenvalue of $B \in S_n$ is denoted by $m_B(\beta)$. The eigenvalue β is *simple* if $m_B(\beta) = 1$. The *maximum multiplicity* of G is

$$M(G) = \max\{m_B(\beta) \mid \beta \in \sigma(B), B \in \mathcal{S}(G)\}$$

and the *minimum rank* of G is

$$\text{mr}(G) = \min\{\text{rank } B \mid B \in \mathcal{S}(G)\}.$$

Since $m_B(0) = \dim \ker B$, and one eigenvalue can be shifted to another by translation by a multiple of the identity matrix, the maximum multiplicity of G is also the maximum nullity among matrices in $\mathcal{S}(G)$. Thus, it is clear that

$$M(G) + \text{mr}(G) = |G|,$$

and the problem of determining maximum multiplicity is equivalent to the problem of determining minimum rank.

Most of the progress on IEPG is for trees, and for a tree the maximum multiplicity is easy to determine [34, 25]. The Parter-Wiener Theorem [35, 37] and the interlacing of eigenvalues play important roles.

If $B \in \mathcal{S}(G)$, then $B(k) \in \mathcal{S}(G - k)$. Let $B \in S_n$ and $k \in \{1, \dots, n\}$. If the eigenvalues of B are $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$ and the eigenvalues of $B(k)$ are $\theta_1 \leq \theta_2 \leq \dots \leq \theta_{n-1}$, then by the *Interlacing Theorem* [20, Fact 8.2.5], $\beta_1 \leq \theta_1 \leq \beta_2 \leq \theta_2 \leq \dots \leq \beta_{n-1} \leq \theta_{n-1} \leq \beta_n$.

Corollary 2.2. *If $\beta \in \sigma(B)$, $m_{B(k)}(\beta) \in \{m_B(\beta) - 1, m_B(\beta), m_B(\beta) + 1\}$.*

We say k is a *Parter-Wiener (PW) vertex* of B for eigenvalue β if $m_{B(k)}(\beta) = m_B(\beta) + 1$; k is a *strong PW vertex* of B for β if k is a PW vertex of B for β and β is an eigenvalue of at least three components of $\mathcal{G}(B) - k$. The next theorem is referred to as the *Parter-Wiener Theorem*.

Theorem 2.3. [35, 37, 28] *If T is a tree, $B \in \mathcal{S}(T)$ and $m_B(\beta) \geq 2$, then there is a strong PW vertex of B for β .*

Corollary 2.4. *If T is a tree, $B \in \mathcal{S}(T)$ and $\sigma(B) = (\beta_1, \dots, \beta_n)$, then β_1 and β_n are simple eigenvalues.*

As many people have observed, the Parter-Wiener Theorem need not be true for graphs that are not trees.

Example 2.5. For \mathcal{A} the adjacency matrix of C_4 , $m_{\mathcal{A}}(0) = 2$ but there is no PW vertex since $C_4 - k$ is P_3 for any vertex k . (See Section 3 for more information about the adjacency matrix.)

In [25] two more parameters of a graph that are related to maximum multiplicity and minimum rank were defined. The path cover number of G , $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G , and $\Delta(G) = \max\{p - q \mid \text{there is a set of } q \text{ vertices whose deletion leaves } p \text{ paths}\}$.

Theorem 2.6. [25] *For any tree T , $M(T) = P(T) = \Delta(T)$.*

These parameters provide an easy way to compute $M(T)$ (see [20, Fact 34.2.8] or [15] for algorithms). A vertex v of a tree T is a *high degree vertex* if $\deg_T v \geq 3$. Only high degree vertices can be strong PW vertices. The possible combinations of eigenvalues and multiplicities for certain families of trees have been determined. Since the first and last eigenvalue of a matrix whose graph is a tree must be simple, clearly the order is important in determining which lists of eigenvalues and multiplicities are possible. If the distinct eigenvalues of B are $\check{\beta}_1 < \cdots < \check{\beta}_r$ with multiplicities m_1, \dots, m_r , respectively, then (m_1, \dots, m_r) is called the ordered multiplicity list of B .

Example 2.7. The star on $n+1$ vertices, $K_{1,n}$, has only one high degree vertex, say vertex 1. Thus this vertex must be the strong PW vertex for any multiple eigenvalue of B with $\mathcal{G}(B) = K_{1,n}$. By choosing the diagonal elements of B for $2, \dots, n+1$ to be 0, we obtain $m_{B(1)}(0) = n$ and so $m_B(0) = n - 1$, $M(K_{1,n}) = n - 1$ and $\text{mr}(K_{1,n}) = 2$. Recall that the first and last eigenvalues are necessarily simple, and because there is only one PW vertex, by interlacing any multiple eigenvalues must be separated by a simple eigenvalue. Thus the only possible ordered multiplicity lists are $(1, m_1, 1, m_2, \dots, 1, m_r, 1)$ where $\sum_{i=1}^r m_i = n - r - 1$. Although it is not obvious, all such ordered multiplicity lists are attainable and can be realized for any real numbers.

Theorem 2.8. [16, 26, 27, 3] *The possible ordered multiplicity lists of the following families of trees have been determined. Furthermore, if there is a matrix $B \in \mathcal{S}(G)$ with distinct eigenvalues $\check{\beta}_1 < \cdots < \check{\beta}_r$ having multiplicities m_1, \dots, m_r , then for*

any real numbers $\gamma_1 < \dots < \gamma_r$, there is a matrix in $\mathcal{S}(G)$ having eigenvalues $\gamma_1, \dots, \gamma_r$ with multiplicities m_1, \dots, m_r .

- Paths
- Double Paths
- Stars
- Generalized Stars
- Double Generalized Stars

Thus for any of these graphs, determination of the possible ordered multiplicity lists of the graph is equivalent to the solution of the Inverse Eigenvalue Problem of the graph.

However, Barioli and Fallat [2] established that sometimes there are restrictions on which real numbers can appear as the eigenvalues for an attainable ordered multiplicity list.

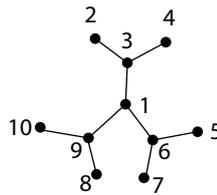


Figure 2: The tree T_{BF} for which an ordered multiplicity list is possible only for certain real numbers

Example 2.9. For the tree T_{BF} shown in Figure 2, the spectrum of the adjacency matrix is $\sigma(\mathcal{A}) = (-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, \sqrt{2}, \sqrt{2}, \sqrt{5})$, so the ordered multiplicity list of \mathcal{A} is $(1, 2, 4, 2, 1)$. But the trace technique in [2] shows that if $B \in \mathcal{S}(T_{BF})$ has the five distinct eigenvalues $\check{\beta}_1 < \check{\beta}_2 < \check{\beta}_3 < \check{\beta}_4 < \check{\beta}_5$ with multiplicities $m_B(\check{\beta}_1) = m_B(\check{\beta}_5) = 1, m_B(\check{\beta}_2) = m_B(\check{\beta}_4) = 2, m_B(\check{\beta}_3) = 4$, then $\check{\beta}_1 + \check{\beta}_5 = \check{\beta}_2 + \check{\beta}_4$.

2.1 Minimum Rank of a Graph

There has recently been extensive interest in the problem of determining the maximum multiplicity, or equivalently, the minimum rank of a graph, and more progress has been made on that problem than on IEPG. While this parameter is straightforward to compute for trees, it is not known how to compute minimum rank of an arbitrary graph. Many additional developments occurred as a result of the American Institute of Mathematics workshop “Spectra of families of matrices described by graphs, digraphs, and sign patterns.” Links to recent papers and additional information is available on the workshop web page [1]. That page also has a link to an on-line catalog of minimum rank of families of graphs [21], and a table listing the minimum ranks of all graphs of order at most seven. See [15] for a more extensive survey of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there. Here we briefly summarize some of the known results for determining minimum rank.

If G is not connected, then any matrix $B \in \mathcal{S}(G)$ is block diagonal, with the diagonal blocks corresponding to the connected components of G , and the spectrum of B is the union of the spectra of the diagonal blocks. Thus we usually restrict our attention to connected graphs.

Characterizations of graphs of order n having minimum rank $1, 2, n - 2$ and $n - 1$ have been obtained: For any graph G that has an edge, any matrix in $\mathcal{S}(G)$ has at least two nonzero entries, so $\text{mr}(G) \geq 1$. By examining the rank 1 matrix J (all of whose entries are 1), we see that $\text{mr}(K_n) = 1$. If G is connected, then for any matrix $B \in \mathcal{S}(G)$, there is no row consisting entirely of zeros. Any rank 1 matrix B with no row of zeros has all entries nonzero, and thus $\mathcal{G}(B) = K_n$. Thus, for G a connected graph of order greater than one, $\text{mr}(G) = 1$ is equivalent to $G = K_n$.

Theorem 2.10. [6] *A connected graph G has $\text{mr}(G) \leq 2$ if and only if G does not contain as an induced subgraph any of: P_4 , Dart, \bowtie , or $K_{3,3,3}$ (the complete tripartite graph), all shown in Figure 3.*

Additional characterizations of graphs having minimum rank 2 can be found in [6]. The situation is, however, very different for minimum rank 3, where Tracy Hall has recently shown there is an infinite family of forbidden induced subgraphs [19].

For any graph G , a matrix $B \in \mathcal{S}(G)$ with $\text{rank } B \leq n - 1$ can always

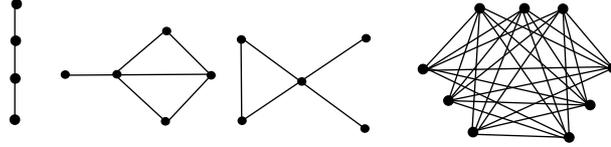


Figure 3: Forbidden induced subgraphs for $\text{mr}(G) = 2$

be obtained by taking $C \in \mathcal{S}(G)$, $\gamma \in \sigma(C)$, and $B = C - \gamma I$. Thus, for any graph G , $\text{mr}(G) \leq n - 1$. If $B \in \mathcal{S}(P_n)$, by deleting the first row and last column, we obtain an upper triangular $(n - 1) \times (n - 1)$ submatrix with nonzero diagonal, so $\text{rank } B \geq n - 1$. Thus $\text{mr}(P_n) = n - 1$.

Theorem 2.11. [16] *Let $|G| = n$. If for all $B \in \mathcal{S}(G)$, all eigenvalues of B are simple, then $G = P_n$. Equivalently, $\text{mr}(G) = n - 1$ implies $G = P_n$.*

Minimum rank $|G| - 2$ was characterized in [22, 29]. Through cut-vertex reduction (see Theorem 2.12 below), the problem can be reduced to the case of a 2-connected graph. A *polygonal path* is a “path” of cycles built from cycles C_{m_1}, \dots, C_{m_k} constructed so that for $i = 2, \dots, k$, $C_{m_{i-1}} \cap C_{m_i}$ has exactly one edge, and for and $j < i - 1$, $C_{m_j} \cap C_{m_i}$ has no edges. An example of a polygonal path is shown in Figure 4. A polygonal path has been called an LSEAC graph, a 2-connected partial linear 2-tree, a 2-connected partial 2-path, or a linear 2-tree by some authors (the last of these terms is unfortunate, since a polygonal path need not be a 2-tree). For a 2-connected graph, $\text{mr}(G) = |G| - 2$ if and only if G is a polygonal path [22] (see Theorem 4.13 below). A complete characterization of graphs G having $\text{mr}(G) = |G| - 2$ is also given in [29].

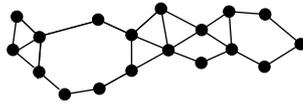


Figure 4: A polygonal path

If the graph G has a cut-vertex, then the problem of computing $\text{mr}(G)$ can be reduced to computing the minimum ranks of several smaller induced subgraphs.

Theorem 2.12. [4] *Let v be a cut-vertex of graph G . For $i = 1, \dots, h$, let*

$W_i \subseteq V(G)$ be the vertices of the i th component of $G - v$ and let $V_i = \{v\} \cup W_i$. Then

$$\text{mr}(G) = \min\left\{\sum_{i=1}^h \text{mr}(G[V_i]), \sum_{i=1}^h \text{mr}(G[W_i] + 2)\right\}.$$

3 Spectral Graph Theory

Spectral Graph Theory has traditionally used the spectra of specific matrices associated with the graph, such as the adjacency matrix, the Laplacian matrix, or their normalized forms, to provide information about the graph. For certain families of graphs it is possible to characterize a graph by the spectrum (of one of these matrices). More generally, this is not possible, but useful information about the graph can be obtained from the spectra of these various matrices. There are also important applications to other fields such as chemistry. Here we present only a very brief introduction to this extensive subject. The reader is referred to several books, such as [12, 11, 13, 8], for a more thorough discussion and lists of references to original papers.

Let G be a graph with vertices $\{1, \dots, n\}$. We will discuss the following matrices associated with G .

- The *adjacency matrix*, $\mathcal{A} = [a_{ij}]$, where $a_{ij} = 1$ if $\{i, j\}$ is an edge of G and $a_{ij} = 0$ otherwise. Let $\sigma(\mathcal{A}) = (\alpha_1, \dots, \alpha_n)$.
- The *diagonal degree matrix*, $\mathcal{D} = \text{diag}(\text{deg}_G 1, \dots, \text{deg}_G n)$.
- The *normalized adjacency matrix*, $\widehat{\mathcal{A}} = \sqrt{\mathcal{D}}^{-1} \mathcal{A} \sqrt{\mathcal{D}}^{-1}$, where $\sqrt{\mathcal{D}} = \text{diag}(\sqrt{\text{deg}_G 1}, \dots, \sqrt{\text{deg}_G n})$. Let $\sigma(\widehat{\mathcal{A}}) = (\hat{\alpha}_1, \dots, \hat{\alpha}_n)$.
- The *Laplacian matrix*, $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Let $\sigma(\mathcal{L}) = (\lambda_1, \dots, \lambda_n)$.
- The *normalized Laplacian matrix*, $\widehat{\mathcal{L}} = \sqrt{\mathcal{D}}^{-1} (\mathcal{D} - \mathcal{A}) \sqrt{\mathcal{D}}^{-1} = I - \widehat{\mathcal{A}}$. Let $\sigma(\widehat{\mathcal{L}}) = (\hat{\lambda}_1, \dots, \hat{\lambda}_n)$.
- The *signless Laplacian matrix*, $|\mathcal{L}| = \mathcal{D} + \mathcal{A}$. Let $\sigma(|\mathcal{L}|) = (\mu_1, \dots, \mu_n)$.
- The *normalized signless Laplacian matrix*, $|\widehat{\mathcal{L}}| = \sqrt{\mathcal{D}}^{-1} (\mathcal{D} + \mathcal{A}) \sqrt{\mathcal{D}}^{-1} = I + \widehat{\mathcal{A}}$. Let $\sigma(|\widehat{\mathcal{L}}|) = (\hat{\mu}_1, \dots, \hat{\mu}_n)$.

Note that $\mathcal{A}, \widehat{\mathcal{A}}, |\mathcal{L}|, |\widehat{\mathcal{L}}|, \mathcal{L}, \widehat{\mathcal{L}} \in \mathcal{S}(G)$.

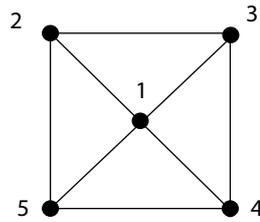


Figure 5: Wheel on 5 vertices

Example 3.1. For the wheel on five vertices, shown in Figure 5, the matrices \mathcal{A} , $\widehat{\mathcal{A}}$, \mathcal{L} , $\widehat{\mathcal{L}}$, $|\mathcal{L}|$, $|\widehat{\mathcal{L}}|$ and their spectra are

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \widehat{\mathcal{A}} = \begin{bmatrix} 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

$$\sigma(\mathcal{A}) = (-2, 1 - \sqrt{5}, 0, 0, 1 + \sqrt{5}) \quad \sigma(\widehat{\mathcal{A}}) = \left(-\frac{2}{3}, -\frac{1}{3}, 0, 0, 1\right)$$

$$\mathcal{L} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix} \quad \widehat{\mathcal{L}} = \begin{bmatrix} 1 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{-1}{2\sqrt{3}} & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 \\ \frac{-1}{2\sqrt{3}} & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} \\ \frac{-1}{2\sqrt{3}} & -\frac{1}{3} & 0 & -\frac{1}{3} & 1 \end{bmatrix}$$

$$\sigma(\mathcal{L}) = (0, 3, 3, 5, 5) \quad \sigma(\widehat{\mathcal{L}}) = \left(0, 1, 1, \frac{4}{3}, \frac{5}{3}\right)$$

$$|\mathcal{L}| = \begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 & 0 \\ 1 & 0 & 1 & 3 & 1 \\ 1 & 1 & 0 & 1 & 3 \end{bmatrix} \quad |\widehat{\mathcal{L}}| = \begin{bmatrix} 1 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & 1 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 1 & \frac{1}{3} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{bmatrix}$$

$$\sigma(|\mathcal{L}|) = \left(1, \frac{9-\sqrt{17}}{2}, 3, 3, \frac{9+\sqrt{17}}{2}\right) \quad \sigma(|\widehat{\mathcal{L}}|) = \left(\frac{1}{3}, \frac{2}{3}, 1, 1, 2\right)$$

Since $\widehat{\mathcal{L}} = I - \widehat{\mathcal{A}}$ and $|\widehat{\mathcal{L}}| = I + \widehat{\mathcal{A}}$, if the spectrum of any one of $\widehat{\mathcal{A}}$, $\widehat{\mathcal{L}}$, $|\widehat{\mathcal{L}}|$, is known, the spectrum of any of the others is readily computed. If G is regular of degree r then $\widehat{\mathcal{A}} = \frac{1}{r} \mathcal{A}$, $\mathcal{L} = rI - \mathcal{A}$, $|\mathcal{L}| = rI + \mathcal{A}$, so if the spectrum of any one of \mathcal{A} , $\widehat{\mathcal{A}}$, \mathcal{L} , $\widehat{\mathcal{L}}$, $|\mathcal{L}|$, $|\widehat{\mathcal{L}}|$ is known so are the spectra of all of these matrices.

The matrices \mathcal{A} , $\widehat{\mathcal{A}}$, $|\mathcal{L}|$, $|\widehat{\mathcal{L}}|$ are all non-negative, and if G is connected, they are all irreducible. The *Perron-Frobenius Theorem* [20] provides the following information about an irreducible non-negative matrix B (where $\rho(B)$ denotes the spectral radius, i.e., maximum absolute value of an eigenvalue of B).

1. $\rho(B)$ is an eigenvalue of B .
2. $\rho(B)$ is a simple eigenvalue of B .
3. There is a positive vector x such that $Bx = \rho(B)x$

Let B be a symmetric non-negative matrix. Eigenvectors for distinct eigenvalues of B are orthogonal. If B has a positive eigenvector x for eigenvalue β , then any eigenvector for a different eigenvalue cannot be positive, and so $\beta = \rho(B)$. Let $e = [1, 1, \dots, 1]^T$. Then since $\widehat{\mathcal{A}} \sqrt{\mathcal{D}}e = \sqrt{\mathcal{D}}e$, $\rho(\widehat{\mathcal{A}}) = 1$, and $\rho(|\widehat{\mathcal{L}}|) = 2$.

The matrices \mathcal{A} , \mathcal{D} , $\widehat{\mathcal{A}}$, \mathcal{L} , $\widehat{\mathcal{L}}$, $|\mathcal{L}|$, $|\widehat{\mathcal{L}}|$ are also connected via the incidence matrix. The *(vertex-edge) incidence matrix* \mathcal{N} of graph G with n vertices and m edges is the $n \times m$ 0,1-matrix with rows indexed by the vertices of G and columns indexed by the edges of G , such that the v, e entry of \mathcal{N} is 1 (respectively, 0) if edge e is (respectively, is not) incident with vertex v . Then

$$\mathcal{N}\mathcal{N}^T = \mathcal{D} + \mathcal{A} = |\mathcal{L}| \text{ and } |\widehat{\mathcal{L}}| = (\sqrt{\mathcal{D}}^{-1} \mathcal{N}) (\sqrt{\mathcal{D}}^{-1} \mathcal{N})^T$$

An *orientation* of graph G is the assignment of a direction to each edge, converting edge $\{i, j\}$ to either arc (i, j) or arc (j, i) . The *oriented incidence matrix* \mathcal{N}' of an oriented graph G' with n vertices and m arcs is the $n \times m$ 0,1,-1-matrix with rows indexed by the vertices of G and columns indexed by the arcs of G such that the $v, (w, v)$ -entry of \mathcal{N}' is 1, the $v, (v, w)$ -entry of \mathcal{N}' is -1, and all other entries are 0. If G' is any orientation of G and \mathcal{N}' is the oriented incidence matrix then

$$\mathcal{N}'\mathcal{N}'^T = \mathcal{D} - \mathcal{A} = \mathcal{L} \text{ and } \widehat{\mathcal{L}} = (\sqrt{\mathcal{D}}^{-1} \mathcal{N}') (\sqrt{\mathcal{D}}^{-1} \mathcal{N}')^T$$

So \mathcal{L} , $|\mathcal{L}|$, $\widehat{\mathcal{L}}$, $|\widehat{\mathcal{L}}|$ are all positive semidefinite, and so have non-negative eigenvalues. The *inertia* of a matrix B is the ordered triple (i_+, i_-, i_0) , where i_+ is the number of positive eigenvalues of B , i_- is the number of negative eigenvalues of B , and i_0 is the number of zero eigenvalues of B . By Sylvester's Law of Inertia [20], the inertia of \mathcal{L} is equal to the inertia of $\widehat{\mathcal{L}}$. Since $\widehat{\mathcal{L}} + |\widehat{\mathcal{L}}| = 2I$, the following facts have been established, provided G is connected.

1. $\sigma(|\widehat{\mathcal{L}}|) \subset [0, 2]$ and $\hat{\mu}_n = 2$ with eigenvector $\sqrt{\mathcal{D}}e$.
2. $\sigma(\widehat{\mathcal{A}}) \subset [-1, 1]$ and $\hat{\alpha}_n = 1$ with eigenvector $\sqrt{\mathcal{D}}e$.

3. $\sigma(\widehat{\mathcal{L}}) \subset [0, 2]$ and $\hat{\lambda}_1 = 0$ with eigenvector $\sqrt{D}e$.
4. $\lambda_1 = 0$.

If G is not connected, the multiplicity of 0 as an eigenvalue of \mathcal{L} is the number of connected components of G . For each of the matrices \mathcal{A} , $\widehat{\mathcal{A}}$, $|\mathcal{L}|$, $|\widehat{\mathcal{L}}|$, \mathcal{L} , $\widehat{\mathcal{L}}$ the spectrum is the union of the spectra of the components.

If A is the adjacency matrix of the line graph $L(G)$ of G (cf. [18]), then $\mathcal{N}^T \mathcal{N} = 2I + A$. It follows from well-known results in matrix theory that the non-zero eigenvalues of $\mathcal{N} \mathcal{N}^T$ and $\mathcal{N}^T \mathcal{N}$ are the same (including multiplicities). Thus the spectrum of $|\mathcal{L}|$ is readily determined from that of the adjacency matrix of $L(G)$. Since $\mathcal{N}^T \mathcal{N}$ is positive semidefinite, the least eigenvalue of the adjacency matrix of $L(G)$ is greater than or equal to -2. See [18] for further discussion of line graphs and graphs with adjacency matrix having all eigenvalues greater than or equal to -2.

We now turn our attention to information about the graph that can be extracted from the spectra of these matrices. This is the approach typically taken in Spectral Graph Theory.

The following parameters of graph G are determined by the spectrum of the adjacency matrix or, equivalently, by its characteristic polynomial $p(x) = x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ (note $a_{n-1} = 0$ since $\text{tr } \mathcal{A} = 0$).

1. The number of edges of $G = -a_{n-2} = \frac{\text{tr} \mathcal{A}^2}{2} = \frac{\sum \alpha_i^2}{2}$.
2. The number of triangles of $G = -\frac{a_{n-3}}{2} = \frac{\text{tr} \mathcal{A}^3}{6} = \frac{\sum \alpha_i^3}{6}$.

The first equality in each of these statements is obtained by viewing the coefficient of $p(x)$ as $(-1)^k$ times the sum of the determinants of principal submatrices of order k , the second is obtained by considering walks, and the third is obtained by using the fact that a real symmetric matrix is unitarily similar to a diagonal matrix. Unfortunately these results do not extend cleanly to longer cycles, as can be seen by considering the 4-cycle.

One use of spectral graph theory is to assist in determining whether two graphs are isomorphic. If two graphs have different spectra (equivalently, different characteristic polynomials) then clearly they are not isomorphic. However, non-isomorphic graphs can be cospectral. Figure 6 shows two graphs having the same spectrum for the adjacency matrix.

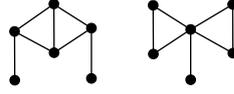


Figure 6: Cospectral graphs with $p(x) = -1 + 4x + 7x^2 - 4x^3 - 7x^4 + x^6$

A graph G is called *spectrally determined* if any graph with the same spectrum is isomorphic to G . Of course, one must identify the matrix (e.g., adjacency, Laplacian, etc.) from which the spectrum is taken. Examples of graphs that are spectrally determined by the adjacency matrix [13]:

- Complete graphs
- Empty graphs
- Graphs with one edge
- Graphs missing only 1 edge
- Regular graphs of degree 2
- Regular graphs of degree $n - 3$, where n is the order of the graph

However, Schwenk found a method for constructing cospectral trees and proved his famous result that almost all trees are not spectrally determined by the adjacency matrix.

Theorem 3.2. [36] *As n goes to infinity, the proportion of trees on n vertices that are determined by the spectrum of the adjacency matrix goes to 0.*

McKay [33] showed that the same is true of the Laplacian spectrum of a tree. For a tree T , it is easy to find a diagonal matrix having diagonal entries in $\{1, -1\}$ such that $|\mathcal{L}|(T) = D^{-1}\mathcal{L}(T)D$, so $\sigma(|\mathcal{L}|(T)) = \sigma(\mathcal{L}(T))$

Theorem 3.3. [36, 33], see also [14, 13] *For almost all trees T there is a non-isomorphic tree T' that T and T' have the same adjacency spectrum, and the same Laplacian spectrum, and the same signless Laplacian spectrum.*

A recent survey of results on cospectral graphs and spectrally determined graphs can be found in [14].

There are many other graph parameters for which information can be extracted from the spectra of the various matrices associated with a graph. Here we mention only two examples, the vertex connectivity and the diameter.

The second smallest eigenvalue of the Laplacian $\mathcal{L}(G)$, $\lambda_2(G)$, is called the algebraic connectivity of G .

Theorem 3.4. [17], see also [18] *If G is not K_n , the vertex connectivity is greater than or equal to the algebraic connectivity, i.e., $\lambda_2(G) \leq \kappa_0(G)$.*

The *distance* between two vertices in a graph is the length of (i.e., number of edges in) a shortest path between them. The diameter of a graph G , $\text{diam}(G)$, is maximum distance between any two vertices of G .

Theorem 3.5. [7] *The diameter of a connected graph G is less than the number of distinct eigenvalues of the adjacency matrix of G .*

The proof of Theorem 3.5 extends to show $\text{diam}(G)$ is less than the number of distinct eigenvalues of any non-negative matrix $B \in \mathcal{S}(G)$. If T is a tree and $B \in \mathcal{S}(T)$, it is possible to find a real number γ and a 1,-1-diagonal matrix S such that $STS^{-1} + \gamma I$ is non-negative. Thus, we have the following theorem.

Theorem 3.6. [31] *If T is a tree, for any $B \in \mathcal{S}(T)$, the diameter of T is less than the number of distinct eigenvalues of B .*

There are many examples of trees T for which the minimum number of distinct eigenvalues is $\text{diam}(T) + 1$. Barioli and Fallat [2] gave an example of a tree for which the minimum number of distinct eigenvalues is strictly greater than this bound, and Kim and Shader [30] recently found a family of trees for whom the diameter can be less than the minimum number of distinct eigenvalues by an arbitrary amount.

There are also several other diameter results involving the Laplacian and normalized Laplacian, see for example [8].

4 Colin de Verdière-type Parameters

Colin de Verdière introduced several parameters defined as the maximum nullity of a subset of matrices in $\mathcal{S}(G)$ (the nullity is often called *corank* in this context). For such a parameter, every matrix M over which the nullity is maximized must satisfy the *Strong Arnold Property*: If X is a symmetric matrix such

that $MX = 0$ and $x_{i,j} \neq 0$ implies $i \neq j$ and i, j is not an edge of G , then $X = 0$. The Strong Arnold Property is the requirement that certain manifolds intersect transversally. See [24] for more details. The Strong Arnold Property is related to minor monotonicity of the graph parameter. A *contraction* of G is obtained by identifying two adjacent vertices of G , and suppressing any loops or multiple edges that arise in this process. A *minor* of G arises by performing a series of deletions of edges, deletions of isolated vertices, and/or contractions of edges. A graph parameter ζ is *minor monotone* if for any minor G' of G , $\zeta(G') \leq \zeta(G)$. Colin de Verdière-type parameters have close connections to both classical spectral graph theory and (via maximum multiplicity) to the Inverse Eigenvalue Problem.

4.1 $\mu(G)$

The symmetric matrix $L = [\ell_{ij}]$ is a generalized Laplacian matrix of G if for all i, j with $i \neq j$, $\ell_{ij} < 0$ if i and j are adjacent in G and $\ell_{ij} = 0$ if i and j are nonadjacent. Clearly any generalized Laplacian L of G is in $\mathcal{S}(G)$, and \mathcal{L} and $\widehat{\mathcal{L}}$ are generalized Laplacians. Note that if L is a generalized Laplacian then $-L$ has non-negative off-diagonal elements, and so there is a real number c such that $cI - L$ is non-negative. Thus, if G is connected, by the Perron-Frobenius Theorem, the least eigenvalue of L is simple.

The graph parameter $\mu(G)$ was introduced by Colin de Verdière in 1990 ([9] in English). A thorough introduction to this important subject is provided by [24]. Here we list only a few of the definitions and results.

The matrix L is a Colin de Verdière matrix for graph G if

1. L is a generalized Laplacian matrix of G .
2. L has exactly one negative eigenvalue (of multiplicity 1).
3. L satisfies the Strong Arnold Property.

The *Colin de Verdière number* $\mu(G)$ is the maximum multiplicity of 0 as an eigenvalue of a Colin de Verdière matrix. A Colin de Verdière matrix realizing this maximum is called optimal. Note that condition 2 ensures that $\mu(G)$ is the multiplicity of $\lambda_2(L)$ for an optimal Colin de Verdière matrix. Clearly $\mu(G) \leq M(G)$, since any Colin de Verdière matrix is in $\mathcal{S}(G)$. There are many examples, such as Example 4.1 below, of graphs G where this inequality is strict, primarily due to

the failure of the Strong Arnold Property for matrices realizing $M(G)$, but these tend to occur in relatively sparse graphs, such as trees, where other methods are available for computation of maximum multiplicity.

Example 4.1. The star $K_{1,n}$ has $M(K_{1,n}) = n - 1$ and this multiplicity is attained (for eigenvalue 0) by the adjacency matrix. For $n > 3$ (if the high degree vertex is 1), the matrix

$X = (\mathbf{e}_2 - \mathbf{e}_3)(\mathbf{e}_4 - \mathbf{e}_5)^T + (\mathbf{e}_4 - \mathbf{e}_5)(\mathbf{e}_2 - \mathbf{e}_3)^T$ (where $\mathbf{e}_k = [0, \dots, 0, 1, 0, \dots, 0]^T$) shows that \mathcal{A} does not have the Strong Arnold Property. In fact, $\mu(K_{1,n}) = 2$ (provided $n > 2$) [24].

Theorem 4.2. [9], see also [24]. *If H is a minor of G then $\mu(H) \leq \mu(G)$.*

The Strong Arnold Property is essential to this minor-monotonicity, as the following example shows.

Example 4.3. Consider the graph \bowtie shown in Figure 3. From [6], $\text{mr}(\bowtie) = 3$, so $M(\bowtie) = 2$, but deletion of the edge that joins the two degree 2 vertices produces $K_{1,4}$ and $M(K_{1,4}) = 3$.

The Robertson-Seymour theory of graph minors asserts that the family of graphs G with $\mu(G) \leq k$ can be characterized by a finite set of forbidden minors [24]. The parameter $\mu(G)$ was introduced to describe planarity. A graph is planar if it can be drawn in the plane without crossing edges. A graph is outerplanar if it has such a drawing with a face that contains all vertices. An embedding of a graph G into \mathbb{R}^3 is *linkless* if no disjoint cycles in G are linked in \mathbb{R}^3 . A graph is *linklessly embeddable* if it has a linkless embedding. See [24] for more detail. Colin de Verdière; Robertson, Seymour and Thomas; and Lovász and Schijver have used this to establish the following characterizations.

Theorem 4.4. (See [24] for original sources.)

1. $\mu(G) \leq 1$ if and only if G is a disjoint union of paths.
2. $\mu(G) \leq 2$ if and only if G is outerplanar.
3. $\mu(G) \leq 3$ if and only if G is planar.
4. $\mu(G) \leq 4$ if and only if G is linklessly embeddable.

Theorem 4.5. (See [18] for original sources.) *Let L be a generalized Laplacian matrix of the graph G with $\sigma(L) = (\omega_1, \omega_2, \dots, \omega_n)$. If G is 2-connected and outerplanar then $m_L(\omega_2) \leq 2$. If G is 3-connected and planar then $m_L(\omega_2) \leq 3$.*

4.2 $\nu(G)$

The parameter $\nu(G)$ [10] is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices $A \in \mathcal{S}(G)$ that satisfy:

1. $A \in \mathcal{S}(G)$.
2. A is positive semidefinite.
3. A satisfies the Strong Arnold Hypothesis.

Theorem 4.6. [10]. *If H is a minor of G then $\nu(H) \leq \nu(G)$.*

The parameter $\nu(G)$ is useful in determining the maximum eigenvalue multiplicity for the family of positive semidefinite matrices described by G . Lovász, Saks and Schrijver [32] showed that vertex connectivity of a graph G is a lower bound to maximum nullity of positive semidefinite matrices described by G and showed for almost all matrices attaining maximum nullity an additional property that implies the Strong Arnold Property. The version in the next theorem was explicitly stated by van der Holst in [23].

Theorem 4.7. [23, Theorem 4], [32, Corollary 1.4] *For every graph G ,*

$$\kappa_0(G) \leq \nu(G).$$

4.3 $\xi(G)$

The Strong Arnold Hypothesis seems to be essential to minor-monotonicity, as noted in Example 4.3. The parameter $\xi(G)$ was introduced in [5] as the Colin de Verdière-type parameter specifically designed for use in computing minimum rank and maximum eigenvalue multiplicity, by removing any unnecessary restrictions but preserving minor monotonicity.

Definition 4.8. [5] For a graph G , $\xi(G)$ is the maximum nullity among matrices $A \in \mathcal{S}(G)$ satisfying the Strong Arnold Property.

Clearly, $\mu(G) \leq \xi(G) \leq M(G)$ and $\nu(G) \leq \xi(G) \leq M(G)$. It is possible to have both $\mu(G) < \xi(G)$ and $\nu(G) < \xi(G)$.

Example 4.9. The graph G shown in Figure 7 has $\mu(G) = \nu(G) = 2 < 3 = \xi(G)$ [5].

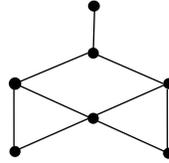


Figure 7: A graph for which $\mu(G) = \nu(G) < \xi(G)$

Theorem 4.10. [5] *The parameter $\xi(G)$ is minor monotone.*

To use minor monotonicity one needs to know $\xi(G)$ for various graphs G .

Theorem 4.11. [5] *The values of $\xi(G)$ are known for the following graphs.*

1. $\xi(K_p) = p - 1$
2. $\xi(K_{p,q}) = p + 1$ if $p \leq q$ and $3 \leq q$.
3. $\xi(P_n) = 1$
4. If T is a tree that is not a path, then $\xi(T) = 2$.

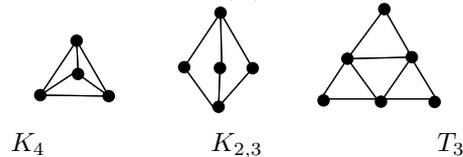
Corollary 4.12.

1. If K_p is a minor of G , then $M(G) \geq p - 1$.
2. If $p \leq q, 3 \leq q$ and $K_{p,q}$ is a minor of G , then $M(G) \geq p + 1$.

In [22] $\xi(G)$ was used to determine the 2-connected graphs having maximum multiplicity 2.

Theorem 4.13. [22] *Let G be a 2-connected graph of order n . The following are equivalent:*

1. $\xi(G) = 2$.
2. $M(G) = 2$.
3. $\text{mr}(G) = n - 2$.
4. G has no K_4 -, $K_{2,3}$ -, or T_3 -minor (see Figure 8).
5. G is a polygonal path.

Figure 8: Forbidden minors for $\text{mr}(G) = n - 2$ (for 2-connected graphs)

5 Conclusion

Clearly there are close connections between the recent work in Spectral Graph Theory on Colin de Verdière-type parameters, and the Inverse Eigenvalue Problem of a Graph. Matrices attaining $M(G)$ for eigenvalue 0 are central to this connection. Equivalently, we are concerned with matrices attaining the minimum rank of G . In particular, matrices that satisfy the Strong Arnold Property and the realize the minimum rank of G are of interest.

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