Chamchuri Journal of Mathematics
Volume 2(2010) Number 1, 11–30
http://www.math.sc.chula.ac.th/cjm

New Fixed Point Iterations with Errors for Nonexpansive Nonself-Mapping

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Received 7 Oct 2009
Revised 9 Feb 2010
Accepted 9 Feb 2010

Abstract: In this paper, a new type of two-step iteration with errors for a nonexpansive nonself-mapping is introduced and studied. Weak and strong convergence theorems of such iterations are established under certain conditions in a uniformly convex Banach space. The results obtained in this paper extend and improve the corresponding results of Shahzad [N. Shahzad, Approximating fixed points of non-self nonexpansive mappings in Banach spaces, Nonlinear Anal., 61(2005), 1031–1039], Tan and Xu [K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl., 178(1993), 301–308] and others.

Keywords: Nonexpansive nonself-mapping, Completely continuous, Condition (A), Opial's condition, Kadec-Klee property

2000 Mathematics Subject Classification: 47H10, 47H09, 46B20

1 Introduction

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ the nonexpansive retraction of X onto C, and $T : C \to X$ a given mapping. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative scheme:

* *The author is supported by the Thailand Research Fund, The Commission on Higher Education (MRG5180036), and Naresuan Phayao University, Phayao, Thailand.*

$$\begin{aligned} y_n &= P((1-a_n-\mu_n)x_n + a_nTP((1-\beta_n)x_n+\beta_nTx_n)+\mu_nw_n), \\ x_{n+1} &= P((1-b_n-\delta_n)x_n + b_nTP((1-\gamma_n)y_n+\gamma_nTy_n)+\delta_nv_n), \end{aligned} \quad (1)$$

$n \geq 1$, where $\{a_n\},\{b_n\},\{\mu_n\},\{\delta_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in $[0,1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C.

If $a_n = \mu_n = \delta_n \equiv 0$, then (1) reduces to the iterative scheme defined by Shahzad [23]:

$$x_1 \in C,\ x_{n+1} = P((1-b_n)x_n + b_nTP((1-\gamma_n)x_n+\gamma_nTx_n)), n \geq 1, \quad (2)$$

where $\{b_n\}$ and $\{\gamma_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$.

If $T : C \to C$ and $a_n = \mu_n = \delta_n \equiv 0$, then (1) reduces to the iterative scheme defined by Tan and Xu [27]:

$$x_{n+1} = (1-b_n)x_n + b_nT((1-\gamma_n)x_n+\gamma_nTx_n), \quad n \geq 1, \quad (3)$$

where $\{b_n\}$ and $\{\gamma_n\}$ are appropriate real sequences in $[0,1]$.

Fixed-point iteration processes for approximating fixed points of nonexpansive mappings in Banach spaces have been studied by various authors (see [2, 4, 6, 7, 11–14, 20]) using the Mann iteration process (see [16]) or the Ishikawa iteration process (see [11, 12, 27, 31]). For nonexpansive nonself-mappings, some authors (see [16–19, 22, 25, 27, 29]) have studied the strong and weak convergence theorems in Hilbert spaces or uniformly convex Banach spaces. In 2000, Noor [2] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. In 2005, Suantai [24] defined a new three-step iteration which is an extension of Noor iteration and gave some weak and strong convergence theorems of such iteration for asymptotically nonexpansive mappings in uniformly convex Banach spaces. In 1998, Takahashi and Kim [26] proved strong convergence of approximants to fixed points of nonexpansive nonself-mappings in reflexive Banach spaces with a uniformly Gâteaux differentiable norm. In the same year, Jung and Kim [13] proved the existence of a fixed point for a nonexpansive nonself-mapping in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

In [27], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive self-mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space X. More preciesely, they proved the following theorem.

Theorem 1.1. *(Tan and Xu [27, Theorem 1, p. 305]). Let X be a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm and C a nonempty closed convex bounded subset of X. Let $T: C \to C$ be a nonexpansive mapping. Let $\{b_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} b_n(1-b_n) = \infty$, $\sum_{n=1}^{\infty} \gamma_n(1-b_n) < \infty$, and $\limsup_{n\to\infty} \gamma_n < 1$. Then the sequence $\{x_n\}$ generated by (3) converges weakly to some fixed point of T.*

In Theorem 1.1, the mapping T remains self-mapping of a nonempty closed convex subset C of a uniformly convex Banach space. If, however, the domain C of T is a proper subset of X (and this is the case in several applications), and T maps C into X then, the sequence $\{x_n\}$ generated by (3) may not be well defined.

Note that each l^p $(1 \le p < \infty)$ satisfies the Opial's condition, while all L^p do not have the property unless $p = 2$ and the dual of reflexive Banach spaces with a Fréchet differentiable norm have the Kadec–Klee property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial property; however, their dual does have the Kadec–Klee property (see [10, 14]).

Recently, Shahzad [23] extended Tan and Xu's result [27] to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. He studied weak convergence of the modified Ishikawa type iteration process (2) in a uniformly convex Banach space whose dual has the Kadec–Klee property. The result applies not only to L^p spaces with $(1 \le p < \infty)$ but also to other spaces which do not satisfy Opial's condition or have a Fréchet differentiable norm. Meanwhile, the results of [23] generalized the results of [27].

Inspired and motivated by research going on in this area, we define and study a new type of two-step iterative scheme with errors (1) for nonexpansive nonself-mapping.

The purpose of this paper is to construct an iteration scheme with errors for approximating a fixed point of nonexpansive nonself-mappings (when such a fixed point exists) and to prove some strong and weak convergence theorems for such mappings in a uniformly convex Banach space. Our results extend and improve the corresponding ones announced by Shahzad [23], Tan and Xu [27], and others.

Now, we recall some well known concepts and results.

Let X be a Banach space with dimension $X \ge 2$. The modulus of X is the function $\delta_X : (0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x-y\|\}.$$

Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$. It is known that a uniformly convex Banach space is reflexive and strictly convex. The norm of X is said to be Fréchet differentiable if for each $x \in X$ with $\|x\| = 1$ the limit

$$\lim_{n\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists and is attained uniformly for y, with $\|y\| = 1$.

A subset C of X is said to be retract if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P. A set C is optimal if each point outside C can be moved to be closer to all points of C. It is well known (see [9]) that

(1) If X is a separable, strictly convex, smooth, reflexive Banach space, and if $C \subset$X is an optimal set with interior, then C is a nonexpansive retract of X.

(2) A subset of l^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. Moreover, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Recall that a Banach space X is said to satisfy *Opial's condition* [19] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A Banach space X is said to have the Kadec–Klee property if for every sequence $\{x_n\}$ in X, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$ for more details on Kadec-Klee property, the reader is referred to [8, 25] and the references therein. The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy *condition*(A) [22] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$; see [21, p.377] for an example of nonexpansive mappings satisfying *condition*(A). Senter and Dotson [22] approximated fixed points of a nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [17] and Tan and Xu [27] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same *condition*(A) which is weaker than the requirement that T is demicompact.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.2 ([27])**.** *Let* $\{a_n\}, \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \ \ \forall n = 1, 2, \ldots .$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$*, then*

(1) $\lim_{n\to\infty} a_n$ *exists .*

(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$*.*

Lemma 1.3 ([29])**.** *Let* $p > 1$*,* $r > 0$ *be two fixed numbers. Then a Banach space* X *is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \to [0, \infty)$*,* $g(0) = 0$ *such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

for all x, y *in* $B_r = \{x \in X : \|x\| \le r\}$*,* $\lambda \in [0, 1]$*, where* $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$*.*

Lemma 1.4 ([7])**.** *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X : \|x\| \le r\}$*,* $r > 0$*. Then there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \to [0, \infty), g(0) = 0$ *such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x-y\|)$$

for all $x, y, z \in B_r$*, and all* $\alpha, \beta, \gamma, \in [0, 1]$ *with* $\alpha + \beta + \gamma = 1$*.*

Lemma 1.5 ([4])**.** *Let* X *be a uniformly convex Banach space,* C *a nonempty closed convex subset of* X*, and* $T : C \to X$ *be a nonexpansive mapping. Then* $I - T$ *is demiclosed at* 0*, i.e., if* $x_n \to x$ *weakly and* $x_n - Tx_n \to 0$ *strongly, then* $x \in F(T)$*, where* $F(T)$ *is the set of fixed point of* T*.*

Lemma 1.6 ([24])**.** *Let* X *be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in* X*. Let* $u, v \in X$ *be such that* $\lim_{n\to\infty} \|x_n - u\|$ *and* $\lim_{n\to\infty} \|x_n - v\|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *are subsequences of* $\{x_n\}$ *which converge weakly to* u *and* v*, respectively, then* $u = v$*.*

Lemma 1.7 ([14]). *Let X be a real reflexive Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $x^*, y^* \in \omega_w(x_n)$; where $\omega_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.*

We denote by Γ the set of strictly increasing, continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. Let C be a convex subset of the Banach space X. A mapping $T : C \to C$ is said to be type (γ) [3] if $\gamma \in \Gamma$ and $0 \leqslant \alpha \leqslant 1$,

$$\gamma(\|\alpha Tx + (1-\alpha)Ty - T(\alpha x + (1-\alpha)y)\|) \leqslant \|x - y\| - \|Tx - Ty\|$$

for all x, y in C. Obviously, every type (γ) mapping is nonexpansive. For more information about mappings of type(γ), see [1, 5, 15].

Lemma 1.8 ([6, 18]). *Let X be a uniformly convex Banach space and C a convex subset of X. Then there exists $\gamma \in \Gamma$ such that for each mapping $S : C \to C$ with* Lipschitz constant L*,*

$$\|\alpha Sx + (1-\alpha)Sy - S(\alpha x + (1-\alpha)y)\| \leqslant L\gamma^{-1}(\|x - y\| - \frac{1}{L}\|Sx - Sy\|)$$

for all $x, y \in C$ and $0 < \alpha < 1$.

2 Main Results

In this section, we prove weak and strong convergence theorems of the new iterative scheme (1) for a nonexpansive nonself-mapping in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. From an arbitrary $x_1 \in C$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1). Then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$.*

Proof. Let $x^* \in F(T)$, and $M = \max\{\sup_{n\geq 1} \|w_n - x^*\|,\ \sup_{n\geq 1} \|v_n - x^*\|\}$. For each $n \geq 1$, using (1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P((1 - b_n - \delta_n)x_n + b_n TP((1 - \gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n) - x^*\| \\
&= \|P((1 - b_n - \delta_n)x_n + b_n TP((1 - \gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n) - P(x^*)\| \\
&\leq \|(1 - b_n - \delta_n)x_n + b_n TP((1 - \gamma_n)y_n + \gamma_n Ty_n) + \delta_n v_n - x^*\| \\
&= \|(1 - b_n - \delta_n)(x_n - x^*) + b_n(TP((1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n) - x^*) + \delta_n(v_n - x^*)\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|TP((1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n) - x^*\| + \delta_n\|v_n - x^*\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|P((1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n) - x^*\| + \delta_n\|v_n - x^*\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|(1 - \gamma_n)y_n \\
&\quad + \gamma_n Ty_n - x^*\| + \delta_n\|v_n - x^*\| \\
&= (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|(1 - \gamma_n)(y_n - x^*) \\
&\quad + \gamma_n(Ty_n - x^*)\| + \delta_n\|v_n - x^*\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n((1 - \gamma_n)\|y_n - x^*\| \\
&\quad + \gamma_n\|y_n - x^*\|) + \delta_n\|v_n - x^*\| \\
&= (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|y_n - x^*\| + \delta_n\|v_n - x^*\| \\
&\leq (1 - b_n - \delta_n)\|x_n - x^*\| + b_n\|y_n - x^*\| + M\delta_n \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - x^*\| &= \|P((1 - a_n - \mu_n)x_n + a_n TP((1 - \beta_n)x_n + \beta_n Tx_n) + \mu_n w_n) - x^*\| \\
&= \|P((1 - a_n - \mu_n)x_n + a_n TP((1 - \beta_n)x_n + \beta_n Tx_n) + \mu_n w_n) - P(x^*)\| \\
&\leq \|(1 - a_n - \mu_n)x_n + a_n TP((1 - \beta_n)x_n + \beta_n Tx_n) + \mu_n w_n - x^*\| \\
&= \|(1 - a_n - \mu_n)(x_n - x^*) + a_n(TP((1 - \beta_n)x_n \\
&\quad + \beta_n Tx_n) - x^*) + \mu_n(w_n - x^*)\| \\
&\leq (1 - a_n - \mu_n)\|x_n - x^*\| + a_n\|TP((1 - \beta_n)x_n \\
&\quad + \beta_n Tx_n) - x^*\| + \mu_n\|w_n - x^*\| \\
&\leq (1 - a_n - \mu_n)\|x_n - x^*\| + a_n\|P((1 - \beta_n)x_n \\
&\quad + \beta_n Tx_n) - x^*\| + \mu_n\|w_n - x^*\|
\end{aligned}$$

$$\begin{aligned}
&\leq (1-a_n-\mu_n)\|x_n-x^*\| + a_n\|(1-\beta_n)x_n \\
&\quad + \beta_n Tx_n - x^*\| + \mu_n\|w_n - x^*\| \\
&= (1-a_n-\mu_n)\|x_n-x^*\| + a_n\|(1-\beta_n)(x_n-x^*) \\
&\quad + \beta_n(Tx_n - x^*)\| + \mu_n\|w_n - x^*\| \\
&\leq (1-a_n-\mu_n)\|x_n-x^*\| + a_n(1-\beta_n)\|x_n-x^*\| \\
&\quad + a_n\beta_n\|x_n - x^*\| + \mu_n\|w_n - x^*\| \\
&= (1-a_n-\mu_n)\|x_n-x^*\| + a_n\|x_n-x^*\| + \mu_n\|w_n-x^*\| \\
&= (1-\mu_n)\|x_n-x^*\| + \mu_n\|w_n-x^*\| \\
&\leq \|x_n-x^*\| + M\mu_n. \qquad (5)
\end{aligned}$$

Using (4) and (5), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &\leq (1-b_n-\delta_n)\|x_n-x^*\| + b_n(\|x_n-x^*\| + M\mu_n) + M\delta_n \\
&= (1-b_n-\delta_n)\|x_n-x^*\| + b_n\|x_n-x^*\| + Mb_n\mu_n + M\delta_n \\
&= (1-\delta_n)\|x_n-x^*\| + Mb_n\mu_n + M\delta_n \\
&\leq \|x_n-x^*\| + k^n_{(1)}, \qquad (6)
\end{aligned}$$

where $k^n_{(1)} = Mb_n\mu_n + M\delta_n$. Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, we have $\sum_{n=1}^{\infty} k^n_{(1)} < \infty$. We obtained from (6) and Lemma 1.2(i) that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. This completes the proof. □

Lemma 2.2. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ such that $a_n+\mu_n$ and $b_n+\delta_n$ are in $[0,1]$ for all $n \geq 1$, and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\delta_n < \infty$, $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$ and $\limsup_{n\to\infty}(a_n+\beta_n) < 1$. From an arbitrary $x_1 \in C$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1). Then $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.*

Proof. Let $x^* \in F(T)$. Then, by Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Set $q_n = P((1-\beta_n)x_n + \beta_n Tx_n)$ and $s_n = P((1-\gamma_n)y_n + \gamma_n Ty_n)$. Since $\{x_n\}$ and $\{y_n\}$ are bounded, it follows that $\{x_n - x^*\}$, $\{Tx_n - x^*\}$, $\{y_n - x^*\}$, $\{Ty_n - x^*\}$,

$\{Ts_n - x^*\}$ and $\{Tq_n - x^*\}$ are all bounded. This allows us to put

$$K = \max\{M, \sup_{n\geq 1}\|x_n - x^*\|, \sup_{n\geq 1}\|Tx_n - x^*\|, \sup_{n\geq 1}\|y_n - x^*\|, \sup_{n\geq 1}\|Ty_n - x^*\|, \sup_{n\geq 1}\|Ts_n - x^*\|, \sup_{n\geq 1}\|Tq_n - x^*\|\}.$$

Since T is a nonexpansive, from Lemma 1.3 and Lemma 1.4 we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P((1 - b_n - \delta_n)x_n + b_nTP((1 - \gamma_n)y_n + \gamma_nTy_n) + \delta_nv_n) - x^*\|^2\\ &= \|P((1 - b_n - \delta_n)x_n + b_nTs_n + \delta_nv_n) - x^*\|^2\\ &\leq \|(1 - b_n - \delta_n)x_n + b_nTs_n + \delta_nv_n - x^*\|^2\\ &= \|(1 - b_n - \delta_n)(x_n - x^*) + b_n(Ts_n - x^*) + \delta_n(v_n - x^*)\|^2\\ &\leq (1 - b_n - \delta_n)\|x_n - x^*\|^2 + b_n\|Ts_n - x^*\|^2 + \delta_n\|v_n - x^*\|^2\\ &\quad - (1 - b_n - \delta_n)b_ng(\|Ts_n - x_n\|)\\ &\leq (1 - b_n - \delta_n)\|x_n - x^*\|^2 + b_n\|Ts_n - x^*\|^2 + K^2\delta_n\\ &\quad - b_n(1 - b_n - \delta_n)g(\|Ts_n - x_n\|), \end{aligned} \tag{7}$$

$$\begin{aligned}\|Ts_n - x^*\|^2 &= \|TP((1 - \gamma_n)y_n + \gamma_nTy_n) - x^*\|^2\\ &\leq \|P((1 - \gamma_n)y_n + \gamma_nTy_n) - x^*\|^2\\ &\leq \|(1 - \gamma_n)y_n + \gamma_nTy_n - x^*\|^2\\ &= \|(1 - \gamma_n)(y_n - x^*) + \gamma_n(Ty_n - x^*)\|^2\\ &\leq (1 - \gamma_n)\|y_n - x^*\|^2 + \gamma_n\|Ty_n - x^*\|^2\\ &\quad - W_2(\gamma_n)g(\|Ty_n - y_n\|)\\ &\leq \|y_n - x^*\|^2 - W_2(\gamma_n)g(\|Ty_n - y_n\|), \end{aligned} \tag{8}$$

$$\begin{aligned}\|y_n - x^*\|^2 &= \|P((1 - a_n - \mu_n)x_n + a_nTP((1 - \beta_n)x_n + \beta_nTx_n) + \mu_nw_n) - x^*\|^2\\ &= \|P((1 - a_n - \mu_n)x_n + a_nTq_n + \mu_nw_n) - x^*\|^2\\ &\leq \|(1 - a_n - \mu_n)x_n + a_nTq_n + \mu_nw_n - x^*\|^2\\ &= \|(1 - a_n - \mu_n)(x_n - x^*) + a_n(Tq_n - x^*) + \mu_n(w_n - x^*)\|^2\\ &\leq (1 - a_n - \mu_n)\|x_n - x^*\|^2 + a_n\|Tq_n - x^*\|^2 + \mu_n\|w_n - x^*\|^2\\ &\quad - a_n(1 - a_n - \mu_n)g(\|Tq_n - x_n\|)\\ &\leq (1 - a_n - \mu_n)\|x_n - x^*\|^2 + a_n\|Tq_n - x^*\|^2 + K^2\mu_n \end{aligned} \tag{9}$$

and

$$\begin{aligned}
\|Tq_n - x^*\|^2 &= \|TP((1-\beta_n)x_n + \beta_n Tx_n) - x^*\|^2 \\
&\leq \|P((1-\beta_n)x_n + \beta_n Tx_n) - x^*\|^2 \\
&\leq \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\|^2 \\
&= \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\
&\quad - W_2(\beta_n)g(\|Tx_n - x_n\|) \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad - W_2(\beta_n)g(\|Tx_n - x_n\|) \\
&\leq \|x_n - x^*\|^2. \qquad (10)
\end{aligned}$$

By using (7), (8), (9) and (10), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n\|Ts_n - x^*\|^2 + K^2\delta_n \\
&\quad - b_n(1-b_n-\delta_n)g(\|Ts_n - x_n\|) \\
&\leq (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n(\|x_n - x^*\|^2 + K^2\mu_n \\
&\quad - W_2(\gamma_n)g(\|Ty_n - y_n\|)) + K^2\delta_n - b_n(1-b_n-\delta_n)g(\|Ts_n - x_n\|) \\
&= (1-b_n-\delta_n)\|x_n - x^*\|^2 + b_n\|x_n - x^*\|^2 + K^2 b_n\mu_n \\
&\quad - b_n W_2(\gamma_n)g(\|Ty_n - y_n\|) + K^2\delta_n - b_n(1-b_n-\delta_n)g(\|Ts_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + K^2 b_n\mu_n + K^2\delta_n \\
&\quad - b_n W_2(\gamma_n)g(\|Ty_n - y_n\|) - b_n(1-b_n-\delta_n)g(\|Ts_n - x_n\|) \\
&= \|x_n - x^*\|^2 - b_n W_2(\gamma_n)g(\|Ty_n - y_n\|) \\
&\quad - b_n(1-b_n-\delta_n)g(\|Ts_n - x_n\|) + k^n_{(2)} \\
&= \|x_n - x^*\|^2 - b_n\gamma_n(1-\gamma_n)g(\|Ty_n - y_n\|) \\
&\quad - b_n(1-b_n-\delta_n)g(\|Ts_n - x_n\|) + k^n_{(2)}, \qquad (11)
\end{aligned}$$

where $k^n_{(2)} = K^2 b_n\mu_n + K^2\delta_n$. Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, we have $\sum_{n=1}^{\infty} k^n_{(2)} < \infty$. From (11), we obtain the following two important inequalities:

$$b_n\gamma_n(1-\gamma_n)g(\|Ty_n - y_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + k^n_{(2)} \qquad (12)$$

and

$$b_n(1-b_n-\delta_n)g(\|Ts_n - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + k^n_{(2)}. \qquad (13)$$

Since $\lim_{n\to\infty}\delta_n = 0$, $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$ and $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$, there exists $n_0 \in \mathbb{N}$ and $\eta_1, \eta_2, \eta_3, \eta_4 \in (0,1)$ such that $0 < \eta_1 < b_n < \eta_2 < 1$ and $0 < \eta_3 < \gamma_n < \eta_4 < 1$ for all $n \ge n_0$. Hence, by (12) and (13), we have

$$\eta_1\eta_3(1-\eta_4)g(\|Ty_n - y_n\|) \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + k^n_{(2)} \tag{14}$$

and

$$\eta_1(1-\eta_2)g(\|Ts_n - x_n\|) \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + k^n_{(2)} \tag{15}$$

for all $n \ge n_0$. By (14) and (15), applying for $m \ge n_0$, we have

$$\begin{aligned}\eta_1\eta_3(1-\eta_4)\sum_{n=n_0}^{m} g(\|Ty_n - y_n\|) &\le \sum_{n=n_0}^{m}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + \sum_{n=n_0}^{m} k^n_{(2)} \\ &= \|x_{n_0} - x^*\|^2 + \sum_{n=n_0}^{m} k^n_{(2)} \end{aligned} \tag{16}$$

and

$$\begin{aligned}\eta_1(1-\eta_2)\sum_{n=n_0}^{m} g(\|Ts_n - x_n\|) &\le \sum_{n=n_0}^{m}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + \sum_{n=n_0}^{m} k^n_{(2)} \\ &= \|x_{n_0} - x^*\|^2 + \sum_{n=n_0}^{m} k^n_{(2)}. \end{aligned} \tag{17}$$

Since $\sum_{n=1}^{\infty} k^n_{(2)} < \infty$, by letting $m \to \infty$ in (16) and (17) we get $\sum_{n=n_0}^{\infty} g(\|Ty_n - y_n\|) < \infty$ and $\sum_{n=n_0}^{\infty} g(\|Ts_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g(\|Ty_n - y_n\|) = 0 = \lim_{n\to\infty} g(\|Ts_n - x_n\|)$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_{n\to\infty}\|Ty_n - y_n\| = 0 = \lim_{n\to\infty}\|Ts_n - x_n\|. \tag{18}$$

Using (1), we have

$$\begin{aligned}\|y_n - x_n\| &= \|P((1-a_n-\mu_n)x_n + a_nTP((1-\beta_n)x_n + \beta_nTx_n) + \mu_nw_n) - x_n\| \\ &\le \|(1-a_n-\mu_n)x_n + a_nTP((1-\beta_n)x_n + \beta_nTx_n) + \mu_nw_n - x_n\| \\ &= \|a_n(TP((1-\beta_n)x_n + \beta_nTx_n) - x_n) + \mu_n(w_n - x_n)\| \\ &= \|a_n(TP((1-\beta_n)x_n + \beta_nTx_n) - Tx_n + Tx_n - x_n) + \mu_n(w_n - x_n)\| \\ &\le a_n\|TP((1-\beta_n)x_n + \beta_nTx_n) - Tx_n + Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\ &\le a_n\|TP((1-\beta_n)x_n + \beta_nTx_n) - Tx_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\ &\le a_n\|P((1-\beta_n)x_n + \beta_nTx_n) - x_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\ &\le a_n\|(1-\beta_n)x_n + \beta_nTx_n - x_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\ &\le a_n\beta_n\|Tx_n - x_n\| + a_n\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\ &\le (a_n + \beta_n)\|Tx_n - x_n\| + \mu_n\|w_n - x_n\|. \end{aligned} \tag{19}$$

Since T is a nonexpansive mapping, from $s_n = P((1-\gamma_n)y_n + \gamma_n Ty_n)$ we have

$$\begin{aligned}
\|Tx_n - x_n\| &= \|Tx_n - Ts_n + Ts_n - x_n\| \\
&\le \|Tx_n - Ts_n\| + \|Ts_n - x_n\| \\
&\le \|x_n - s_n\| + \|Ts_n - x_n\| \\
&\le (1-\gamma_n)\|y_n - x_n\| + \gamma_n\|Ty_n - x_n\| + \|Ts_n - x_n\| \\
&= (1-\gamma_n)\|y_n - x_n\| + \gamma_n\|Ty_n - y_n + y_n - x_n\| + \|Ts_n - x_n\| \\
&\le (1-\gamma_n)\|y_n - x_n\| + \gamma_n\|Ty_n - y_n\| + \gamma_n\|y_n - x_n\| + \|Ts_n - x_n\| \\
&= \|y_n - x_n\| + \gamma_n\|Ty_n - y_n\| + \|Ts_n - x_n\|. \qquad (20)
\end{aligned}$$

It follows from (19) and (20) that

$$\begin{aligned}
\|Tx_n - x_n\| \le (a_n + \beta_n)\|Tx_n - x_n\| + \mu_n\|w_n - x_n\| \\
+ \gamma_n\|Ty_n - y_n\| + \|Ts_n - x_n\|,
\end{aligned}$$

which implies

$$(1 - a_n - \beta_n)\|Tx_n - x_n\| \le \mu_n\|w_n - x_n\| + \gamma_n\|Ty_n - y_n\| + \|Ts_n - x_n\|.$$

Since $\limsup_{n\to\infty}(a_n + \beta_n) < 1$, there exists a positive integer N_0 and $\eta \in (0,1)$ such that $a_n + \beta_n < \eta < 1$ for all $n \ge N_0$. Then for $n \ge N_0$, we have

$$(1-\eta)\|Tx_n - x_n\| \le \mu_n\|w_n - x_n\| + \gamma_n\|Ty_n - y_n\| + \|Ts_n - x_n\|.$$

This together with (18) and $\lim_{n\to\infty}\mu_n = 0$ imply that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. This completes the proof. □

Theorem 2.3. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a completely continuous nonexpansive nonself-mapping with $F(T) \ne \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ such that $a_n + \mu_n$ and $b_n + \delta_n$ are in $[0,1]$ for all $n \ge 1$, and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\delta_n < \infty$, $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$, $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$ and $\limsup_{n\to\infty}(a_n + \beta_n) < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1) converge strongly to a fixed point of T.*

Proof. By Lemma 2.2, we have

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \qquad (21)$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (21), $\{x_{n_k}\}$ converges. Let $q = \lim_{k\to\infty} x_{n_k}$. By the continuity of T and (21) we have that $Tq = q$, so q is a fixed point of T. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - q\|$ exists. Then $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$. Since $\lim_{n\to\infty} \mu_n = 0$, using (19) and (21), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

It follows that $\lim_{n\to\infty} \|y_n - q\| = 0$. This completes the proof. □

Next, we prove a strong convergence theorem for nonexpansive nonself-mapping in a uniformly convex Banach space satisfying *condition*(A).

Theorem 2.4. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ such that $a_n+\mu_n$ and $b_n+\delta_n$ are in $[0,1]$ for all $n \geq 1$, and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$ and $\limsup_{n\to\infty}(a_n + \beta_n) < 1$. Suppose that T satisfies condition(A). Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1) converge strongly to a fixed point of T.*

Proof. Let $x^* \in F(T)$. Then, as in Lemma 2.1, $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + k_{(1)}^n,$$

where $\sum_{n=1}^{\infty} k_{(1)}^n < \infty$. This implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + k_{(1)}^n$ and so, by Lemma 1.2(i), $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 2.2, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since T satisfies *condition*(A), we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} k_{(1)}^n < \infty$, given any $\epsilon < 0$, there exists a natural number n_0 such that $d(x_n, F(T)) < \frac{\epsilon}{4}$ and $\sum_{i=n_0}^{n} k_{(1)}^i < \frac{\epsilon}{2}$ for all $n \geq n_0$. So we can find $y^* \in F(T)$ such that $\|x_{n_0} - y^*\| < \frac{\epsilon}{4}$. For $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &= \|x_{n+m} - y^*\| + \|x_n - y^*\| \\
&\leq \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| + \sum_{i=n_0}^{n} k_{(1)}^i \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u$. Then $d(u, F(T)) = 0$. It follows that $u \in F(T)$. As in the proof of Theorem 2.3, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0,$$

it follows that $\lim_{n\to\infty} y_n = u$. This completes the proof. □

If $a_n = \mu_n = \delta_n \equiv 0$, then the iterative scheme (1) reduces to that of (2) and the following result is directly obtained by Theorem 2.4.

Theorem 2.5. *(Shahzad [23] Theorem 3.6, p.1037). Let X be a real uniformly convex Banach space and C a nonempty closed convex subset of X which is also a nonexpansive retract of X. Let $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{b_n\}$ and $\{\gamma_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (2). Suppose T satisfies condition (A). Then $\{x_n\}$ converges strongly to some fixed point of T.*

In the remainder of this section, we deal with the weak convergence of the new iterative scheme (1) for nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ such that $a_n + \mu_n$ and $b_n + \delta_n$ are in $[0,1]$ for all $n \geq 1$, and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$ and $\limsup_{n\to\infty}(a_n + \beta_n) < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1) converge weakly to a fixed point of T.*

Proof. By using the same proof as in Lemma 2.2, it can be shown that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_{n_i} \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.5, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.5, $u, v \in F(T)$. By Lemma 1.2(i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 1.6 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T. As in the proof of Theorem 2.3, we have $\lim_{n\to\infty}\|y_n - x_n\| = 0$ and $x_n \to u$ weakly as $n \to \infty$, it follows that $y_n \to u$ weakly as $n \to \infty$. The proof is completed. □

Next, we deal with the weak convergence of the sequence $\{x_n\}$ defined by (1) in a uniformly convex Banach space X whose dual X^* has the Kadec-Klee property.

Theorem 2.7. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{a_n\}, \{b_n\}, \{\mu_n\}, \{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ such that $a_n+\mu_n$ and $b_n+\delta_n$ are in $[0,1]$ for all $n \geq 1$, and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\delta_n < \infty$, $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$, $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$ and $\limsup_{n\to\infty}(a_n + \beta_n) < 1$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (1). Then for all $u, v \in F(T)$, the limit $\lim_{n\to\infty}\|tx_n - (1-t)u - v\|$ exists for all $t \in [0,1]$.*

Proof. It follows from Lemma 2.1 that the sequence $\{x_n\}$ is bounded. Then there exists $R > 0$ such that $\{x_n\} \subset B_R(0)\cap C$. Let $a_n(t) := \|tx_n+(1-t)u-v\|$ where $t \in (0,1)$. Then $\lim_{n\to\infty} a_n(0) = \|u - v\|$ and by Lemma 2.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}\|x_n - v\|$ exists. Without loss of the generality, we may assume that $\lim_{n\to\infty}\|x_n - v\| = r$ for some positive number r. Let $x \in C$,

$$y_n(x) = P((1 - a_n - \mu_n)x + a_nTP((1 - \beta_n)x + \beta_nTx) + \mu_nw_n).$$

Define $T_n : C \to C$ by

$$T_nx = P((1 - b_n - \delta_n)x + b_nTP((1 - \gamma_n)y_n(x) + \gamma_nTy_n(x)) + \delta_nv_n)$$

for all $x \in C$. For $x, z \in C$, we have

$$\begin{aligned}
\|T_nx - T_nz\| &= \|P((1 - b_n - \delta_n)x + b_nTP((1 - \gamma_n)y_n(x) + \gamma_nTy_n(x)) + \delta_nv_n) \\
&\quad - (P((1 - b_n - \delta_n)z + b_nTP((1 - \gamma_n)y_n(z) + \gamma_nTy_n(z)) + \delta_nv_n))\| \\
&\leq (1 - b_n - \delta_n)\|x - z\| + b_n\|TP((1 - \gamma_n)y_n(x) + \gamma_nTy_n(x)) \\
&\quad - TP((1 - \gamma_n)y_n(z) + \gamma_nTy_n(z))\| \\
&\leq (1 - b_n)\|x - z\| + b_n\|(1 - \gamma_n)(y_n(x) - y_n(z)) + \gamma_n(Ty_n(x) - Ty_n(z))\| \\
&\leq (1 - b_n)\|x - z\| + b_n(1 - \gamma_n)\|y_n(x) - y_n(z)\| + b_n\gamma_n\|y_n(x) - y_n(z)\| \\
&\leq (1 - b_n)\|x - z\| + b_n\|y_n(x) - y_n(z)\| \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}\|y_n(x)-y_n(z)\| &= \|P((1-a_n-\mu_n)x+a_nTP((1-\beta_n)x+\beta_nTx)+\mu_nw_n)\\ &\quad -P((1-a_n-\mu_n)z+a_nTP((1-\beta_n)z+\beta_nTz)+\mu_nw_n)\|\\ &\le (1-a_n-\mu_n)\|x-z\|+a_n\|TP((1-\beta_n)x+\beta_nTx)\\ &\quad -TP((1-\beta_n)z+\beta_nTz)\|\\ &\le (1-a_n)\|x-z\|+a_n\|(1-\beta_n)(x-z)+\beta_n(Tx-Tz)\|\\ &\le (1-a_n)\|x-z\|+a_n(1-\beta_n)\|x-z\|+a_n\beta_n\|Tx-Tz\|\\ &\le (1-a_n)\|x-z\|+a_n\|x-z\|\\ &= \|x-z\|. \end{aligned} \tag{23}$$

Using (22) and (23), we have $\|T_nx-T_nz\|\le\|x-z\|$. Set $S_{n,m}:=T_{n+m-1}T_{n+m-2}\cdots T_n, n,m\ge 1$, and $b_{n,m}=\|S_{n,m}(tx_n+(1-t)u)-(tS_{n,m}x_n+(1-t)u)\|$, where $0\le t\le 1$. Then $\|S_{n,m}x-S_{n,m}y\|\le\|x-y\|$, $S_{n,m}x_n=x_{n+m}$ and $S_{n,m}x^*=x^*$ for all $x^*\in F(T)$. It follows from Lemma 1.8 that

$$\begin{aligned}b_{n,m} &= \|S_{n,m}(tx_n+(1-t)u)-(tS_{n,m}x_n+(1-t)u)\|\\ &\le \gamma^{-1}(\|x_n-u\|-\|S_{n,m}x_n-S_{n,m}u\|)\\ &\le \gamma^{-1}(\|x_n-u\|-\|x_{n+m}-u\|). \end{aligned} \tag{24}$$

Hence $\gamma(b_{n,m})\le\|x_n-u\|-\|x_{n+m}-u\|$. This implies that $\lim_{n,m\to\infty}\gamma(b_{n,m})=0$. By the property of γ, we obtain that $\lim_{n,m\to\infty}b_{n,m}=0$. Observe that

$$\begin{aligned}a_{n+m}(t) &= \|tx_{n+m}+(1-t)u-v\|\\ &= \|tS_{n,m}x_n+(1-t)u-S_{n,m}v\|\\ &= \|S_{n,m}v-(tS_{n,m}x_n+(1-t)u)\|\\ &= \|S_{n,m}v-S_{n,m}(tx_n+(1-t)u)+S_{n,m}(tx_n+(1-t)u)\\ &\quad -(tS_{n,m}x_n+(1-t)u)\|\\ &\le \|S_{n,m}v-S_{n,m}(tx_n+(1-t)u)\|+b_{n,m}\\ &= \|S_{n,m}(tx_n+(1-t)u)-S_{n,m}v\|+b_{n,m}\\ &\le \|tx_n+(1-t)u-v\|+b_{n,m}\\ &= a_n(t)+b_{n,m}. \end{aligned}$$

Consequently,

$$\begin{aligned}\limsup_{m\to\infty} a_m(t) &= \limsup_{m\to\infty} a_{n+m}(t)\\ &\le \limsup_{m\to\infty}(b_{n,m}+a_n(t))\\ &\le \gamma^{-1}(\|x_n-u\| - \lim_{m\to\infty}\|x_m-u\|)+a_n(t)\end{aligned}$$

and

$$\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t).$$

This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t\in[0,1]$. This completes the proof. □

Theorem 2.8. *Let X be a uniformly convex Banach space such that its dual X^* has the Kaded-Klee property and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T: C\to X$ be a nonexpansive mapping with $F(T)\neq\emptyset$. Suppose that $\{a_n\},\{b_n\},\{\mu_n\},\{\delta_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ such that $a_n+\mu_n$ and $b_n+\delta_n$ are in $[0,1]$ for all $n\ge 1$, and $\{w_n\}$, $\{v_n\}$ are bounded sequences in C such that $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\delta_n<\infty$, $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$, $0<\liminf_{n\to\infty}b_n\le\limsup_{n\to\infty}b_n<1$ and $\limsup_{n\to\infty}(a_n+\beta_n)<1$. Then the sequence $\{x_n\}$ defined by the iterative scheme (1) converges weakly to a fixed point of T.*

Proof. It follows from Lemma 2.1 that the sequence $\{x_n\}$ is bounded. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to a point $x^*\in C$. By Lemma 2.2, we have $\lim_{n\to\infty}\|x_{n_j}-Tx_{n_j}\|=0$. Now using Lemma 1.5, we have $(I-T)x^*=0$, that is $Tx^*=x^*$. Thus $x^*\in F(T)$. It remains to show that $\{x_n\}$ converges weakly to x^*. Suppose that $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then $y^*\in C$ and so $x^*,y^*\in\omega_w(x_n)\cap F(T)$. By Theorem 2.7,

$$\lim_{n\to\infty}\|tx_n-(1-t)x^*-y^*\|$$

exists for all $t\in[0,1]$. It follows from Lemma 1.7 that $x^*=y^*$. As a result, $\omega_w(x_n)$ is a singleton, and so $\{x_n\}$ converges weakly to a fixed point of T. □

If $a_n=\mu_n=\delta_n\equiv 0$, then the iterative scheme (1) reduces to that of (2) and the following result is directly obtained by Theorem 2.8.

Theorem 2.9. *(Shahzad [23] Theorem 3.5, p.1036). Let X be a real uniformly convex Banach space such that its dual X^* has the Kadec-Klee property and C a* *nonempty closed convex subset of X which is also a nonexpansive retract of X. Let $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{b_n\}$ and $\{\gamma_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (2). Then $\{x_n\}$ converges weakly to some fixed point of T.*

Acknowledgements: The author would like to thank the Thailand Research Fund, The Commission on Higher Education (MRG5180036), and Naresuan Phayao University, Phayao, Thailand, for financial support during the preparation of this paper. Thanks are also extended to the anonymous referees for their helpful comments which improved the presentation of the original version of this paper.

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