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On the Criteria for Linear Independence of Nesterenko, Fischler and Zudilin

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Abstract: In 1985, Yu. V. Nesterenko produced a criterion for linear independence, which is a variant of Siegel's. While Siegel used upper bounds on full systems of forms, Nesterenko used upper and lower bounds on sufficiently dense sequences of individual forms. The proof of Nesterenko's criterion was simplified by F. Amoroso and P. Colmez in 2003. More recently, S. Fischler and W. Zudilin produced a refinement, together with a much simpler proof. This new proof rests on a simple argument which we expand here. We get a new result, which contains Nesterenko's criterion, as well as criteria for algebraic independence.

Keywords: Linear independence criterion, Siegel's method, Nesterenko's criterion, criteria for transcendence, criteria for algebraic independence

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1 Introduction

In his fundamental paper [9] in 1929, C.L. Siegel introduced the following result.

Theorem 1.1 (Siegel's linear independence criterion). *Let* $\underline{\vartheta} = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{R}^m$*. Assume that, for all* $\varepsilon > 0$*, there exists a complete system of* $m+1$ *linearly independent linear forms in* $m+1$ *variables*

$$L_i = L_i(\underline{X}) = \sum_{j=0}^{m} b_{ij} X_j, \quad i = 0, 1, \ldots, m, \quad b_{ij} \in \mathbb{Z},$$

such that

$$\max_{0\le i\le m} |L_i(1,\underline{\vartheta})| \le \frac{\varepsilon}{A^{m-1}} \qquad \textit{where} \qquad A = \max_{0\le i,j\le m} |b_{ij}|.$$

Then the numbers $1, \vartheta_1, \ldots, \vartheta_m$ *are linearly independent over* $\mathbb{Q}$.

This result is discussed in [3] Chap. 2 § 1.4. We reproduce the proof here. As pointed out by Siegel, this argument yields not only linear independence results, but also quantitative refinements (measures of linear independence) which we do not consider here. Studying only qualitative aspects of the subject enables us to simplify the situation.

Proof. Let

$$L(\underline{X}) = a_0X_0 + \cdots + a_mX_m, \quad a_j \in \mathbb{Z},$$

be a non–zero linear form with integer coefficients in $m+1$ variables. The aim is to prove, under the assumptions of Theorem 1.1, that $L(1,\underline{\vartheta}) \ne 0$. Set

$$H = \max_{0\le j\le m} |a_j|.$$

Let $\varepsilon < 1/(m!\cdot mH)$ be a positive real number. Among the forms $\{L_0,\ldots,L_m\}$ satisfying the assumptions of Theorem 1.1 for fixed ε, there exist m of them, say $L_{k_1},\ldots,L_{k_m}$, along with L that make up a complete system of linearly independent forms. Denote by Δ the determinant of the coefficient matrix of the system of linear forms $L, L_{k_1},\ldots,L_{k_m}$ and, for $0\le i,j\le m$, by $\Delta_{i,j}$ the (i,j)-minor of this matrix. Then

$$\Delta = L(1,\underline{\vartheta})\cdot\Delta_{0,0} + \sum_{i=1}^{m} L_{k_i}(1,\underline{\vartheta})\cdot\Delta_{i,0}.$$

Since $\Delta\in\mathbb{Z}$ and $\Delta\ne 0$, we have $|\Delta|\ge 1$. One easily estimates that, for $0\le j\le m$,

$$|\Delta_{0,j}| \le m!A^m \qquad \text{and} \qquad \max_{1\le i\le m} |\Delta_{i,j}| \le m!HA^{m-1}.$$

It follows that

$$\begin{aligned}(m!)^{-1} &\le |L(1,\underline{\vartheta}))|\cdot A^m + \sum_{i=1}^{m} |L_{k_i}(1,\underline{\vartheta})|\cdot HA^{m-1}\\ &\le |L(1,\underline{\vartheta})|\cdot A^m + \varepsilon\cdot mH\\ &< |L(1,\underline{\vartheta})|\cdot A^m + (m!)^{-1}.\end{aligned}$$

Thus $L(1,\underline{\vartheta}) \ne 0$. $\square$

Siegel used this approach to prove some transcendence results (linear independence of $1, \vartheta, \ldots, \vartheta^m, \ldots$).

In 1985, Yu. V. Nesterenko (see the corollary of the theorem in [7]) introduced a different type of criterion, involving a sequence of linear forms (and not a sequence of complete systems of linear forms). For each of them, he requires not only an upper bound, but also a lower bound.

Theorem 1.2 (Nesterenko's linear independence criterion). *Let c_1, c_2, τ_1, τ_2 be positive real numbers and $\sigma(n)$ a non–decreasing positive function such that*

$$\lim_{n\to\infty} \sigma(n) = \infty \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$

Let $\underline{\vartheta} = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{R}^m$. Assume that, for all sufficiently large integers n, there exists a linear form with integer coefficients in $m+1$ variables

$$L_n(\underline{X}) = \ell_{0n}X_0 + \ell_{1n}X_1 + \cdots + \ell_{mn}X_m,$$

which satisfies the conditions

$$\sum_{i=0}^{m} |\ell_{in}| \le e^{\sigma(n)} \quad \text{and} \quad c_1 e^{-\tau_1\sigma(n)} \le |L_n(1, \underline{\vartheta})| \le c_2 e^{-\tau_2\sigma(n)}.$$

Then $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\vartheta_1 + \cdots + \mathbb{Q}\vartheta_m) \ge (1+\tau_1)/(1+\tau_1-\tau_2)$.

The original proof by Nesterenko was rather involved; it had been simplified by F. Amoroso [1] and this simplification was revisited (and translated from Italian to French) by P. Colmez [2]. Recently, S. Fischler and W. Zudilin [5] obtained a refinement of Nesterenko's criterion. Their proof of this refinement was much easier than the previous proofs. We develop this refinement in the second section, also we show that this result contains Nesterenko's criterion. Then we deduce criteria for algebraic independence and transcendence criteria in the third and the fourth sections, respectively.

2 Main theorem and its proof

Here is our main result.

Theorem 2.1. *Let $\underline{\xi} = (\xi_i)_{i\ge 0}$ be a sequence of real numbers with $\xi_0 = 1$, $(r_n)_{n\ge 0}$ a non–decreasing sequence of positive integers, $(Q_n)_{n\ge 0}$, $(A_n)_{n\ge 0}$ and*

$(B_n)_{n\geq 0}$ *sequences of positive real numbers such that* $\lim_{n\to\infty} A_n^{1/r_n} = \infty$ *and, for all sufficiently large integers* n,

$$Q_n B_n \leq Q_{n+1} B_{n+1}.$$

For any integer $n \geq 0$, *let*

$$L_n(\underline{X}) = \ell_{0n} X_0 + \ell_{1n} X_1 + \cdots + \ell_{r_n n} X_{r_n}$$

be a linear form with integer coefficients in $r_n + 1$ *variables. Assume that, for any sufficiently large integer* n,

$$\sum_{i=0}^{r_n} |\ell_{in}| \leq Q_n, \quad 0 < |L_n(\underline{\xi})| \leq \frac{1}{A_n} \quad \text{and} \quad \frac{|L_{n-1}(\underline{\xi})|}{|L_n(\underline{\xi})|} \leq B_n.$$

Then

$$A_n \leq 2^{r_n+1} (B_n Q_n)^{r_n}$$

for all sufficiently large integers n.

Proof. For n a sufficiently large integer, let C_n be the convex symmetric compact set in $\mathbb{R}^{1+r_n}$ defined by

$$|x_0| \leq \frac{1}{2|L_n(\underline{\xi})|}, \quad |x_0\xi_i - x_i| \leq |2L_n(\underline{\xi})|^{1/r_n} \quad (1 \leq i \leq r_n). \tag{1}$$

The volume of C_n is 2^{r_n+1}. From Minkowski's Convex Body Theorem (Theorem 2B of Chapter II, [8]), there is a non-zero integer point in C_n. We fix such a $\underline{x}(n) = (x_0(n), \ldots, x_{r_n}(n)) \in \mathbb{Z}^{r_n+1} \setminus \{0\}$. Since $A_n^{1/r_n} \to \infty$ as $n \to \infty$, we have $x_0(n) \neq 0$ for n sufficiently large. Indeed, if we had $x_0(n) = 0$ with $A_n > 2$, we would have $|x_i(n)| \leq (2/A_n)^{1/r_n} < 1$, and then $x_i(n) = 0$, for each $i = 1, ..., r_n$, which contradicts the choice of $x(n)$.

From the assumptions and the estimate $|2L_n(\underline{\xi})|^{1/r_n} \leq (2/A_n)^{1/r_n}$ for n sufficiently large, it follows that the sequence $|2L_n(\underline{\xi})|^{1/r_n}$ tends to zero as $n \to \infty$. Since $r_n \geq 1$, $|L_n(\underline{\xi})|$ also tends to zero.

Now if the sequence $|x_0(n)|$ did not tend to infinity, it admits a bounded subsequence, so a constant subsequence, and we would have $x_0(n) = y$, with $y \in \mathbb{Z}$, $y \neq 0$, for all n belonging to an infinite subset A of $\mathbb{N}$. For all integers $i \leq \sup r_n$, we would have $\lim_{n\in A, n\to\infty}(y\xi_i - x_i(n)) = 0$. Therefore $x_i(n) = y\xi_i$ for $n \in A$ sufficiently large. Thus $y\xi_i \in \mathbb{Z}$ for all i, hence $yL_n(\xi) \in \mathbb{Z}$ for all n.

Since $L_n(\xi) \neq 0$, we would have $|L_n(\xi)| \geq 1/|y|$ for all n, which contradicts the fact that $|L_n(\underline{\xi})|$ tends to zero. Therefore $\lim_{n\to\infty} |x_0(n)| = \infty$.

We fix n sufficiently large and we write $\underline{x}$ instead of $\underline{x}(n)$. Let k denote the least positive integer such that

$$|x_0| \leq \frac{1}{2|L_k(\underline{\xi})|}\cdot$$

Since $|x_0|$ tends to infinity with n, it follows that k also tends to infinity with n. Moreover we have $k \leq n$ and

$$|x_0| > \frac{1}{2|L_{k-1}(\underline{\xi})|}\cdot$$

Now we can write

$$\sum_{i=0}^{r_k} \ell_{ik} x_i = x_0 \sum_{i=0}^{r_k} \ell_{ik}\xi_i + \sum_{i=0}^{r_k} \ell_{ik}(x_i - x_0\xi_i). \tag{2}$$

The term on the left-hand side of (2) is an integer. On the right-hand side, the first term has absolute value equal to $|x_0 L_k(\underline{\xi})|$. Therefore it is bounded above by $1/2$. We now use the following remark (compare with [5]): *if an integer can be written as a sum $x + y$ of two real numbers with $|x| \leq 1/2$, then $|y| \geq |x|$*. Hence

$$\left|\sum_{i=0}^{r_k} \ell_{ik}(x_i - x_0\xi_i)\right| \geq \left|x_0 \sum_{i=0}^{r_k} \ell_{ik}\xi_i\right|. \tag{3}$$

The term on the left-hand side of (3) is bounded above by

$$\sum_{i=0}^{r_k} |\ell_{ik}| \max_{1\leq i\leq r_n} |x_i - x_0\xi_i| \leq Q_k (2|L_n(\underline{\xi})|)^{1/r_n} \leq Q_k \left(\frac{2}{A_n}\right)^{1/r_n},$$

while the term on the right-hand side of (3) is bounded below by

$$|x_0|\cdot |L_k(\underline{\xi})| \geq \frac{1}{2|L_{k-1}(\underline{\xi})|}\cdot |L_k(\underline{\xi})| \geq \frac{1}{2B_k}\cdot$$

Thus

$$A_n \leq 2(2B_kQ_k)^{r_n}.$$

Since $B_kQ_k \leq B_nQ_n$, this implies $A_n \leq 2(2B_nQ_n)^{r_n}$. $\square$

Remark 2.2. Theorem 2.1 does not contain Theorem 6 of [5] (see below), because the latter introduces refinements involving gcd's, but its proof relies on the same arguments.

Theorem 2.3 (Theorem 6 of [5]). *Let $\xi_0,\dots,\xi_r$ be real numbers, with $r\ge 1$. Let $\tau,\gamma_1,\dots,\gamma_r>0$. For any $n\ge 1$ and any $i\in\{0,\dots,r\}$, let $\ell_{i,n}\in\mathbb{Z}$. For $n\ge 1$ and $i\in\{1,\dots,r\}$, let $\delta_{i,n}$ be a positive divisor of $\ell_{i,n}$ such that*

(i) *$\delta_{i,n}$ divides $\delta_{i+1,n}$ for any $n\ge 1$ and any $i\in\{1,\dots,r-1\}$, and*

(ii) *$\dfrac{\delta_{j,n}}{\delta_{i,n}}$ divides $\dfrac{\delta_{j,n+1}}{\delta_{i,n+1}}$ for any $n\ge 1$ and any $0\le i<j\le r$, with $\delta_{0,n}=1$.*

Assume that there exists an increasing sequence $(Q_n)_{n\ge 1}$ of integers such that, as $n\to\infty$, the following conditions are met:

$$Q_{n+1}=Q_n^{1+o(1)},$$
$$\max_{0\le i\le r}|\ell_{i,n}|\le Q_n^{1+o(1)},$$
$$\left|\sum_{i=0}^{r}\ell_{i,n}\xi_i\right|=Q_n^{-\tau+o(1)},$$
$$\delta_{i,n}=Q_n^{\gamma_i+o(1)}\quad \textit{for any } i\in\{1,\dots,r\}.$$

Let $s=\dim_{\mathbb{Q}}\operatorname{Span}_{\mathbb{Q}}(\xi_0,\dots,\xi_r)-1$. Then we have

$$s\ge\tau+\gamma_1+\cdots+\gamma_s.$$

We deduce from Theorem 2.1 a slight refinement of Theorem 1.2 (Nesterenko's linear independence criterion).

Corollary 2.4. *Let τ_1,τ_2 be positive real numbers and $\sigma(n)$ a non–decreasing positive function such that $\lim_{n\to\infty}\sigma(n)=\infty$. Let $\underline{\vartheta}=(\vartheta_1,\dots,\vartheta_m)\in\mathbb{R}^m$. Assume that, for all sufficiently large integers n, there exists a linear form with integer coefficients in $m+1$ variables*

$$L_n(\underline{X})=\ell_{0n}X_0+\ell_{1n}X_1+\cdots+\ell_{mn}X_m$$

which satisfies the conditions

$$\sum_{i=0}^{m}|\ell_{in}|\le e^{\sigma(n)}\qquad\textit{and}\qquad e^{-(\tau_1+o(1))\sigma(n)}\le|L_n(1,\underline{\vartheta})|\le e^{-(\tau_2+o(1))\sigma(n+1)}.$$

Then $\dim_{\mathbb{Q}}(\mathbb{Q}+\mathbb{Q}\vartheta_1+\cdots+\mathbb{Q}\vartheta_m)\ge(1+\tau_1)/(1+\tau_1-\tau_2)$.

Remark 2.5. 1. If ℓ is an integer and if the assumptions of Corollary 2.4 are satisfied for parameters τ_1 and τ_2 such that $\tau_1>(\ell-1)+(\tau_2-\tau_1)\ell$, then the conclusion yields $\dim_{\mathbb{Q}}(\mathbb{Q}+\mathbb{Q}\vartheta_1+\cdots+\mathbb{Q}\vartheta_m)\ge\ell+1$. In particular, if

$\tau_1 > (m-1) + m(\tau_1 - \tau_2)$, then $1, \vartheta_1, \dots, \vartheta_m$ are linearly independent over $\mathbb{Q}$.
2. The hypotheses of Corollary 2.4 imply that $L_n(1, \underline{\vartheta}) \neq 0$ and $\lim_{n\to\infty} L_n(1, \underline{\vartheta}) = 0$, therefore $\underline{\vartheta} \notin \mathbb{Q}^m$. If the aim is to prove $\underline{\vartheta} \notin \mathbb{Q}^m$, then there is no need of a lower bound for $|L_n(1, \underline{\vartheta})|$. However, assuming a lower bound for $|L_n(1, \underline{\vartheta})|$ enables Nesterenko in [7] to reach a quantitative estimate. Consider for instance the special case $m = 1$ with $\vartheta_1 = \vartheta \in \mathbb{R}$: under the assumptions of Theorem 1.2, ϑ is not a Liouville number. Conversely, Theorem 1 of [4] (see below) shows that for a real number $\vartheta = \vartheta_1$ which is not a Liouville number, the assumptions of Corollary 2.4 with $m = 1$ are satisfied for suitable values of the parameters.

Theorem 2.6 (Theorem 1 of [4]). *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$, and let (Q_n) and (ε_n) be sequences of positive real numbers with*

$$\lim_{n\to+\infty} Q_n = +\infty, \qquad \lim_{n\to+\infty} \varepsilon_n = 0 \quad \textit{and} \quad \varepsilon_n \geq Q_n^{-\frac{1}{\mu-1}+o(1)},$$

where μ is a real number such that $\mu > \mu(\xi)$ and $\mu(\xi)$ is the irrationality exponent of ξ, defined as the infimum of all real numbers μ such that the inequality $|\xi - \frac{p}{q}| > \frac{1}{q^\mu}$ holds for all integers p, q, with q sufficiently large. Then there exist integer sequences (u_n) and (v_n) such that

$$\lim_{n\to+\infty} \frac{u_n}{Q_n} = \lim_{n\to+\infty} \frac{u_n\xi - v_n}{\varepsilon_n} = 1.$$

Proof of Corollary 2.4. Set $\vartheta_0 = 1$. Let $\{\xi_0, \xi_1, \dots, \xi_r\}$ with $\xi_0 = 1$ be a basis of the vector space spanned by $\vartheta_0, \dots, \vartheta_m$ over $\mathbb{Q}$. Let $\underline{\xi} = (\xi_1, \dots, \xi_r) \in \mathbb{R}^r$. Then for $0 \leq i \leq m$, there are integers $d, c_{i0}, \dots, c_{ir}$ with $d > 0$ such that $d\vartheta_i = \sum_{j=0}^{r} c_{ij}\xi_j$. For any $n \in \mathbb{N}$, define a linear form in $r+1$ variables $Y_0, \dots, Y_r$ with integer coefficients by

$$\begin{aligned}\Lambda_n(Y_0, \dots, Y_r) &= L_n\left(\sum_{j=0}^{r} c_{0j}Y_j, \dots, \sum_{j=0}^{r} c_{mj}Y_j\right)\\ &= \sum_{i=0}^{m} c_{i0}\ell_{in}Y_0 + \cdots + \sum_{i=0}^{m} c_{ir}\ell_{in}Y_r.\end{aligned}$$

From the definition of Λ_n we infer $\Lambda_n(1, \underline{\xi}) = d \cdot L_n(1, \underline{\vartheta})$. We apply Theorem 2.1 with a finite sequence $\xi_0, \xi_1, \dots, \xi_r$ with $r_n = r$ for all n. Let τ_1' and τ_2' satisfy

$\tau_1' > \tau_1$ and $\tau_2' < \tau_2$. We take

$$Q_n = ce^{\sigma(n)}, \quad A_n = e^{\tau_2'\sigma(n+1)} \quad \text{and} \quad B_n = e^{(\tau_1'-\tau_2')\sigma(n)},$$

where $c > 0$ is a suitable constant. We obtain

$$e^{\tau_2'\sigma(n+1)} \leq 2^{r+1}c^r e^{r(\tau_1'-\tau_2'+1)\sigma(n)}.$$

Since $\sigma(n)$ is non–decreasing and tends to infinity, this estimate implies

$$\tau_2' \leq r(1+\tau_1'-\tau_2').$$

Since this last inequality holds for all (τ_1', τ_2') with $\tau_1' > \tau_1$ and $\tau_2' < \tau_2$, we deduce $r \geq \tau_2/(1+\tau_1-\tau_2)$. Therefore

$$\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\vartheta_1 + \cdots + \mathbb{Q}\vartheta_m) = r + 1 \geq \frac{\tau_2}{1+\tau_1-\tau_2} + 1 = \frac{1+\tau_1}{1+\tau_1-\tau_2}$$

as desired. □

3 Criteria for algebraic independence

Definition 3.1. Let $P(X) = a_0 + a_1X + \cdots + a_nX^n$ be a polynomial with complex coefficients. We define the height $H(P)$ of P as the maximum of the absolute values of its coefficients, that is,

$$H(P) = \max_{0 \leq i \leq n} |a_i|.$$

Definition 3.2. Let P as in the definition 3.1. The length $L(P)$ of P is defined as the sum of the absolue values of its coefficients, that is,

$$L(P) = \sum_{i=0}^{n} |a_i|.$$

Note that

$$L(P) \leq (\deg P + 1)H(P). \tag{4}$$

Definition 3.3. We define the size $t(P)$ of a polynomial $P \in \mathbb{C}[X]$ as

$$t(P) := \log H(P) + (\log 2)\deg P.$$

We deduce from Theorem 2.1 the following criterion for algebraic independence:

Corollary 3.4. *Let $\vartheta_1, \dots, \vartheta_t$ be real numbers, $(d_n)_{n\geq 0}$ a sequence of positive integers, $(\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ sequences of positive real numbers such that $\lim_{n\to\infty} d_n^{-t}\gamma_n = \infty$ and, for all sufficiently large integers n,*

$$\alpha_n + \beta_n - \gamma_{n-1} \leq \alpha_{n+1} + \beta_{n+1} - \gamma_n.$$

Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X_1, \dots, X_t]$, where P_n has total degree at most d_n and length at most e^{β_n}, satisfying

$$e^{-\alpha_n} \leq |P_n(\vartheta_1, \dots, \vartheta_t)| \leq e^{-\gamma_n}.$$

Then

$$\gamma_n \leq \log 2 + \left(\binom{d_n + t}{t} - 1\right)(\alpha_n + \beta_n - \gamma_{n-1} + \log 2)$$

for all sufficiently large integers n.

Proof. The number of integer tuples $\underline{i} = (i_1, \dots, i_t)$ in $\mathbb{Z}^t_{\geq 0}$ with $i_1 + \dots + i_t \leq d_n$ is $\binom{d_n + t}{t}$. For each of these $\underline{i}$, let $\xi_{\underline{i}} = \vartheta_1^{i_1} \dots \vartheta_t^{i_t}$. Let $n \in \mathbb{N}$. We write

$$P_n(X_1, \dots, X_t) = \sum_{\underline{i}} a_{\underline{i},n} X_1^{i_1} \dots X_t^{i_t}.$$

Put $L_n(\underline{X}) = \sum_{\underline{i}} a_{\underline{i},n} X_{\underline{i}}$. Then $L_n(\underline{\xi}) = P_n(\vartheta_1, \dots, \vartheta_t)$. Applying Theorem 2.1 with $r_n = \binom{d_n + t}{t} - 1$, $Q_n = e^{\beta_n}$, $A_n = e^{\gamma_n}$ and $B_n = e^{\alpha_n - \gamma_{n-1}}$, we obtain

$$\gamma_n \leq \log 2 + \left(\binom{d_n + t}{t} - 1\right)(\alpha_n + \beta_n - \gamma_{n-1} + \log 2)$$

for all sufficiently large integers n. □

We deduce from Corollary 3.4 the following special cases. In the first one, we consider a sequence of bounded degree polynomials. Since our method relies on linear elimination, our results are sharp when the dimension of the space is not too large. As far as criteria for algebraic independence are concerned, we get sharper results when the degree is bounded.

Corollary 3.5. *Let $\vartheta_1, \dots, \vartheta_t$ be real numbers, d a positive integer, τ_1, τ_2 positive real numbers and $\sigma(n)$ a non–decreasing positive function such that* $\lim_{n\to\infty} \sigma(n) =$

∞. *Assume that there exists a sequence* $(P_n)_{n\geq 0}$ *of polynomials in* $\mathbb{Z}[X_1,\ldots,X_t]$, *where* P_n *has total degree at most* d *and length at most* $e^{\sigma(n)}$, *satisfying*

$$e^{-(\tau_1+o(1))\sigma(n)} \leq |P_n(\vartheta_1,\ldots,\vartheta_t)| \leq e^{-(\tau_2+o(1))\sigma(n+1)}.$$

Then

$$\tau_2 \leq \left(\binom{d+t}{t} - 1\right)(1+\tau_1-\tau_2).$$

Proof. Let ε be a positive real number $< \tau_2$. Set $\tau_1' = \tau_1+\varepsilon$ and $\tau_2' = \tau_2-\varepsilon$. We apply Corollary 3.4 with $d_n = d$, $\alpha_n = \tau_1'\sigma(n)$, $\beta_n = \sigma(n)$ and $\gamma_n = \tau_2'\sigma(n+1)$. The conclusion follows by letting ε tend to 0. □

Here are examples with sequences of polynomials whose degrees could tend to infinity.

Corollary 3.6. *Let* $\vartheta_1,\ldots,\vartheta_t$ *be real numbers,* β, δ, κ *and* λ *positive real numbers,* α *and* γ *real numbers. Assume* $\kappa \leq 1/t$. *Furthermore, assume that there exists a sequence* $(P_n)_{n\geq 0}$ *of polynomials in* $\mathbb{Z}[X_1,\ldots,X_t]$, *where* P_n *has total degree at most* $\delta n^\kappa + o(n^\kappa)$ *and length at most* $e^{\beta n+o(n)}$, *satisfying*

$$e^{-\lambda n^{1+t\kappa}-(\alpha+o(1))n} \leq |P_n(\vartheta_1,\ldots,\vartheta_t)| \leq e^{-\lambda n^{1+t\kappa}-(\gamma+o(1))n}.$$

Then

$$\lambda(1-2\delta^t/t!) \leq \delta^t(\alpha+\beta-\gamma)/t! \quad \textit{when} \quad \kappa = 1/t,$$

and

$$\lambda \leq \delta^t(\alpha+\beta-\gamma)/t! \quad \textit{when} \quad \kappa < 1/t.$$

Proof. Let ε be a positive real number. Set $\alpha' = \alpha+\varepsilon$, $\beta' = \beta+\varepsilon$, $\gamma' = \gamma-\varepsilon$ and $\delta' = \delta+\varepsilon$. Let $d_n = \delta' n^\kappa$, $\beta_n = \beta' n$, $\gamma_n = \lambda n^{1+t\kappa} + \gamma' n$,

$$\alpha_n = \begin{cases} \lambda n^2 + \alpha' n & \text{when} \quad \kappa = 1/t, \\ \lambda(n-1)^{1+t\kappa} + \alpha' n & \text{when} \quad \kappa < 1/t. \end{cases}$$

We obtain $e^{-\alpha_n} \leq |P_n(\vartheta_1,\ldots,\vartheta_t)| \leq e^{-\gamma_n}$ for n sufficiently large. Besides,

$$\alpha_n + \beta_n - \gamma_{n-1} = \begin{cases} (2\lambda+\alpha'+\beta'-\gamma')n - \lambda + \gamma' & \text{when} \quad \kappa = 1/t, \\ (\alpha'+\beta'-\gamma')n + \gamma' & \text{when} \quad \kappa < 1/t. \end{cases}$$

Since the hypothesis implies that $\alpha - \gamma \geq 0$, it follows that the sequence

$$(\alpha_n + \beta_n - \gamma_{n-1})_{n\geq 1}$$

is non–decreasing for sufficiently large n. The conclusion follows by applying Corollary 3.4 and letting ε tend to 0. □

Corollary 3.7. *Let $\vartheta_1, \dots, \vartheta_t$ be real numbers, $(d_n)_{n\geq 0}$ a sequence of positive integers and β, κ and λ positive real numbers. Assume*

$$d_n = o(n^{1/t}) \quad \textit{and} \quad d_n^t - d_{n-1}^t \leq \kappa$$

for all sufficiently large integers n. Furthermore, assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X_1, \dots, X_t]$, where P_n has total degree at most d_n and length at most $e^{\beta n + o(n)}$ satisfying

$$|P_n(\vartheta_1, \dots, \vartheta_t)| = e^{-(\lambda n d_n^t + o(n))}.$$

Then

$$\lambda(t! - \kappa) \leq \beta.$$

Proof. Let $\varepsilon > 0$, $\alpha_n = \lambda(n-1)(d_{n-1}^t + \kappa) + \varepsilon n$, $\gamma_n = \lambda n d_n^t - \varepsilon n$ and $\beta_n = \beta n + \varepsilon n$. For n sufficiently large, since

$$\lambda n(d_n^t - d_{n-1}^t) + \lambda(d_{n-1}^t + \kappa) + o(n) \leq \lambda n\kappa + o(n) \leq \lambda n\kappa + \varepsilon n,$$

we have

$$\lambda n d_n^t + o(n) \leq \lambda(n-1)(d_{n-1}^t + \kappa) + \varepsilon n.$$

It follows that $|P_n(\vartheta_1, \dots, \vartheta_t)| \geq e^{-\alpha_n}$.

Also, $|P_n(\vartheta_1, \dots, \vartheta_t)| \leq e^{-\gamma_n}$ for n sufficiently large.

Since $\alpha_n + \beta_n - \gamma_{n-1} = (\lambda\kappa + \beta + 3\varepsilon)n - \lambda\kappa - \varepsilon$, it follows that the sequence $(\alpha_n + \beta_n - \gamma_{n-1})_{n\geq 1}$ is non–decreasing for sufficiently large n. The conclusion follows by applying Corollary 3.4 and letting ε tend to 0. □

Algebraic independence results follow from our criteria:

Corollary 3.8. *Let $\vartheta_1, \dots, \vartheta_t$ be real numbers, d, δ two positive integers with $d \geq \delta$ and τ, η two positive real numbers satisfying*

$$\frac{\tau}{1+\eta} > \binom{t+d}{t} - \binom{t+d-\delta}{t} - 1.$$

Let $\big(\sigma(n)\big)_{n\geq 1}$ be a non–decreasing sequence of real numbers which tends to infinity. Assume that there is a sequence $(P_n)_{n\geq n_0}$ of polynomials in $\mathbb{Z}[X_1, \dots, X_t]$, where P_n has total degree $\leq d$ and length $\leq e^{\sigma(n)}$, such that

$$e^{-(\tau+\eta+o(1))\sigma(n)} \leq |P_n(\vartheta_1, \dots, \vartheta_t)| \leq e^{-(\tau+o(1))\sigma(n+1)}.$$

Then $\vartheta_1, \ldots, \vartheta_t$ *do not satisfy any algebraic dependence relation with rational coefficients of degree* $\leq \delta$*.*

Proof. We use Corollary 2.4 with $\tau_1 = \tau + \eta$, $\tau_2 = \tau$, and $1, \vartheta_1, \ldots, \vartheta_m$ in Corollary 2.4 replaced by $\vartheta_1^{i_1} \cdots \vartheta_t^{i_t}$ with $i_1 + \cdots + i_t \leq d$. Then the dimension of the subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by $\{\vartheta_1^{i_1} \cdots \vartheta_t^{i_t} \mid i_1 + \cdots + i_t \leq d\}$ is bounded below by

$$1 + \frac{\tau}{1+\eta}\cdot$$

Now, we suppose that there exists a polynomial $Q \in \mathbb{Z}[X_1, \ldots, X_t]$ of total degree $\leq \delta$ such that $Q(\vartheta_1, \ldots, \vartheta_t) = 0$. Let ψ be the linear transformation from the set of all polynomials of total degree $\leq d$ in $\mathbb{Q}[X_1, \ldots, X_t]$ to $\mathbb{R}$ defined by $\psi(P(X_1, \ldots, X_t)) = P(\vartheta_1, \ldots, \vartheta_t)$ for $P(X_1, \ldots, X_t) \in \mathbb{Q}[X_1, \ldots, X_t]$. Then

$$\{X_1^{j_1} \cdots X_t^{j_t} \cdot Q \mid j_1 + \cdots + j_t \leq d - \delta\} \subset \ker \psi.$$

It follows that

$$\dim(\ker \psi) \geq \binom{t+d-\delta}{t}.$$

Therefore, the dimension of the image of ψ, which is equal to

$$\binom{t+d}{t} - \dim(\ker \psi),$$

is bounded above by

$$\binom{t+d}{t} - \binom{t+d-\delta}{t}.$$

Since the image of ψ is the subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by

$$\{\vartheta_1^{i_1} \cdots \vartheta_t^{i_t} \mid i_1 + \cdots + i_t \leq d\},$$

we have

$$1 + \frac{\tau}{1+\eta} \leq \binom{t+d}{t} - \binom{t+d-\delta}{t},$$

which is a contradiction. $\square$

When the assumptions of Corollary 3.8 are satisfied for any $\delta > 0$, we deduce the algebraic independence of the numbers $\vartheta_1, \ldots, \vartheta_t$.

Corollary 3.9. *Let $\vartheta_1, \ldots, \vartheta_t$ be real numbers and $(\tau_d)_{d\geq 1}$, $(\eta_d)_{d\geq 1}$ two sequences of positive real numbers satisfying*

$$\frac{\tau_d}{d^{t-1}(1+\eta_d)} \longrightarrow +\infty.$$

Furthermore, let $\big(\sigma(n)\big)_{n\geq 1}$ be a non–decreasing sequence of real numbers which tends to infinity. Assume that for all sufficiently large d, there is a sequence $(P_n)_{n\geq n_0(d)}$ of polynomials in $\mathbb{Z}[X_1, \ldots, X_t]$, where P_n has total degree $\leq d$ and length $\leq e^{\sigma(n)}$, such that, for $n \geq n_0(d)$,

$$e^{-(\tau_d+\eta_d)\sigma(n)} \leq |P_n(\vartheta_1, \ldots, \vartheta_t)| \leq e^{-\tau_d\sigma(n+1)}.$$

Then $\vartheta_1, \ldots, \vartheta_t$ are algebraically independent.

Proof. Let δ be a positive integer. For sufficiently large d, we have

$$\binom{t+d}{t} - \binom{t+d-\delta}{t} - 1 < cd^{t-1} < \frac{\tau_d}{1+\eta_d}$$

where $c > 0$ is a suitable constant. Then the hypotheses of Corollary 3.8 are satisfied with $\tau = \tau_d$ and $\eta = \eta_d$. Hence $\vartheta_1, \ldots, \vartheta_t$ do not satisfy any algebraic dependence relation with rational coefficients of degree $\leq \delta$. This is true for all δ. Therefore $\vartheta_1, \ldots, \vartheta_t$ are algebraically independent. $\square$

4 Transcendence criteria

The special case $t = 1$ of Corollary 3.4 yields a criterion for transcendence, which is similar to a well known result due to A.O. Gel'fond (see for instance Lemma 6.3, § 1.3, Chap. 6 of [3]).

Corollary 4.1. *Let ϑ be a real number, $(d_n)_{n\geq 0}$ a sequence of positive integers, $(\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ sequences of positive real numbers such that $\lim_{n\to\infty} \gamma_n/d_n = \infty$ and, for all sufficiently large integers n, $\alpha_n + \beta_n - \gamma_{n-1} \leq \alpha_{n+1} + \beta_{n+1} - \gamma_n$. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where P_n has degree at most d_n and length at most e^{β_n}, satisfying*

$$e^{-\alpha_n} \leq |P_n(\vartheta)| \leq e^{-\gamma_n}.$$

Then

$$\gamma_n \leq \log 2 + d_n(\alpha_n + \beta_n - \gamma_{n-1} + \log 2)$$

for all sufficiently large integers n.

Here is the special case $t = 1$ of Corollary 3.5.

Corollary 4.2. *Let ϑ be a real number, d a positive integer, τ_1, τ_2 positive real numbers and $\sigma(n)$ a non–decreasing positive function such that $\lim_{n\to\infty} \sigma(n) = \infty$. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where P_n has degree at most d and length at most $e^{\sigma(n)}$, satisfying*

$$e^{-(\tau_1+o(1))\sigma(n)} \leq |P_n(\vartheta)| \leq e^{-(\tau_2+o(1))\sigma(n+1)}.$$

Then

$$\tau_2 \leq d + d(\tau_1 - \tau_2).$$

Note that if $\tau_1 = \tau_2$, then the conclusion is $\tau_1 \leq d$.

It is interesting to compare Corollary 4.2 with the results following from the proof of Gel'fond's criterion: in our present paper, we use only linear elimination, while Gel'fond's proof relies on algebraic elimination. In Gel'fond's criterion, there is no need of a lower bound for $|P_n(\vartheta)|$, but the conclusion is not sharp. We recall Lemma 14 of [6] that

Lemma 4.3 (Lemma 14 of [6]). *Let a and b be real numbers ≥ 1, and let $(t_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ be sequences of positive real numbers such that*

$$t_n \leq t_{n+1} \leq at_n, \quad d_n \leq d_{n+1} \leq bd_n, \quad (n \in \mathbb{N}).$$

Moreover, assume that t_n tends to infinity with n. Let $\theta \in \mathbb{C}$ and let γ be a real number $> a+b+1$. If there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of nonzero polynomials in $\mathbb{Z}[X]$ satisfying

$$\deg P_n \leq d_n, \quad t(P_n) \leq t_n, \quad |P_n(\theta)| \leq e^{-\gamma d_n t_n}, \quad (n \in \mathbb{N}),$$

then $\theta \in \overline{\mathbb{Q}}$ (and $P_n(\theta) = 0$ for each n sufficiently large).

Then we deduce the following proposition.

Proposition 4.4. *Let ϑ be a complex number, d a positive integer, α and β positive real numbers. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where P_n has degree at most d and length at most $e^{\beta n+o(n)}$, satisfying $0 < |P_n(\vartheta)| \leq e^{-\alpha n+o(n)}$. Then $\alpha \leq 3d\beta$.*

Under the assumptions of Proposition 4.4, if we assume that ϑ is a real number and

$$|P_n(\vartheta)| = e^{-\alpha n+o(n)},$$

then Corollary 4.2 gives the stronger conclusion $\alpha \leq d\beta$.

We conclude with the special case $t = \kappa = 1$ of Corollary 3.6, which is also the special case where δ_n, α_n, β_n and γ_n satisfy

$$\delta_n = \delta n + o(n),\ \alpha_n = \lambda n^2 + \alpha n + o(n),\ \beta_n = \beta n + o(n) \text{ and } \gamma_n = \lambda n^2 + \gamma n + o(n)$$

of Corollary 4.1.

Corollary 4.5. *Let ϑ be a real number, β, δ and λ positive real numbers such that $\delta < 1/2$, α and γ real numbers. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where P_n has degree at most $\delta n + o(n)$ and length at most $e^{\beta n + o(n)}$, satisfying*

$$e^{-(\lambda n^2 + \alpha n + o(n))} \leq |P_n(\vartheta)| \leq e^{-(\lambda n^2 + \gamma n + o(n))}.$$

Then $\lambda(1 - 2\delta) \leq \delta(\alpha + \beta - \gamma)$.

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