A Class of Harmonic Meromorphic Functions with Negative Coefficients

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Abstract: A class of complex-valued harmonic meromorphic functions of the form $f(z) = h(z) + \overline{g(z)}, |z| > 1$, is introduced. It is shown that the functions in this class are orientation preserving and univalent outside the unit disk. Necessary and Sufficient coefficient conditions are obtained for functions in this class, which are also shown to be necessary when the co-analytic part g has negative coefficients. Distortion bounds, extreme points, convolution condition and convex combination are also obtained.

Keywords: Harmonic functions,Meromorphic functions, Starlike functions

2000 Mathematics Subject Classification: Primary 30C45,30C50

1 Introduction

Harmonic univalent functions are known to be important in the study of minimal surfaces and have found applications in different fields such as Engineering, Operations research and applied Mathematics [1]. A complex-valued harmonic function is a continuous function $f = u + iv$ in a complex domain $D \subseteq C$ with both u

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and v real harmonic in D. Several classes of complex-valued harmonic univalent functions have been introduced and investigated following the basic work of Clunie and Sheil-Small [2]. Hengartner and Schober [3], among other things, investigated the family M of functions $f(z) = h(z) + \overline{g(z)}$ which are harmonic, meromorphic orientation-preserving and univalent in $\overline{U} = \{z : |z| > 1\}$ where

$$h(z) = z + \Sigma_{n=1}^{\infty} a_n z^{-n}; \quad g(z) = \Sigma_{n=1}^{\infty} b_n z^{-n}; \quad z \in \overline{U}. \tag{1}$$

Jahangiri [4] and Jahangiri and Silverman [5] have also investigated harmonic, meromorphic functions which are starlike in $\overline{U}$.

Here we introduce another class $G_H(\alpha, \beta, t)$ of harmonic, meromorphic functions defined as follows: For $0 \le \beta < 1$, let $G_H(\alpha, \beta, t)$ consist of functions $f \in M$ so that

$$\Re\left\{(1 + e^{i\alpha}) \frac{(zh'(z) - \overline{zg'(z)})}{f_t(z)} - e^{i\alpha}\right\} \ge \beta \tag{2}$$

where $f_t(z) = (1 - t)z + tf(z)$, $0 \le t \le 1$, $0 \le \beta < 1$, and α real.

It is of interest to note that when $t = 1$, the class $G_H(\alpha, \beta, 1)$ has been investigated in [6] and in the special case $\alpha = 0, t = 1$, the class $G_H(0, \beta, 1)$ is related to the class $\Sigma_{\mathbb{H}}^*(\gamma)$ [4] by the relation $G_H(0, \beta, 1) = \Sigma_{\mathbb{H}}^*(\frac{1+\beta}{2})$.

Also let $G_{\overline{H}}(\alpha, \beta, t)$ be the subclass of $G_H(\alpha, \beta, t)$ consisting of functions $f = h + \overline{g}$ in which h and g are of the form

$$h(z) = z + \Sigma_{n=1}^{\infty} a_n z^{-n}; \quad g(z) = -\Sigma_{n=1}^{\infty} b_n z^{-n} \quad a_n \ge 0, b_n \ge 0 \tag{3}$$

We determine sufficient coefficient condition for harmonic meromorphic functions $f = h + \overline{g}$ to be in the class $G_H(\alpha, \beta, t)$. We then show that this coefficient condition is also necessary for $f \in G_{\overline{H}}(\alpha, \beta, t)$. Moreover, we obtain distortion bounds, extreme points, convolution condition and convex combination for functions in $G_{\overline{H}}(\alpha, \beta, t)$.

2 Coefficient Conditions

First we prove a sufficient coefficient condition for harmonic functions in $G_H(\alpha, \beta, t)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are of the form (1). If*

$$\Sigma_{n=1}^{\infty}[2n + t(1 + \beta)]|a_n| + \Sigma_{n=1}^{\infty}[2n - t(1 + \beta)]|b_n| \le 1 - \beta \tag{4}$$

when $0 \le \beta < 1$ and $0 \le t < 1$, then f is univalent, sense preserving, harmonic mapping in $\Delta = \{z : |z| < 1\}$ and $f \in G_H(\alpha, \beta, t)$.

Proof. Consider the function $f = h + \overline{g}$, where h and g are given by (1). In [5] it has been proved that if $\Sigma_{n=1}^{\infty} n|a_n| + \Sigma_{n=1}^{\infty} n|b_n| \leq 1$, $|b_1| < 1$ then f is harmonic, orientation preserving and univalent in Δ.

For $0 \leq \beta < 1$, we note that

$$n \leq \frac{2n + t(1 + \beta)}{1 - \beta} \quad \text{and} \quad n \leq \frac{2n - t(1 + \beta)}{1 - \beta}.$$

Therefore f is harmonic, orientation preserving and univalent in $\overline{U}$ due to (4). In order to show that $f \in G_H(\alpha, \beta, t)$, in view of (2), we have to show that $\Re\left[\frac{A(z)}{B(z)}\right] > \beta$ where

$$A(z) = (1 + e^{i\alpha})[zh'(z) - \overline{zg'(z)}] - e^{i\alpha}[(1 - t)z + t(h(z) + \overline{g(z)})],$$

$$B(z) = (1 - t)z + t(h(z) + \overline{g(z)})$$

Using the fact that $\Re w \geq \beta$ if and only if $|1 - \beta + w| \geq |1 + \beta - w|$ it is enough to show that

$$|A(z) + (1 - \beta)B(z)| - |A(z) - (1 + \beta)B(z)| \geq 0 \quad \text{for } 0 \leq \beta < 1.$$

Differentiating h and g and substituting in the above inequality we obtain

$$
\begin{aligned}
&|A(z) + (1 - \beta)B(z)| - |A(z) - (1 + \beta)B(z)| \\
&= \Big|(1 - \beta - e^{i\alpha})(1 - t)z + (1 - \beta)t(z) + (1 + e^{i\alpha})zh'(z) - e^{i\alpha}th(z) \\
&\quad + \overline{(1 - \beta)tg(z) - (1 + e^{i\alpha})zg'(z) - e^{i\alpha}tg(z)}\Big| \\
&\quad - \Big|(1 + \beta + e^{i\alpha})(1 - t)z + (1 + \beta)th(z) - (1 + e^{i\alpha})zh'(z) + e^{i\alpha}th(z) \\
&\quad + \overline{(1 + \beta)tg(z) + (1 + e^{i\alpha})zg'(z) + e^{i\alpha}tg(z)}\Big| \\
&\geq (2 - \beta)|z| - \Sigma_{n=1}^{\infty}(2n + \beta t)|a_n||z|^{-n} - \Sigma_{n=1}^{\infty}(2n - \beta t)|b_n||z|^{-n} \\
&\quad - \beta|z| - \Sigma_{n=1}^{\infty}(2n + \beta t + 2t)|a_n||z|^{-n} - \Sigma_{n=1}^{\infty}(2n - \beta t - 2t)|b_n||z|^{-n} \\
&= 2\left\{(1 - \beta)|z| - \Sigma_{n=1}^{\infty} 2n + t(1 + \beta)|a_n||z|^{-n} - \Sigma_{n=1}^{\infty} 2n - t(1 + \beta)|b_n||z|^{-n}\right\} \\
&\geq 2\left\{(1 - \beta) - \Sigma_{n=1}^{\infty}\left\{[2n + t(1 + \beta)]|a_n| + [2n - t(1 + \beta)]|b_n|\right\}\right\}
\end{aligned}
$$

Now by (4), this last expression is never negative and so $f \in G_H(\alpha, \beta, t)$. We give an example of a function in the class $G_H(\alpha, \beta, t)$.

Example 2.2. The harmonic function $f = h + \overline{g}$ where

$$h(z) = z + \frac{1-\beta}{4[2+t(1+\beta)]}\frac{1}{z}, \; g(z) = \frac{1-\beta}{4[2-t(1+\beta)]}\frac{1}{\overline{z}}$$

where $0 \leq \beta < 1$ and $0 \leq t < 1$, satisfies the sufficient condition of Theorem 2.1 and hence belongs to the class $G_H(\alpha, \beta, t)$.

Next we show that the coefficient condition (4) is also necessary for functions in $G_{\overline{H}}(\alpha, \beta, t)$.

Theorem 2.3. *Let $f = h + \overline{g}$ be so that h and g are of the form (3). A necessary and sufficient condition for f to be in $G_{\overline{H}}(\alpha, \beta, t)$ is that*

$$\Sigma_{n=1}^{\infty}\left\{[2n+t(1+\beta)]|a_n| + [2n-t(1+\beta)]|b_n|\right\} \leq 1 - \beta \tag{5}$$

Proof. In the view of Theorem 2.1, we need only show that $f \notin G_{\overline{H}}(\alpha, \beta, t)$ if the coefficient inequality (5) does not hold. We note that if $f \in G_{\overline{H}}(\alpha, \beta, t)$ we must have

$$\Re\frac{(1+e^{i\alpha})(zh'(z) - \overline{zg'(z)}) - e^{i\alpha}[(1-t)z + t(h(z) + \overline{g(z)})]}{(1-t)z + t[h(z) + \overline{g(z)}]} - \beta$$

$$= \Re\left\{\frac{(1-\beta)z - \Sigma_{n=1}^{\infty}[n(1+e^{i\alpha}) + t(\beta+e^{i\alpha})]a_n z^{-n}}{z + t\Sigma_{n=1}^{\infty}a_n z^{-n} - t\Sigma_{n=1}^{\infty}b_n \overline{z}^{-n}}\right\} \geq 0$$

This inequality must hold for all $z \in \overline{U}$ and for all real α. Letting $z = r > 1$ and $\alpha = 0$, we have

$$\Re\frac{(1-\beta) - \Sigma_{n=1}^{\infty}[2n + t(\beta+1)]a_n r^{-(n+1)} - \Sigma_{n=1}^{\infty}[2n - t(1+\beta)]b_n r^{-(n+1)}}{1 + t\Sigma_{n=1}^{\infty}a_n r^{-(n+1)} - t\Sigma_{n=1}^{\infty}b_n r^{-(n+1)}}$$

$$= \frac{A(r)}{B(r)} \geq 0.$$

If the condition (5) does not hold, then $A(r)$ is negative for r sufficiently close to 1. Thus there exists a $z_0 = r_0 > 1$ for which the quotient $\frac{A(r)}{B(r)}$ is negative. This contradicts that $\frac{A(r)}{B(r)} \geq 0$ and so the proof is complete. $\qquad\blacksquare$

The distortion bounds for functions in $G_{\overline{H}}(\alpha, \beta, t)$ are now given.

Theorem 2.4. *If $f \in G_{\overline{H}}(\alpha, \beta, t)$ then*

$$r - (1-\beta)r^{-1} \leq |f(z)| \leq r + (1-\beta)r^{-1}, |z| = r > 1.$$

Proof. We prove the right hand inequality. The argument for the left hand inequality is similar and is omitted. Let $f \in G_{\overline{H}}(\alpha, \beta, t)$. Taking the absolute value of f we obtain

$$
\begin{aligned}
|f(z)| &= |z + \Sigma_{n=1}^{\infty} a_n z^{-n} - \Sigma_{n=1}^{\infty} b_n z^{-n}| \\
&\leq r + \Sigma_{n=1}^{\infty}(|a_n| + |b_n|)r^{-n} \\
&\leq r + \Sigma_{n=1}^{\infty}\{[2n + t(1 + \beta)]|a_n| + [2n - t(1 + \beta)]|b_n|\} r^{-1} \\
&\leq r + (1 - \beta)r^{-1}
\end{aligned}
$$

3 Extreme Points

We use the coefficient bounds obtained in section 2 to determine the extreme points for functions in $G_{\overline{H}}(\alpha, \beta, t)$.

Theorem 3.1. $f \in G_{\overline{H}}(\alpha, \beta, t)$ *if and only if* f *can be expressed as*

$$
f = \Sigma_{n=0}^{\infty}(x_n h_n + y_n g_n)
$$

where $z \in \overline{U}$,

$$
h_0(z) = z, h_n(z) = z + \frac{1 - \beta}{2n + t(1 + \beta)} z^{-n} (n = 1, 2, \dots)
$$

$$
g_0(z) = z, g_n(z) = z - \frac{1 - \beta}{2n - t(1 + \beta)} (\overline{z})^{-n} (n = 1, 2, \dots)
$$

and

$$
\Sigma_{n=0}^{\infty}(x_n + y_n) = 1, x_n \geq 0, y_n \geq 0.
$$

Proof. Note that for f we may write

$$
\begin{aligned}
f(z) &= \Sigma_{n=0}^{\infty}(x_n h_n + y_n g_n) \\
&= x_0 h_0 + y_0 g_0 + \Sigma_{n=1}^{\infty} x_n \left[z + \frac{1 - \beta}{2n + t(1 + \beta)} z^{-n} \right] \\
&\qquad + y_n \left[z - \frac{1 - \beta}{2n - t(1 + \beta)} (\overline{z})^{-n} \right] \\
&= \Sigma_{n=0}^{\infty}(x_n + y_n)z + \Sigma_{n=1}^{\infty} \frac{1 - \beta}{2n + t(1 + \beta)} x_n z^{-n} \\
&\qquad - \Sigma_{n=1}^{\infty} \frac{1 - \beta}{2n - t(1 + \beta)} y_n (\overline{z})^{-n}
\end{aligned}
$$

Now, by Theorem 2.3,

$$\Sigma_{n=1}^{\infty}\left\{[2n+t(1+\beta)]\left[\frac{(1-\beta)x_n}{2n+t(1+\beta)}\right]+[2n-t(1+\beta)]\left[\frac{(1-\beta)y_n}{2n-t(1+\beta)}\right]\right\}$$

$$=(1-\beta)\Sigma_{n=1}^{\infty}(x_n+y_n)\leq(1-\beta)$$

Conversely, suppose that $f \in G_{\overline{H}}(\alpha,\beta,t)$ then

$$\Sigma_{n=1}^{\infty}\left[\frac{2n+t(1+\beta)}{1-\beta}a_n+\frac{2n-t(1+\beta)}{1-\beta}b_n\right]\leq 1.$$

Setting

$$x_n=\frac{2n+t(1+\beta)}{1-\beta}a_n, y_n=\frac{2n-t(1+\beta)}{1-\beta}b_n, 0\leq x_0\leq 1$$

$y_0=1-x_0-\Sigma_{n=1}^{\infty}(x_n+y_n)$, we obtain $f(z)=\Sigma_{n=0}^{\infty}(x_n h_n+y_n g_n)$ as required.

4   Convolution and Convex Combination

In this section we show that the class $G_{\overline{H}}(\alpha,\beta,t)$ is invariant under convolution and convex combinations of its members.

For harmonic functions

$$f(z)=z+\Sigma_{n=1}^{\infty}a_n z^{-n}-\Sigma_{n=1}^{\infty}b_n(\overline{z})^{-n}, F(z)=z+\Sigma_{n=1}^{\infty}A_n z^{-n}-\Sigma_{n=1}^{\infty}B_n(\overline{z})^{-n}$$

we define the convolution of f and F as

$$
\begin{aligned}
(f*F)(z) &= f(z)*F(z) \\
&= z+\Sigma_{n=1}^{\infty}a_n A_n z^{-n}-\Sigma_{n=1}^{\infty}b_n B_n(\overline{z})^{-n}
\end{aligned}
$$

Theorem 4.1. *For $0\leq\gamma\leq\beta\leq 1$. Let $f\in G_{\overline{H}}(\alpha,\beta,t)$ and $F\in G_{\overline{H}}(\alpha,\gamma,t)$. Then $f*F\in G_{\overline{H}}(\alpha,\beta,t)\subset G_{\overline{H}}(\alpha,\gamma,t)$*

Proof. Suppose f and F are so that $f*F$ is given by the above convolution. Since $f\in G_{\overline{H}}(\alpha,\beta,t)$ and $F\in G_{\overline{H}}(\alpha,\gamma,t)$, the coefficients of f and F must satisfy conditions given by Theorem 2.3. So far the coefficients of $f*F$ we can write

$$\Sigma_{n=1}^{\infty}\left\{[2n+t(1+\beta)]a_n A_n+[2n-t(1+\beta)]b_n B_n\right\}$$

$$\leq\Sigma_{n=1}^{\infty}\left\{[2n+t(1+\beta)]a_n+[2n-t(1+\beta)]b_n\right\}$$

The right hand side of the above inequality is bounded by $1 - \beta$ because $f \in G_{\overline{H}}(\alpha, \beta, t)$.

Thus $f * F \in G_{\overline{H}}(\alpha, \beta, t) \subset G_{\overline{H}}(\alpha, \gamma, t)$.

Finally, we examine the convex combinations of $G_{\overline{H}}(\alpha, \beta, t)$.

Theorem 4.2. *The family $G_{\overline{H}}(\alpha, \beta, t)$ is closed under convex combination.*

Proof. Suppose $f_i(z) = z + \Sigma_{n=1}^{\infty} a_{i_n} z^{-n} - \Sigma_{n=1}^{\infty} b_{i_n} (\overline{z})^{-n} \in G_{\overline{H}}(\alpha, \beta, t)$ where $a_{i_n} \geq 0, b_{i_n} \geq 0$ and $i = 1, 2, 3, \ldots$.

Then by Theorem 2.3,

$$\Sigma_{n=1}^{\infty} \left[(2n + t(1 + \beta)) a_{i_n} + (2n - t(1 + \beta)) b_{i_n} \right] \leq 1 - \beta.$$

For $\Sigma_{i=1}^{\infty} t_i = 1$ and $0 \leq t_i \leq 1$, the convex combinations of f_i, may be written as

$$\Sigma_{i=1}^{\infty} t_i f_i(z) = z + \Sigma_{n=1}^{\infty} \left(\Sigma_{i=1}^{\infty} t_i a_{i_n} \right) z^{-n} - \Sigma_{n=1}^{\infty} \left(\Sigma_{i=1}^{\infty} t_i b_{i_n} \right) (\overline{z})^{-n}$$

Thus $\Sigma_{i=1}^{\infty} t_i f_i(z) \in G_{\overline{H}}(\alpha, \beta, t)$, since

$$\Sigma_{n=1}^{\infty} [2n + t(1 + \beta)] \left(\Sigma_{i=1}^{\infty} t_i a_{i_n} \right) + [2n - t(1 + \beta)] \left(\Sigma_{i=1}^{\infty} t_i b_{i_n} \right)$$
$$= \Sigma_{i=1}^{\infty} t_i \left\{ \Sigma_{n=1}^{\infty} (2n + t(1 + \beta)) a_{i_n} + (2n - t(1 + \beta)) b_{i_n} \right\}$$
$$\leq \Sigma_{i=1}^{\infty} t_i (1 - \beta) = (1 - \beta).$$

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