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Survey Article

The Polya Theory and Permutation Groups

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Abstract: This paper presents a thorough exposition of the Polya Theory in its enumerative applications to permutations groups. The discussion includes the notion of the power group, the Burnside's Lemma along with the notions on groups, stabilizer, orbits and other group theoretic terminologies which are so fundamentally used for a good introduction to the Polya Theory. These in turn, involve the introductory concepts on weights, patterns, figure and configuration counting series along with the extensive discussion of the cycle indexes associated with the permutation group at hand. In order to realize the applications of the Polya Theory, the paper shows that the special figure series $c(x) = 1 + x$ is useful to enumerate the number of G-orbits of r-subsets of any arbitrary set X. Furthermore, the paper also shows that this special figure series simplifies the counting of the number of orbits determined by any permutation group which consequently determines whether or not the said permutation group is transitive.

Keywords: Permutation Groups, Cycle Index Polynomial, G-orbits, Power Group, Stabilizer, Figure Counting Series, Configuration Counting Series, Transitive Group

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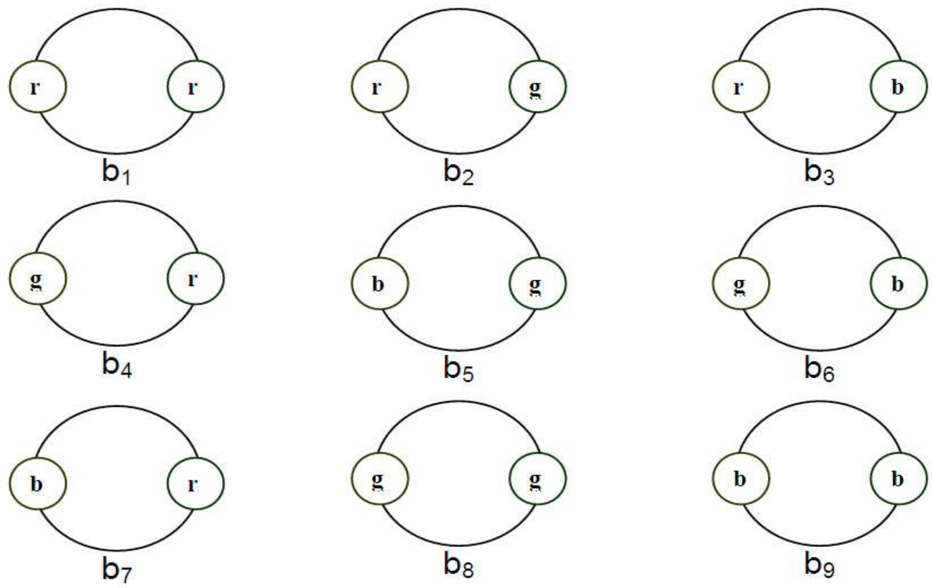
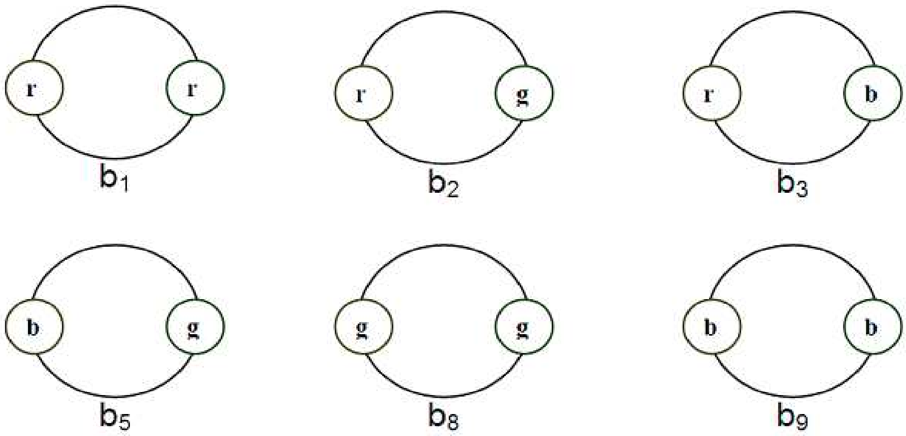
1 Introduction

Many people have difficulty in doing some counting problems probably because sometimes, a situation comes wherein distinct objects are often considered equal. If a teacher for instance is interested in knowing the number of families represented by her class, then she will consider two children to be "equal" if and only if they are siblings. Suppose next we consider the problem of counting non-equivalent bracelets with two beads of three different colors; red (r), blue (b) and green (g). By simple combinational analysis, there will be exactly $3^2 = 9$ possible faces of the bracelets with the above specified colors. They are illustrated below.



Let us now divide these nine bracelets into groups of bracelets by considering two bracelets similar if one can be obtained from the other by rotation. Then we see that b_2 is rotationally equivalent to b_4, hence both should belong to the same group. Likewise, b_3 and b_7 are equivalent; b_5 and b_6 are also equivalent. On the other hand, we see that b_1 belongs to a group that contains itself and so does b_8. Thus, in the sense of grouping these bracelets, we are led to have classified six different bracelets that are non-equivalent under rotation. They are shown below.



In this paper, certain enumerative techniques like the one illustrated above will be developed and used for the solutions of some counting problems specifically those that call the notion of permutation groups. A thorough exposition of a powerful tool in the said enumeration will be the central feature of study from a point of view first developed by George Polya in 1938.

2 The Power Group

Consider two permutation groups G_1, of order m acting on $X = \{x_1, x_2, \ldots, x_d\}$ and another permutation group G_2 of order n acting on $Y = \{y_1, y_2, \ldots, y_e\}$. Here, we refer to the sets X and Y as the object set of G_1 and G_2 respectively.

Definition 2.1. The power group of G_1 and G_2, denoted by $G_2^{G_1}$, is a permutation group which acts on Y^X, the set of all functions from X into Y. For each pair of permutation $\pi \in G_1$ and $\beta \in G_2$, there is a permutation, denoted by $(\pi;\beta)$ in the power group $G_2^{G_1}$ such that for each $f \in Y^X$ and each $x \in X$, $(\pi;\beta)f(x) = \beta(f(\pi(x)))$.

Example 2.2. The cyclic group $C_3 = \{\pi_1, \pi_2, \pi_3\}$ which acts on $X = \{1, 2, 3\}$ where

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

and with $S_2 = \{\sigma_1, \sigma_2\}$, the symmetric group of degree 2 acting on $Y = \{a, b\}$ we have the permutations

$$\sigma_1 = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \qquad \text{and} \qquad \sigma_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Notice that the set of all functions from X into$Y$, that is $Y^X = \{f_1, f_2, \dots, f_8\}$ consists of eight different mappings, where in Table 1, the action of these mappings are described.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
1	a	a	a	b	b	a	b	b
2	a	a	b	a	a	b	b	b
3	a	b	a	a	b	b	a	b

Table 1

To obtain the different permutations of $S_2^{C_3}$, the process requires us to associate each of the two permutations in S_2 with the three permutations in C_3. Hence the power group $S_2^{C_3}$ consists of the following six permutations,

$$\begin{aligned} \alpha_1 &= (\pi_1 \,;\, \sigma_1) & \qquad \alpha_4 &= (\pi_2 \,;\, \sigma_2) \\ \alpha_2 &= (\pi_1 \,;\, \sigma_2) & \qquad \alpha_5 &= (\pi_3 \,;\, \sigma_1) \\ \alpha_3 &= (\pi_2 \,;\, \sigma_1) & \qquad \alpha_6 &= (\pi_3 \,;\, \sigma_2) \end{aligned}$$

and going through all $f \in Y^X$ we have the following

$$\alpha_1 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_8 & f_7 & f_5 & f_6 & f_3 & f_4 & f_2 & f_1 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_3 & f_4 & f_2 & f_6 & f_7 & f_5 & f_8 \end{pmatrix}$$

$$\alpha_4 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_8 & f_5 & f_6 & f_7 & f_4 & f_2 & f_3 & f_1 \end{pmatrix}$$

$$\alpha_5 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_4 & f_2 & f_3 & f_7 & f_5 & f_6 & f_8 \end{pmatrix}$$

$$\alpha_6 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_8 & f_6 & f_7 & f_5 & f_2 & f_3 & f_4 & f_1 \end{pmatrix}.$$

In Table 2, we describe the action of these permutations by applying these to each $x_i \in X$.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
α_1	$a\ a\ a$	$a\ a\ b$	$a\ b\ a$	$b\ a\ a$	$b\ a\ b$	$a\ b\ b$	$b\ b\ a$	$b\ b\ b$
α_2	$b\ b\ b$	$b\ b\ a$	$b\ a\ b$	$a\ b\ b$	$a\ b\ a$	$b\ a\ a$	$a\ a\ b$	$a\ a\ a$
α_3	$a\ a\ a$	$a\ b\ a$	$b\ a\ a$	$a\ a\ b$	$a\ b\ b$	$b\ b\ a$	$b\ a\ b$	$b\ b\ b$
α_4	$b\ b\ b$	$b\ a\ b$	$a\ b\ b$	$b\ b\ a$	$b\ a\ a$	$a\ a\ b$	$a\ b\ a$	$a\ a\ a$
α_5	$a\ a\ a$	$b\ a\ a$	$a\ a\ b$	$a\ b\ a$	$b\ b\ a$	$b\ a\ b$	$a\ b\ b$	$b\ b\ b$
α_6	$b\ b\ b$	$a\ b\ b$	$b\ b\ a$	$b\ a\ b$	$a\ a\ b$	$a\ b\ a$	$b\ a\ a$	$a\ a\ a$

Table 2

The power group has order and degree (mn) and e^d respectively.

3 The Cycle Index Polynomial

Let us begin this section by considering the disjoint cycles of a particular length on every permutation $\pi \in G$.

Definition 3.1. Let G be a permutation group which acts on X with n elements. Suppose further $\pi \in G$. If π can be written as a product of k_1 disjoint cycles of length 1, k_2 disjoint cycles of length 2, $\ldots$, k_n disjoint cycles of length n, then π is said to be of type $(k_1, k_2, \ldots, k_n)$.

Example 3.2. Consider Example 2.2 with $X = \{1, 2, 3\}$, $G = C_3 = \{\pi_1, \pi_2, \pi_3\}$ where

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)\ (2)\ (3)$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ \ 2\ \ 3)$$

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ \ 3\ \ 2).$$

Then π_1 is of type $(3,0,0)$ and π_2 is of type $(0,0,1)$. Similarly, π_3 is of type $(0,0,1)$.

Note that $k_1+2k_2+\cdots+nk_n = n$ since the sum of the length of the cycle is the total number of elements in S. Now, since each $\pi \in G$ can be written uniquely as a product of disjoint cycles, let us denote by $j_k(\pi)$, the number of cycles of length k in the disjoint cycle decomposition of π for each $k = 1, 2, \ldots, n$. We are now ready to define the cycle index of G.

Definition 3.3. Let G be a permutation group on a set X of n elements. If $\pi \in G$ is of type $(j_1(\pi), j_2(\pi), \ldots, j_n(\pi))$ form the product $s_1^{j_1(\pi)} s_2^{j_2(\pi)} \cdots s_n^{j_n(\pi)}$ where the s_i are indeterminates. Then the cycle index of G is the polynomial defined by

$$Z(G; s_1, s_2, \ldots, s_n) = \frac{1}{\mid G \mid} \left\{ \sum_{\pi \in G} s_1^{j_1(\pi)} s_2^{j_2(\pi)} \cdots s_n^{j_n(\pi)} \right\}.$$

Example 3.4. Consider $G = S_3 \{\pi_1, \pi_2, \ldots, \pi_6\}$ where

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)\ \ (2)\ \ (3)$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)\ \ (2\ \ 3)$$

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\ \ 2)\ \ (3)$$

$$\pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ \ 2\ \ 3)$$

$$\pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ \ 3\ \ 2)$$

$$\pi_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ \ 3)\ \ (2).$$

Observe that π_1 is of type $(3,0,0)$ since it has 3 cycles of length 1. This results to a term s_1^3. The permutations π_2, π_3 and π_6 each have one cycle of length 1 and one cycle of length 2. Hence these are of type $(1,1,0)$ which results to a term $3s_1s_2$. Finally, the permutations π_4 and π_5 each have one cycle of length 3. These are of type $(0,0,1)$ which contribute $2s_3$. Hence, the cycle index of S_3 is given by

$$Z(S_3; s_1, s_2, s_3) = \frac{1}{6}\left(s_1^3 + 3s_1s_2 + 2s_3\right).$$

4 The Cycle Index of Special Permutation Groups

Let S_n be the symmetric group of degree n. Without loss of generality, let us assume that an element of S_n is of type $(k_1, k_2, \ldots, k_n)$. Our aim first is to find p – the number of elements of S_n having the above type. Then one of the terms of $Z(S_n)$ is

$$\frac{1}{n!} p\,(s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}).$$

Claim: $p = \dfrac{n!}{(1^{k_1}k_1!)(2^{k_2}k_2!)\cdots(n^{k_n}k_n!)}$

Consider a permutation in S_n say

$$\pi = (z_1\, z_2\, \ldots\, z_r), \quad r \leq n$$

with cycle structure $s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$. We know that each permutation of n objects is associated with a partition of n, that is

$$1k_1 + 2k_2 + \cdots + nk_n = n.$$

Since π is a permutation of n objects, then the number of ways of rearranging the elements in π is $n!$. But among these the k_i cycles of length i contribute a rearrangement which can be done by interchanging cycles of the same length. They make a contribution of $k_1!k_2!\cdots k_n!$ arrangement by the product rule.

Now a cycle of length r such as in the above can be represented by r different ways by cyclically permuting the z_i that is

$$(z_1\ z_2 \cdots\ z_r) = (z_r\ z_1 \cdots\ z_{r-1}) = \cdots = (z_{r-1}\ z_r \cdots\ z_1).$$

Hence, we have to consider the contribution they make to the $n!$ rearrangements. But there are $1^{k_1}2^{k_2}\cdots n^{k_n}$ of them because there are k_i cycles of length i for each i. Thus,

$$p = \frac{n!}{(1^{k_1}k_1!)(2^{k_2}k_2!)\cdots(n^{k_n}k_n!)}$$

and so,

$$Z(S_n) = \frac{1}{n!}\left(\frac{n!}{1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!}\right)s_1^{k_1}s_2^{k_2}\cdots s_n^{k_n}.$$

Going through all $\pi \in S_n$, we have the following result.

Theorem 4.1. *The cycle index of the symmetric group $Z(S_n)$ is given by*

$$Z(S_n) = \frac{1}{n!}\sum p(s_{1_{k_1}}^{k_1}s_2^{k_2}\cdots s_n^{k_n})$$

where

$$p = \frac{n!}{(1^{k_1}k_1!)(2^{k_2}k_2!)\cdots(n^{k_n}k_n!)}$$

and the summation is over all n-tuple $(k_1, k_2, \ldots, k_n)$ of non-negative integers k_i satisfying

$$1k_1 + 2k_2 + \cdots + nk_n = n.$$

Now let $\pi \in S_n$ and suppose π is of type $(k_1, k_2, \ldots, k_n)$. Notice that whenever the sum $k_2 + k_4 + k_6 + \cdots$ is even, then $\pi \in A_n$ since this sum totals all disjoint cycles having lengths $2, 4, 6, \ldots$. Thus, we have come up with the following result which gives the cycle index of the alternating group A_n.

Corollary 4.2. *The cycle index of the alternating group is given by*

$$Z(A_n) = Z(S_n) + Z(S_n; s_1, -s_2, s_3, -s_4, \dots).$$

Example 4.3. From Example 3.4, we know that

$$Z(S_3) = \frac{1}{6}\left(s_1^3 + 3s_1s_2 + 2s_3\right).$$

Solving for $Z(A_3)$, we obtain

$$\begin{aligned} Z(A_3) &= Z(S_3) + Z(S_3; s_1, -s_2, s_3) \\ &= \left(\frac{1}{3}\right)(s_1^3 + 2s_3). \end{aligned}$$

In order to derive the cycle index of the cyclic group, we are going to need the following lemma.

Lemma 4.4. *Let G be a cyclic group of order n generated by x. Suppose that the greatest common divisor of the positive integer k $(k \leq n)$ and n is d. Then x^k and x^d generate the same cyclic subgroup G, its order is n/d.*

Now, suppose C_n is the cyclic group of permutations of order n generated by π acting on a set of S of n elements, where

$$\pi = (1\ 2\ 3\ \dots\ n).$$

Also, let $g_k = \pi^k$ $(k = 0, 1, 2, \dots, n-1)$. By the above lemma, g_k and $g_{(k,n)}$ generate the same cyclic subgroup. In fact, if $(k,n) = (j,n) = d$, then g_j, g_k and g_d will generate the same cyclic subgroup, and in addition, they are of the same type since if $s \in S$, then s belongs to the cycle obtained from

$$s \to g_d(s) \to g_d^2(s) \to \cdots \to g_d^{n/d}(s) = s.$$

This means that g_d is the product of d cycles of length n/d.

Now, to obtain the cycle index of C_n, we shall first find the number of solutions of $(k,n) = d$ or equivalently $\left(\frac{k}{d}, \frac{n}{d}\right) = 1$, since for all such k, the corresponding g_k are having the same type. By definition of the Euler -ϕ function, the number of solutions of $(k/d, n/d) = 1$ is $\phi(n/d)$. Hence, we have the following result.

Theorem 4.5. *The cycle index of the cyclic group C_n is given by*

$$Z(C_n) = \frac{1}{n} \sum_{d|n} \phi(n|d) s_{n|d}^{d}.$$

Example 4.6. To compute $Z(C_6)$, let us first find the possible candidates for the number d, satisfying $d \mid 6$. By inspection, we know that $d = 1, 2, 3, 6$. Now, computing for $\phi(6/d)$, we have

$$\begin{array}{ll} \phi(6/1) = \phi(6) = 2 & \phi(6/2) = \phi(3) = 2 \\ \phi(6/3) = \phi(2) = 1 & \phi(6/6) = \phi(1) = 1 \end{array}$$

and hence,

$$Z(C_6) = \left(\frac{1}{6}\right)(s_1^6 + s_2^3 + 2s_3^2 + s_6).$$

Similarly, if we compute for $Z(C_4)$, we have $d = 1, 2, 4$ with

$$\phi(4/1) = \phi(4) = 2 \qquad \phi(4/2) = \phi(2) = 1 \qquad \phi(4/4) = \phi(1) = 1.$$

Thus,

$$Z(C_4) = \left(\frac{1}{4}\right)(1.s_1^4 + 1.s_2^2 + 2.s_4^1) = \left(\frac{1}{4}\right)(s_1^4 + s_2^2 + 2s_4).$$

The following corollary gives the cycle index of the dihedral group D_n.

Corollary 4.7. *The cycle index of the dihedral group D_n is given by*

$$Z(D_n) = \left(\frac{1}{2}\right) Z(C_n) + \left(\frac{1}{4}\right)(s_2^{n/2} + s_1^2 s_2^{(n-2)/2}).$$

Let us now characterize the cycle index of the identity permutation group.

Theorem 4.8. $Z(E_n; s_1, s_2, \ldots, s_n) = s_1^n$.

Proof. Since the identity permutation group is of type $(n,\ 0,\ 0, \ldots,\ 0)$ then the result is immediate. Here, the cycle index is dependent on the values of n. And so, for $n = 1$, $Z(E_1) = s_1$; for $n = 2$, $Z(E_2) = s_1^2 \ldots$ and so on. This fact shows that the cycle index depends not only in the structure of G as an abstract group but also on the way in which its elements as permutations are interpreted. $\square$

5 The Burnside's Lemma

It makes sense to begin this section by studying what it means to say two objects are the same.

Definition 5.1. Let G be a permutation group on a set X. For $a, b \in X$, define a relation $a \sim b$ to mean that for some $\pi \in G$; $\pi(a) = b$. This relation is called the **binary relation** on X induced by the permutation group G.

Theorem 5.2. *If G is a permutation group, then $\sim$ defines an equivalence relation.*

Definition 5.3. Let G be a permutation group on a set X. Suppose that $\sim$ defines an equivalence relation on X. Then the equivalence class containing the element x, denoted by $\mathrm{Orb}(x)$ is called the G-orbit of x and is defined by

$$\mathrm{Orb}(x) = \{\pi(x) : \pi \in G\}.$$

Lemma 5.4. $\mathrm{Orb}(x) = \mathrm{Orb}(y)$ *if and only if* $x \sim y$.

The problem of determining the number of equivalent objects in X reduces to the problem of counting the number of distinct G-orbits established by $\sim$ on X induced by G. One way of doing this is simply to count, that is to compute all G-orbits and enumerate them. However, this method seems impractical and even more tedious for complex situations. Fortunately, the Burnside's Lemma which we are going to develop next gives an analytical formula for such counting of G-orbits.

The Burnside's Lemma is a powerful technique in the counting of G-orbits induced by a permutation group. This technique which is particularly efficient when the order of the group is small is considered one of the essential parts in the development of the Polya Theory.

Definition 5.5. Let G be a permutation group on X. An element x in X is said to be **invariant** under a permutation $\pi \in G$ if and only if $\pi(x) = x$. To each $x \in X$, the **stabilizer** of x, denoted by $\mathrm{Stab}(x)$ is defined to be the set

$$\mathrm{Stab}(x) = \{\pi \in G : \pi(x) = x\}.$$

Also, for each $\pi \in G$, let $\phi(\pi)$ denote the number of elements of X that are invariant under π that is

$$\phi(\pi) = \mid \{x \in X : \pi(x) = x\} \mid .$$

Lemma 5.6. $\forall x \in X$, $\mathrm{Stab}(x) \leq G$.

Proof. Let $\pi_1, \pi_2 \in \mathrm{Stab}(x)$. Then $\pi_1\pi_2 \in \mathrm{Stab}(x)$ since $\pi_1\pi_2(x) = \pi_1(x) = x$. Also, if $\pi_1 \in \mathrm{Stab}(x)$, then $\pi_1^{-1} \in \mathrm{Stab}(x)$ since $\pi_1(x) = x$ and by multiplying π_1^{-1} to both sides of the equation, we obtain $x = \pi_1^{-1}(x)$. Hence, $\mathrm{Stab}(x) \leq G \;\forall\, x$. $\square$

The following lemma is immediate.

Lemma 5.7. $\forall x \in X$, $\mid \mathrm{Stab}(x) \mid\mid \mathrm{Orb}(x) \mid = \mid G \mid$.

We are now prepared to state the Burnside's Lemma which provides a formula for the number of G-orbits in terms of the average number of fixed points of the permutation in G.

Theorem 5.8. ***(Burnside's Lemma)*** *Let G be a permutation group acting on the set X and suppose $\sim$ is an equivalence relation on X induced by G. If θ is the number of G-orbits in X, then,*

$$\theta = \frac{1}{|G|}\left(\sum_{\pi \in G} \phi(\pi)\right).$$

Proof. Suppose $G = \{\, \pi_1, \pi_2, \ldots, \}$ and $X = \{1, 2, \ldots\}$. Then notice that

$$\sum_{\pi \in X} \phi(\pi) = \sum_{x \in X} |\mathrm{Stab}(x)|$$

because both sides of the equation count the number of pairs satisfying $\pi(x) = x$. Using Lemma 5.7,

$$\sum_{x \in X} |\mathrm{Stab}(x)| = |G| \left(\sum_{x \in X} \frac{1}{|\mathrm{Orb}(x)|}\right).$$

By assumption, the sum in the right hand side of the above equation is just θ since this sum counts the number of distinct G-orbits $\forall x$. Hence,

$$\sum_{\pi \in G} \phi(\pi) \quad = \quad \sum_{x \in X} \mathrm{Stab}(x) \quad = \quad |G|(\theta)$$

and the proof is completed on division by $|G|$. □

6 The Polya Counting Theory

In many instances, the direct application of the Burnside's Lemma is not practically efficient to permit us to enumerate the distinct G-orbits induced by a permutation group. The difficulty perfectly stems from the computation of the number of invariances for a large ordered group.

The Polya's Theorem provides a tool necessary to facilitate this computation. To formulate and prove Polya's Theorem in an abstract and more concise manner, it is somehow convenient to require the notion of functions and patterns as its enumerations are basically performed over sets whose elements are functions.

In the rest of the discussion, we consider X be a set of elements called "places"; and let Y be a set of elements called "figures". Also, we consider the usual permutation group G acting on X, which we call the **configuration group**. Moreover, an element f in Y^X will be called **configuration**.

Definition 6.1. On Y^X, the set of all configurations from X to Y, define the relation $f_1 \sim f_2$ to mean that for some $\pi \in G; f_1(\pi(x)) = f_2(x) \quad \forall x \in X$.

Example 6.2. Suppose $X = \{1, 2\}, Y = \{a, b\}$ and $G = \{\pi_1,\ \pi_2\}$ where

$$\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

There will be $2^2 = 4$ configurations from X into Y, namely

$$\begin{array}{llll} f_1: & 1 \to a & \qquad f_2: & 1 \to a \\ & 2 \to a & & 2 \to b \\ f_3: & 1 \to b & \qquad f_4: & 1 \to b \\ & 2 \to a & & 2 \to b \end{array}$$

Observe that $f_2(\pi_2(1)) = f_2(2) = b = f_3(1)$ and $f_2(\pi_2(2)) = f_2(1) = a = f_3(2)$. Hence, $f_2 \sim f_3$ since $\pi_2 \in G$ implies that $f_2(\pi_2(x)) = f_3(x)$ for all x. With the identity permutation $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ obviously, we have $f_1 \sim f_1$ and $f_4 \sim f_4$.

The relation $\sim$ is an equivalence relation by virtue of which the set Y^X splits into distinct G-orbits. These G-orbits are called **patterns**.

Example 6.3. In Example 6.2, the set Y^X splits into three distinct patterns, namely $P_1 = \{f_1\}$, $P_2 = \{f_2, f_3\}$ and $P_3 = \{f_4\}$.

In order to lead to the classical Polya Theory, we consider the identity group E on Y, the power group E^G acting on Y^X as well as the weight assignment to each $y \in Y$.

Definition 6.4. Let $w : Y \to \{0, 1, 2, \ldots\}$ called the **weight function** whose range is the set of non-negative integers. For each $k = 0, 1, 2, \ldots$ let $c_k = |w^{-1}(k)|$ be the number of figures with weight k. Further, the series in the indeterminate x,

$$c(x) = \sum_{k=0}^{\infty} c_k x^k$$

which enumerates the elements of Y by weight is called the **figure counting series**.

Example 6.5. Consider Example 6.2 with $Y = \{a, b\}$. Suppose we have $w(a) = 0$ and $w(b) = 1$. Then, $c_0 = 1, c_1 = 1$ and the figure counting series is

$$c(x) = \sum_{k=0}^{1} c_k x^k = c_0 x^0 + c_1 x^1 = 1 + x.$$

Definition 6.6. For each $f \in Y^X$, the **weight of** f, denoted by $w(f)$ is defined by

$$w(f) = \sum_{x \in X} = w(f(x)).$$

Example 6.7. With reference to Example 6.2, we obtain

$$\begin{aligned} w(f_1) &= w(f_1(1)) + w(f_1(2)) = w(a) + w(a) = 0 \\ w(f_2) &= w(f_2(1)) + w(f_2(2)) = w(a) + w(b) = 1 \\ w(f_3) &= w(f_3(1)) + w(f_3(2)) = w(b) + w(a) = 1 \\ w(f_4) &= w(f_4(1)) + w(f_4(2)) = w(b) + w(b) = 2. \end{aligned}$$

In particular, notice that the weight of a function is obtained only once the weights have been chosen and assigned. As illustrated in the above example, $w(f_2) = w(f_3)$ and as was in Example 6.3, these configurations belong to the same pattern. We shall formally contain this result in the following lemma.

Lemma 6.8. *On the power group E^G, if f_1 and f_2 are equivalent then $w(f_1) = w(f_2)$.*

Proof. f_1 and f_2 are equivalent means that $\exists\ \pi \in G$ such that $f_1(\pi(x)) = f_2(x)\ \forall\ x \in X$. Now,

$$w(f_1) = \sum_{x\in X} w(f_1(x)) = \sum_{x\in X} w(f_1(\pi(x)) = \sum_{x\in X} w(f_2(x)) = w(f_2)$$

because the second and the third sum have the same terms the order of which is immaterial. $\square$

Lemma 6.8 gives rise to the following definition.

Definition 6.9. The **weight of a pattern** P, denoted by $w(P)$ is the weight of any $f \in P$.

Example 6.10. Using Example 6.3, we have $w(P_1) = 0; w(P_2) = 1$ and $w(P_3) = 2$.

Definition 6.11. Let C_k be the number of patterns of weight k. Then the series in the inderterminate x,

$$C(x) = \sum_{k=0}^{\infty} C_k x^k$$

is called the **configuration counting series**.

Example 6.12. Again, using Example 6.3, we have $C_0 = 1, C_1 = 1$ and $C_2 = 1$. Then the configuration counting series is given by

$$C(x) = \sum_{k=0}^{2} C_k x^k = C_0 x^0 + C_1 x^1 + C_2 x^2$$
$$C(x) = \sum_{k=0}^{\infty} C_k x^k x^k = 1 + x + x^2.$$

The main thrust of Polya's Theorem is to seek a devise of computing $C(x)$ without determining C_k. This is done by expressing $C(x)$ in terms of $c(x)$ and G. Let us denote by $X(G, c(x))$ to mean $Z(G; c(x), c(x^2), c(x^3), \ldots, c(x^n))$.

Theorem 6.13. ***(Polya's Theorem)*** *The configuration counting series $C(x)$ is obtained by substituting the figure counting series $c(x^k)$ for each indeterminate s_k into the cycle index $Z(G)$ of the configuration group. In symbols, $C(x) = Z(G; c(x))$.*

Let us illustrate the theorem by means of an example.

Example 6.14. We wish to verify $C(x) = 1 + x + x^2$ obtained in Example 6.12. Now, $G = S_2 = \{(1)(2),\ (1 \quad 2)\}$ and

$$Z(S_2; s_1, s_2) \quad = \quad \frac{1}{2}\left(s_1^2 + s_2\right).$$

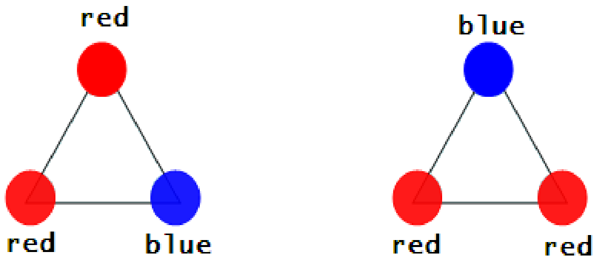
But in Example 6.5, $c(x) = 1 + x \to c(x^2) = 1 + x^2$. Using Polya's Theorem, we obtain

$$\begin{aligned} c(x) &= Z(G;\ \ c(x),\ c(x^2)) \\ &= \frac{1}{2}((1+x)^2 + (1+x^2)) \\ &= \frac{1}{2}(1 + 2x + x^2 + 1 + x^2) \\ &= \frac{1}{2}(2 + 2x + 2x^2) \\ &= 1 + x + x^2 \end{aligned}$$

as desired.

Below, we illustrate Polya's Theorem in a coloring problem.

Example 6.15. Let $X = \{1,2,3\}$ be the set of vertices of an equilateral triangle and suppose $Y = \{\text{red}(r), \text{blue}(b)\}$ be the set of colors to be painted in each of the vertices of the triangle. The total number of ways in which we color the three vertices by red and blue is $2^3 = 8$. Our aim is to determine the number of distinct triangles which are different from another by rotation of the vertices. The group G, appropriate to this problem is S_3 since any rotation of the vertices gives an equivalent triangle. For example, the triangles



are equivalent by a clockwise rotation of the vertices. Now, each $f \in Y^X$ corresponds to a colored vertex in which vertex $k \in X$ has a color $f(k)$. Hence, it can be said that the triangle represented by f has $|f^{-1}(\text{red})|$ vertex of red color and $|f^{-1}(\text{blue})|$ vertex of blue color.

Let E_2 be the identity permutation group acting on Y. If we assign $w(\text{red}) = 0$ and $w(\text{blue}) = 1$, then $c_0 = 1$ and $c_1 = 1$. Hence,

$$c(x) = \sum_{k=0}^{1} c_k x^k = c_o x^0 + c_1 x^1 = 1 + x$$

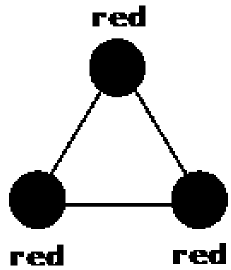
is the counting series for Y. Let f be a function of weight k represents a triangle with $(3-k)$ red vertex and k blue vertex. With in this condition, the configuration counting series $C(x)$ will enumerate the distinct triangles and the coefficient of the term x^k is the number of such triangles with k blue vertex. Now, $c(x) = 1 + x$, then $c(x^2) = 1 + x^2$ and $c(x^3) = 1 + x^3$. Recall that the cycle index polynomial for S_3 is

$$Z(S_3; s_1, s_2, s_3) = \frac{1}{6}\left(s_1^3 + 3s_1s_2 + 2s_3\right).$$

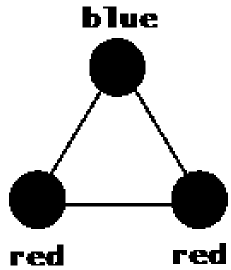
Thus, using Polya's Theorem, we obtain:

$$\begin{aligned} C(x) = Z(S_3; c(x)) &= \frac{1}{6}\left((1+x)^3 + 3(1+x)(1+x^2) + 2(1+x^3)\right) \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

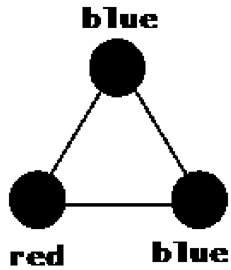
Now, in the polynomial $1 + x + x^2 + x^3$, the term: $1 = x^0$ with coefficient 1 and exponent 0 ($k = 0$) means that there is one triangle with zero blue vertex and $3 - k = 3 - 0 = 3$ red vertices.



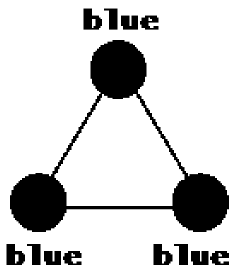
x with coefficient 1 and exponent 1 ($k = 1$) means that there is one triangle which contains one blue vertex and $3 - k = 3 - 1 = 2$ red vertices.



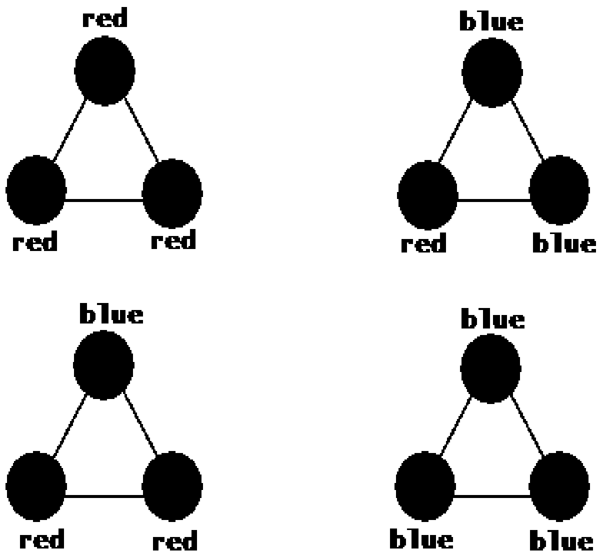
x^2 with coefficient 1 and exponent 2 ($k = 2$) means that there is one triangle with two blue vertices and $3 - k = 3 - 2 = 1$ red vertex.



x^3 with coefficient 1 and exponent 3 ($k = 3$) means that there is one triangle with three blue vertices and $k - 3 = 3 - 3 = 0$ red vertices.



Thus, we have four distinct triangles with blue and red vertices. They are shown below:



7 Some Corollaries

Corollary 7.1. *The number of patterns of configurations determined by the power group* E_m^G *is obtained by substituting the integer* m *for each variable in the cycle index of* G*, where* $m = |Y|$*.*

Example 7.2. The cycle index associated with G in Example 6.2 is

$$Z(G; s_1, s_2) = \frac{1}{2}(s_1^2 + s_2).$$

Hence, with $m = |Y| = 2$, the number of patterns of configuration determined by E_2^G is

$$Z(G; 2,\ 2) = \frac{1}{2}(2^2 + 2) = \frac{1}{2}(4 + 2) = 3.$$

This result is in conformity with Example 6.3 in which we exhibited the 3 distinct patterns.

Corollary 7.3. *The coefficient of x^r in $Z(G; 1 + x)$ is the number of G-orbits of r-subsets of X.*

Now on setting $r = 1$, the above corollary suggests that the coefficient of x in $Z(G; 1+x)$ is the number of G-orbits of the singleton sets of X. But we can think of these sets just like individual elements of X. Thus, in the sense of Burnside's Lemma, the following corollary is immediate.

Corollary 7.4. *The number of orbits determined by G is the coefficient of x in $Z(G; 1 + x)$.*

Definition 7.5. Let G be a permutation group which acts on the set X. G is called **transitive** if it determines only one orbit, namely X itself. Otherwise, G is called **intransitive**.

The following corollary results with the above definition.

Corollary 7.6. *A permutation group G is transitive if and only if the coefficient of x in $Z(G; 1 + x)$ is 1.*

8 The Transitivity of Special Permutation Groups

This section extends the application of Polya's Theorem to a situation in which we characterize the permutation groups: symmetric, alternating, cyclic, dihedral and identity permutation groups by means of substituting the figure series $c(x) = 1 + x$ into their respective cycle indexes. In effect, the resulting polynomial will determine the transitivity of these permutation groups.

Example 8.1. In Example 3.4 and Example 4.3, we have shown that

$$Z(S_3;\ s_1, s_2, s_3) = \frac{1}{6}\left(s_1^3 + 3s_1s_2 + 2s_3\right) \quad \text{and}$$
$$Z(A_3) = \frac{1}{3}(s_1^3 + 2s_3).$$

Then using $c(x) = 1 + x$, notice that

$$Z(S_3;\ 1+x) = 1 + x + x^2 + x^3 \quad \text{and}$$
$$Z(A_3;\ 1+x) = 1 + x + x^2 + x^3.$$

These computations show that the permutation groups S_3 and A_3 are transitive in view of Corollary 7.6. In fact, for any n it can be verified that

$$Z(S_n;\ 1+x) = 1 + x + x^2 + \cdots + x^n \quad \text{and}$$
$$Z(A_n;\ 1+x) = 1 + x + x^2 + \cdots + x^n.$$

Hence, using Polya's Theorem, S_n and A_n are transitive groups. Moreover, it can be verified that

$$Z(C_4;\ 1+x) = 1 + x + 2x^2 + x^3 + x^4$$
$$Z(D_4;\ 1+x) = 1 + x + 2x^2 + x^3 + x^4$$
$$Z(C_6;\ 1+x) = 1 + x + 3x^2 + 4x^3 + 3x^4 + x^5 + x^6$$

and

$$Z(D_6;\ 1+x) = 1 + x + 3x^2 + 3x^3 + 3x^4 + x^5 + x^6.$$

Thus, C_4, D_4, C_6 and D_6 are transitive. In general, C_n and D_n are both transitive.

Finally, since $Z(E_n) = s_1^n$, then

$$\begin{aligned} Z(E_n;\ 1+x) &= (1+x)^n \\ &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n. \end{aligned}$$

Notice that E_n is transitive only if $n = 1$. For $n = 2$, E_n is not transitive.

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