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2-Domination Number of Butterfly Graphs

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Abstract: A new family of graphs called Butterfly Graphs is introduced recently and study of its parameters is under progress. Butterfly Graphs are undirected graphs and are widely used in interconnection networks. In this paper, we find 2-dominating sets of Butterfly graph and show that 2-domination number of $BF(n)$ is

$$\gamma_{\times 2}(BF(n)) = 2^n \qquad \text{if } n = 2$$
$$\gamma_{\times 2}(BF(n)) = k \times 2^n + r \times 2^{n-1} \qquad \text{if } n = 3k + r.$$

Keywords: Butterfly Graphs, Dominating set, 2-Dominating set

AMS (MOS) Subject Classification: 68R10

1 Introduction

Butterfly Networks are interconnection networks which form the back bone of distributed memory parallel architecture. They have very good symmetry in structure and are regular graphs. One of the current interests of researchers is Butterfly graphs, because they are studied as a topology of parallel machine architecture.

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The vertices model processors and the edges represent communication links between processors.

A Butterfly Network of dimension n denoted by $BF(n)$ is defined as a graph whose vertices are pairs $(\alpha; v)$ where $\alpha \in \{0, 1, 2, \dots, n\}$ and $v = (x_{n-1}, x_{n-2}, \dots, x_1, x_0)$ is binary string of length n where $x_i = 0$ or 1. For $\alpha = 0, 1, 2, \dots, (n-1)$ there is an edge from vertex $(\alpha; v)$ to $(\alpha + 1; v')$ where v and v' are identical in all n-bits except $x_{\alpha-1} \neq x'_{\alpha+1}$. When $\alpha = n - 1, \alpha + 1$ becomes 0 and there are edges between the vertices in the level L_{n-1} and the vertices in the level L_0, subject to above condition. And these edges are drawn in such a way that they look like wings of a Butterfly. The edges from $(\alpha; v)$ to $(\alpha+1; v)$ are called *straight edges*, the edges from $(\alpha; v)$ to $(\alpha + 1; v')$ where $v' = v + 2^{k+1}$ are called *forward slant edges* and the edges from $(\alpha; v)$ to $(\alpha+1; v')$ where $v' = v - 2^{k+1}$ are called *backward slant edges*.

Berge [3] and Ore [6] were the first to define dominating sets. Domination in a graph along with its many variations provides an extremely rich area of study.

Let $G(V, E)$ be a graph. A subset D of $V(G)$ is called a *dominating set* if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. Then we say that this vertex in D dominates the vertices to which it is adjacent in G. A subset D of G is called a *2-Dominating set of* G if every vertex in $V \setminus D$ is dominated by at least two vertices in D.

Cardinality of the minimum 2-dominating set is called the *2-domination number of* G and is denoted by $\gamma_{\times 2}(G)$.

Generally compilation of $\gamma_{\times 2}(G)$ is difficult. But there are methods and proofs available for special graphs like complete graph, cycle graph, path graph, etc. Butterfly graphs are also a very special family of graphs, which can be defined recursively. This recursive nature of the Butterfly graph renders an easy step by step method to construct a 2-dominating set for $BF(n)$ and to prove its minimality. Symmetric structure of $BF(n)$ is an advantage to compute 2-domination number of $BF(n)$.

2 Recursive Construction

Here we present recursive construction given by Bharath and Raspaud [1] for $BF(n)$ from $BF(n-1)$.

Step 1 : Consider two copies of $BF(n-1)$ and place them adjacent to each other. There are $(n-2)$ levels $L_0, L_1, \ldots, L_{n-2}$. Adjacency of the vertices between levels L_0 to L_{n-2} in each copy is preserved as in the original graph $BF(n-1)$.

Step 2 : Add a new level L_{n-1} with 2^n vertices. The vertices of level L_{n-2} in both the copies are joined to vertices of L_{n-1} by straight and slant edges. Winged edges from vertices of level L_0 are extended to join vertices of L_{n-1} in both the copies.

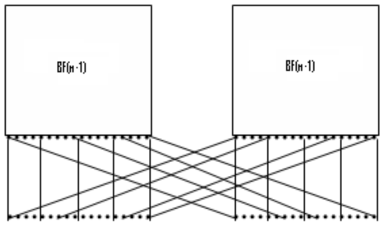

Figure 1: Recursive Construction of $BF(n)$

Theorem 2.1. *The 2-domination number of $BF(n)$ is*

$$\gamma_{\times 2}(BF(n)) = 2^n \qquad \text{if } n = 2$$
$$\gamma_{\times 2}(BF(n)) = k \times 2^n + r \times 2^{n-1} \qquad \text{if } n = 3k + r.$$

Proof. **Case (i):** Consider butterfly graph $BF(2)$. $BF(2)$ is a bipartite graph with 4 vertices in each part, which are independent sets called as levels L_0 and L_1. Domination number of $BF(2)$ is 2 so $\gamma_{\times 2}(BF(2)) \geq \gamma(BF(2)) = 2$. By definition of edges in $BF(2)$, a vertex from one level is adjacent to three vertices of the other level. So to 2-dominate this vertex we need to choose any two of these three adjacent vertices. We also observe that edges in $BF(2)$ without wings can be partitioned into two $K_{2,2}$ and in each bipartite graph $K_{2,2}$ a pair of vertices from one part 2-dominate the vertices of other part. As these two $K_{2,2}$ in $BF(2)$ are disjoint, we must choose two vertices from each $K_{2,2}$ for 2-domination of $BF(2)$. This also gives that less than 4 vertices can not 2-dominate vertices of the bipartite graph $BF(2)$. Hence minimal 2-dominating set of $BF(2)$ has at least 4 vertices.

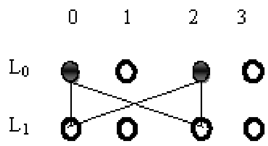

Figure 2: 2-domination in a $K_{2,2}$

The dark black circles in figure 2 form a 2-dominating set of a $K_{2,2}$ which is {(0,0), (0,2), (1,0),(1,2)}. Similarly remaining four vertices also form a $K_{2,2}$ which contributes 2 more vertices to a 2-dominating set. So minimal 2-dominating set of $BF(2)$ has 4 vertices and $\gamma_{\times 2}(BF(2)) = 4$.

Case (ii): Consider butterfly graph $BF(3)$. From recursive construction we know that $BF(3)$ has 2 copies of $BF(2)$ and a level L_2 with 2^3 vertices. From Theorem 4.2 of Chapter 4 of Ph.D. Thesis of Indrani Kelkar [4], the domination number of $BF(3)$ is 6. Minimal dominating set of $BF(3)$ has one vertex from each level in the left part of $BF(3)$ and similarly 3 vertices from right part of $BF(3)$. We know that $BF(3)$ has three levels L_0, L_1 and L_2 with 8 vertices each and degree of each vertex is 4. By the choice of vertices in a minimal dominating set, six vertices of a minimal dominating set cannot 2-dominate all 24 vertices of $BF(3)$. So 2-domination number is > 6. By the symmetry of butterfly graph, 2-domination number must be even. So 2-domination number of $BF(3)$ must be ≥ 8. Now we show the existence of a 2-dominating set of cardinality 8. From Case (i) we observe that in $BF(3)$, between every pair of levels there are 4 disjoint $K_{2,2}$. So if we select all 8 vertices from any one level they will belong to every $K_{2,2}$ in pairs which 2-dominate all vertices of the remaining two levels. Thus $BF(3)$ has a 2-dominating set of cardinality 8. Also it is clear that any minimal 2-dominating set for $BF(3)$ must have all 8 vertices taken from any one level $L_(0)$, $L_(1)$ or $L_(2)$ as each of these levels has two vertices from a $K_{2,2}$ and so 2-dominate all the vertices of remaining two levels. Any other choice will need more than 8 vertices for 2-domination of all the vertices. Hence $\gamma_{\times 2}(BF(3)) = 8$.

Case(iii): Consider Butterfly graph $BF(3k + r)$. For $r = 0$, let us divide all the levels in $BF(3k)$ into k sets of 3 levels each. From the proof in Case(ii) for a minimal dominating set we must include all 2^n vertices from one level (say middle level of three consecutive levels) as they 2-dominate all the vertices of preceding and succeeding levels. So a minimal 2-dominating set of $BF(3k)$ has $k2^n$ vertices. Now for 2-domination of the remaining r levels for $r = 1$ and 2 for minimality of 2-dominating set we must make a change in selection of vertices. For $r = 1$, we have an extra level along with k sets of 3 levels each. For 2-domination of vertices of last level we use winged edges as they form $K_{2,2}$ between first and last level as well as first and second level. If we follow selection as shown in figure 2 where all the black vertices form a 2-dominating set of $BF(4)$, then we get a minimal 2-dominating set for $BF(3k+1)$ by adding extra $2n-1$ vertices to the 2-dominating set for first $3k$ levels. This gives that a minimal 2-dominating set of $BF(3k+1)$ has $k2^n + 2^{n-1}$ vertices.

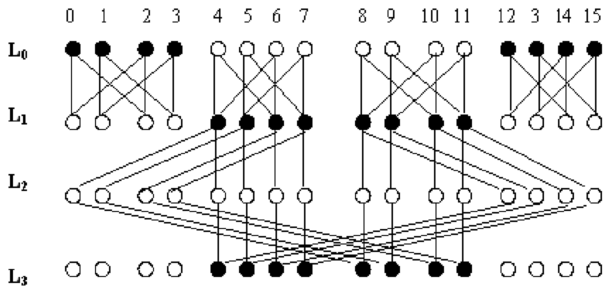

Figure 3: 2-Dominating set of $BF(4)$

In figure 3 black vertices indicate a 2-dominating set for $BF(4)$. Here observe that vertices from last level, not included into dominating set, are 2-dominated by winged edges from first level.

Next for $r = 2$, $BF(3k+2)$ is divided into $k-1$ sets of 3 levels and last $3+2 = 5$ levels are considered together for 2-domination. From first $k-1$ sets of 3 levels each we get $(k-1)2^{n-1}$ vertices into 2-dominating set. From last 5 levels we need to choose 2^{n-1} vertices from specific $K_{2,2}$ sets so as to have minimality and get pattern for last 5 levels as shown below in figure 4.

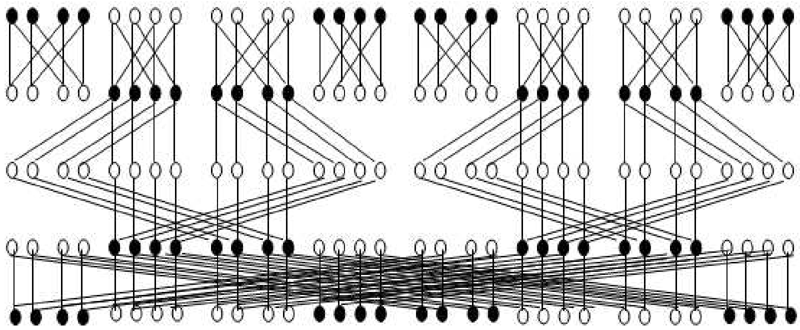

Figure 4: 2-Dominating set of BF(5)

Thus a minimal 2-dominating set of $BF(3k+2)$ has $(k-1)2^n + 4 \times 2^{n-1} = k2^n + 2 \times 2^{n-1}$ vertices.

Thus we get cardinality of minimal 2-dominating sets as follows

$$\begin{aligned} |D| &= k2^n + 1 \times 2^{n-1} && \text{for } BF(3k+1) \\ &= k2^n + 2 \times 2^{n-1} && \text{for } BF(3k+2) \\ |D| &= k2^n + r \times 2^{n-1} && \text{for } BF(3k+r) \end{aligned}$$

where $k = 1, 2, 3, \ldots$ and $r = 1, 2$. □

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