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A New Subclass of Salagean-Type Harmonic Univalent Functions

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Abstract: A new subclass of Salagean-type harmonic univalent function is introduced. Coefficient conditions, extreme points, distortion bounds, convolution conditions, convex combinations and radii of convexity for this subclass are obtained.

Keywords: Harmonic functions, univalent functions, Salagean operator

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1 Introduction

Harmonic mappings in the plane are univalent complex-valued harmonic functions of a complex variable. Although conformal mappings are a special case, harmonic functions in general need not be analytic. Harmonic mappings are important in different applied fields of study [1]. Complex analysts in the recent past have actively investigated harmonic mappings as generalizations of univalent analytic functions.

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Denote by S_H the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0)-1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

Clunie and Sheil-Small [2] investigated the class S_H. The differential operator D^m was introduced by Salagean [6]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [3] defined the modified salagean operator f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \tag{2}$$

$$\text{where} \quad D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \quad \text{and} \quad D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

In this paper, we introduce a new class $S_H^*(m, n, \lambda, \gamma)$ of harmonic functions. For $0 \le \gamma < 1$, $0 \le \lambda \le 1$, α real, $m \in N$, $n \in N_0$, $m > n$ and $z \in U$, we let $S_H^*(m, n, \lambda, \gamma)$ denote the family of harmonic functions f of the form (1) such that

$$Re\left\{ \frac{(1 + e^{i\alpha})D^m f(z)}{\lambda D^m f(z) + (1-\lambda)D^n f(z)} - e^{i\alpha} \right\} \ge \gamma \tag{3}$$

where $D^m f$ is defined by (2).

We let the subclass $TS_H^*(m, n, \lambda, \gamma)$ of $S_H^*(m, n, \lambda, \gamma)$ consisting of harmonic functions $f_m = h + \overline{g}_m$ such that h and $\overline{g}_m$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k, \quad a_k, b_k \ge 0 \tag{4}$$

Remark : The class $TS_H^*(m, n, \lambda, \gamma)$ includes a variety of well-known subclasses for specific values of m, n and λ.

1. $TS_H^*(m, n, 0, \gamma) = G_{\overline{H}}(m, n, \gamma)$ [7].
2. α being real, when $\alpha = 0, \lambda = 0$
   $TS_H^*(m, n, \lambda, \gamma) = \overline{S_H}(m, n, \beta)$ studied in [8] where $\beta = \frac{1+\gamma}{2}$.
3. α being real, when $\alpha = 0$, $TS_H^*(1, 0, \lambda, \gamma) = TS_H^*(\lambda, \beta)$, $\beta = \frac{1+\gamma}{2}$ [4].
4. $TS_H^*(1, 0, 0, \gamma) = G_{\overline{H}}(\gamma)$ [5].

In this note, we obtain coefficient conditions, extreme points, distortion bounds, convolution conditions, and convex combinations for $TS_H^*(m, n, \lambda, \gamma)$.

2 Main Results

We begin with a sufficient coefficient condition for functions in $S_H^*(m,n,\lambda,\gamma)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1). Furthermore, let*

$$\sum_{k=1}^{\infty}\Bigg(\frac{k^m(2-\lambda(1+\gamma)) - k^n(1-\lambda)(1+\gamma)}{1-\gamma}|a_k| + \frac{k^m(2-\lambda(1+\gamma)) - (-1)^{m-n}k^n(1+\gamma)(1-\lambda)}{1-\gamma}|b_k|\Bigg) \le 2 \qquad (5)$$

where $a_1 = 1$, $m \in N$, $n \in N_0$, $m > n$ and $0 \le \gamma < 1$, $0 \le \lambda \le 1$, α real then f is sense preserving, harmonic univalent in U and for $\lambda \le \frac{1-\gamma}{1+\gamma}$, $f \in S_H^*(m,n,\lambda,\gamma)$.

Proof. First we establish that f is sense preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\ge 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty}\frac{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}{1-\gamma}|a_k| \\
&\ge \sum_{k=1}^{\infty}\frac{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1+\gamma)(1-\lambda)k^n}{1-\gamma}|b_k| \\
&\ge \sum_{k=1}^{\infty}\frac{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1+\gamma)(1-\lambda)k^n}{1-\gamma}|b_k||z|^{k-1} \\
&\ge \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \ge |g'(z)|
\end{aligned}$$

To show that f is univalent in U, we show that $f(z_1) \ne f(z_2)$ whenever $z_1 \ne z_2$. Suppose $z_1, z_2 \in U$ so that $z_1 \ne z_2$. Since the unit disc U is simply connected and convex, we then have $z(t) = (1-t)z_1 + tz_2$ in U where $0 \le t \le 1$. Then we write

$$f(z_2) - f(z_1) = \int_0^1 [(z_2 - z_1)h'(z(t)) + \overline{(z_2 - z_1)g'(z(t))}]dt.$$

On dividing throughout by $z_2 - z_1 \ne 0$ and taking only the real parts we obtain

$$\begin{aligned}
Re\,\frac{f(z_2) - f(z_1)}{z_2 - z_1} &= \int_0^1 Re\,[h'(z(t)) + \frac{\overline{(z_2 - z_1)}}{z_2 - z_1}\overline{g'(z(t))}]dt \\
&> \int_0^1 [Re\,h'(z(t)) - |g'(z(t))|]dt \qquad (6)
\end{aligned}$$

On the other hand

$$Re\,h'(z) - |g'(z)| \geq Re\,h'(z) - \sum_{n=1}^{\infty} n|b_n| \geq 1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n|$$

$$\geq 1 - \sum_{k=2}^{\infty} \frac{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}{1-\gamma}|a_k|$$

$$-\sum_{k=1}^{\infty} \frac{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma}|b_k| \geq 0 \quad \text{by (5)}$$

Therefore this together with inequality (6) implies the univalence of f. Next we show that $f \in S_H^*(m,n,\lambda,\gamma)$. To do so, we need to show that when (5) holds then (3) also holds true.

Using the fact that $Re\ w > \gamma$ if and only if $|1-\gamma+w| > |1+\gamma-w|$, it suffices to show that

$$|((1-\gamma-e^{i\alpha})\lambda + (1+e^{i\alpha}))D^m f(z) + (1-\gamma-e^{i\alpha})(1-\lambda)D^n f(z)|$$
$$- |((1+\gamma+e^{i\alpha})\lambda - (1+e^{i\alpha}))D^m f(z) + (1+\gamma+e^{i\alpha})(1-\lambda)D^n f(z)| \geq 0 \quad (7)$$

Substituting for $D^n f(z)$ and $D^m f(z)$ in (7), we obtain

$$\Bigg|z(2-\gamma) + \sum_{k=2}^{\infty}[(\lambda-\lambda\gamma+1)k^m + (1-\lambda-\gamma+\gamma\lambda)k^n$$
$$+ e^{i\alpha}((1-\lambda)k^m - (1-\lambda)k^n)]a_k z^k$$
$$- (-1)^n \sum_{k=1}^{\infty}[((\gamma-1)-\lambda(\gamma-1))k^n - (-1)^{m-n}(\lambda(1-\gamma)+1)k^m$$
$$+ e^{i\alpha}((1-\lambda)k^n - (-1)^{m-n}(1-\lambda)k^m)]\overline{b_k z_k}\Bigg|$$
$$- \Bigg|\gamma z - \sum_{k=2}^{\infty}[(1-\lambda-\lambda\gamma)k^m - (1+\gamma-\lambda-\lambda\gamma)k^n$$
$$+ e^{i\alpha}((1-\lambda)k^m - (1-\lambda)k^n)]a_k z^k$$
$$+ (-1)^n \sum_{k=1}^{\infty}[(1+\gamma)(1-\lambda)k^n - (-1)^{m-n}(1-(1+\gamma)\lambda)k^m$$
$$+ e^{i\alpha}((1-\lambda)k^n - (-1)^{m-n}(1-\lambda)k^m)]\overline{b_k z^k}\Bigg|$$

$$\geq 2(1-\gamma)|z| - \sum_{k=2}^{\infty} 2[(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n]|a_k||z^k|$$

$$- \sum_{k=1}^{\infty} |(\gamma-\lambda\gamma)k^n + (-1)^{m-n}(\lambda\gamma-2)k^m||b_k||z^k|$$

$$- \sum_{k=1}^{\infty} |(1-\lambda)(2+\gamma)k^n - (-1)^{m-n}(2\lambda+\lambda\gamma+2)k^m||b_k||z|^k$$

$$= \begin{cases} 2(1-\gamma)|z| - 2\sum_{k=2}^{\infty}[(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n]|a_k||z|^k \\ -2\sum_{k=1}^{\infty}[(2-\lambda(1+\gamma))k^m + (1-\lambda)(1+\gamma)k^n)]|b_k||z^k|, m-n \text{ is odd} \\ 2(1-\gamma)|z| - 2\sum_{k=2}^{\infty}[(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n]|a_k||z|^k \\ -2\sum_{k=1}^{\infty}[(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n]|b_k||z^k|, m-n \text{ is even} \end{cases}$$

$$= 2(1-\gamma)|z|\Bigg\{1 - \sum_{k=2}^{\infty}\left[\frac{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}{1-\gamma}\right]|a_k||z^{k-1}|$$

$$- \sum_{k=1}^{\infty}\left[\frac{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma}\right]|b_k||z^{k-1}|\Bigg\}$$

$$> 2(1-\gamma)\Bigg\{1 - \sum_{k=2}^{\infty}\left[\frac{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}{1-\gamma}\right]|a_k|$$

$$+ \sum_{k=1}^{\infty}\left[\frac{(2-\lambda(1+\gamma))k^m + (-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma}\right]|b_k|\Bigg\}$$

This last expression is non-negative by (5) and so the proof is complete. □

The harmonic univalent functions

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\gamma}{(2-\lambda(1+\gamma))k^m - (1+\gamma)(1-\lambda))k^n} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\gamma}{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1+\gamma)(1-\lambda))k^n} \overline{y_k z^k}, \tag{8}$$

where $m \in N$, $n \in N_0$, $m > n$ and $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$, show that the coefficient bound given by (5) is sharp.

In the following theorem, it is shown that the condition (5) is also necessary for functions $f_m = h + \overline{g}_m$ where h and g_m are of the form (4).

Theorem 2.2. *Let $f_m = h + \overline{g}_m$ be given by (4). Then $f_m \in TS_H^*(m,n,\lambda,\gamma)$ if and only if*

$$\sum_{k=1}^{\infty}((2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n)a_k + ((2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1-\lambda)(1+\gamma)k^n)b_k \le 2(1-\gamma) \quad (9)$$

where $a_1 = 1$, $0 \le \gamma < 1$, $0 \le \lambda \le \frac{1-\gamma}{1+\gamma}$.

Proof. Since $TS_H^*(m,n,\lambda,\gamma) \subset S_H^*(m,n,\lambda,\gamma)$, we only need to prove the 'only if' part of the theorem. To this end, for functions f_m of the form (4), we notice that the condition $Re\left\{\frac{(1+e^{i\alpha})D^m f(z)}{\lambda D^m f(z)+(1-\lambda)D^n f(z)} - e^{i\alpha}\right\} > \gamma$ is equivalent to

$$Re\left\{\frac{\begin{array}{c}(1-\gamma)z - \sum_{k=2}^{\infty}((1+e^{i\alpha}-(e^{i\alpha}+\gamma)\lambda)k^m \\ -(1-\lambda)(e^{i\alpha}+\gamma)k^n)a_k z^k \\ +(-1)^{2m-1}\sum_{k=1}^{\infty}[(1+e^{i\alpha}-\lambda(e^{i\alpha}+\gamma))k^m \\ -(-1)^{m-n}(e^{i\alpha}+\gamma)(1-\lambda)k^n]b_k\overline{z}^k\end{array}}{\begin{array}{c}z - \sum_{k=2}^{\infty}(\lambda k^m + (1-\lambda)k^n)a_k z^k \\ +(-1)^{m+n-1}\sum_{k=1}^{\infty}((-1)^{m-n}\lambda k^m + (1-\lambda)k^n)b_k\overline{z}^k\end{array}}\right\} \ge 0. \quad (10)$$

The above required condition (10) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, we must have

$$\frac{\begin{array}{c}(1-\gamma) - \sum_{k=2}^{\infty}[(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n]a_k r^{k-1} \\ -\sum_{k=1}^{\infty}[(2-\lambda(1-\gamma))k^m - (-1)^{m-n}(1+\gamma)(1-\lambda)k^n]b_k r^{k-1}\end{array}}{\begin{array}{c}1 - \sum_{k=2}^{\infty}(\lambda k^m + (1-\lambda)k^n)a_k r^{k-1} \\ -(-1)^{m-n}\sum_{k=1}^{\infty}((-1)^{m-n}\lambda k^m + (1-\lambda)k^n)b_k r^{k-1}\end{array}} \ge 0 \quad (11)$$

If the condition (9) does not hold, then the numerator in (11) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0, 1) for which the quotient in (11) is negative. This contradicts the required condition for $f_m \in TS_H^*(m,n,\lambda,\gamma)$ and so the proof is complete. □

Theorem 2.3. *Let f_m be given by (4). Then $f_m \in TS_H^*(m,n,\lambda,\gamma)$ if and only if $f_m(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_{m_k}(z))$ where $h_1(z) = z$,*

$$h_k(z) = z - \frac{1-\gamma}{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}z^k, (k = 2, 3, \dots)$$

$g_{m_k} = z+(-1)^{m-1}\dfrac{1-\gamma}{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1+\gamma)(1-\lambda)k^n}\overline{z}^k, (k=1,2,\dots)$

$x_k \geq 0$, $y_k \geq 0$, $x_1 = 1-\sum_{k=2}^{\infty}(x_k+y_k) \geq 0$. *In particular, the extreme points of* $TS_H^*(m,n,\lambda,\gamma)$ *are* $\{h_k\}$ *and* $\{g_{m_k}\}$.

Proof. Suppose

$$\begin{aligned} f_m(z) &= \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_{m_k}(z)) \\ &= \sum_{k=2}^{\infty}(x_k+y_k)z - \sum_{k=2}^{\infty}\frac{1-\gamma}{(2-\lambda(1+\gamma)k^m-(1-\lambda)(1+\gamma)k^n}x_k z^k \\ &\quad + (-1)^{m-1}\sum_{k=1}^{\infty}\frac{1-\gamma}{(2-\lambda(1+\gamma)k^m-(-1)^{m-n}(1-\lambda)(1+\gamma)k^n}y_k\overline{z}^k \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{(2-\lambda(1+\gamma))k^m-(1-\lambda)(1+\gamma)k^n}{1-\gamma} \\ &\qquad\left(\frac{1-\gamma}{(2-\lambda(1+\gamma))k^m-(1-\lambda)(1+\gamma)k^n}x_k\right) \\ &+\sum_{k=1}^{\infty}\frac{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma} \\ &\qquad\left(\frac{1-\gamma}{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1-\lambda)(1+\gamma)k^n}y_k\right) \end{aligned}$$

$$= \sum_{k=2}^{\infty}x_k + \sum_{k=1}^{\infty}y_k = 1-x_1 \leq 1$$

and so $f_m \in TS_H^*(m,n,\lambda,\gamma)$.

Conversely, if $f_m \in clco\ TS_H^*(m,n,\lambda,\gamma)$, then

$$a_k \leq \frac{1-\gamma}{(2-\lambda(1+\gamma))k^m-(1-\lambda)(1+\gamma)k^n} \text{ and}$$

$$b_k \leq \frac{1-\gamma}{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1-\lambda)(1+\gamma)k^n}$$

Set

$$\begin{aligned} x_k &= \frac{(2-\lambda(1+\gamma))k^m-(1-\lambda)(1+\gamma)k^n}{1-\gamma}a_k, (k=2,3,\dots) \text{ and} \\ y_k &= \frac{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1+\gamma)(1-\lambda)k^n}{1-\gamma}b_k, (k=1,2,\dots) \end{aligned}$$

Then note that by Theorem 2.2, $0 \leq x_k \leq 1$, $(k = 2, 3, \dots)$ and $0 \leq y_k \leq 1$, $(k = 1, 2, \dots)$. We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$ and note that, by theorem 2.2, $x_1 \geq 0$. Consequently, we obtain $f_m(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z))$ as required. □

We now obtain the distortion bounds for functions in $TS_H^*(m, n, \lambda, \gamma)$.

Theorem 2.4. *Let* $f_m \in TS_H^*(m, n, \lambda, \gamma)$. *Then for* $|z| = r < 1$ *we have*

$$\begin{aligned}|f_m(z)| \leq &(1 + b_1)r + \frac{1}{2^n}\left(\frac{1-\gamma}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)}\right.\\ &\left. - \frac{2-\lambda(1+\gamma) - (-1)^{m-n}(1+\gamma)(1-\lambda)}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)} b_1\right) r^2, |z| = r < 1\end{aligned}$$

and

$$\begin{aligned}|f_m(z)| \geq &(1 - b_1)r - \frac{1}{2^n}\left(\frac{1-\gamma}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)}\right.\\ &\left. - \frac{2-\lambda(1+\gamma) - (-1)^{m-n}(1+\gamma)(1-\lambda)}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)} b_1\right) r^2, |z| = r < 1\end{aligned}$$

Proof. We only prove the first inequality. The proof for the second is similar. Let $f_m \in TS_H^*(m, n, \lambda, \gamma)$. We have

$$\begin{aligned}|f_m(z)| &\leq (1 + b_1)r + \sum_{k=2}^{\infty}(a_k + b_k)r^k \leq (1 + b_1)r + \sum_{k=2}^{\infty}(a_k + b_k)r^2\\ &= (1 + b_1)r + \frac{1-\gamma}{2^n[(2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)]}\\ &\quad \sum_{k=2}^{\infty} \frac{2^n[(2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)]}{1-\gamma}(a_k + b_k)r^2\\ &\leq (1 + b_1)r + \frac{(1-\gamma)r^2}{2^n[(2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)]}\\ &\quad \sum_{k=2}^{\infty}\left(\frac{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}{1-\gamma}a_k\right.\\ &\quad \left. + \frac{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma}b_k\right)\\ &\leq (1 + b_1)r + \frac{1}{2^n}\left(\frac{1-\gamma}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)}\right.\\ &\quad \left. - \frac{2-\lambda(1+\gamma) - (-1)^{m-n}(1+\gamma)(1-\lambda)}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)} b_1\right) r^2\end{aligned}$$

The bounds given in Theorem 2.4 for the functions $f = h + \overline{g}$ of the form (4) also hold for functions of the form (1) if the coefficient condition (5) is satisfied. The upper bound given for $f \in TSH^*(m,n,\lambda,\gamma)$ is sharp and the equality occurs for the function

$$\begin{aligned} f(z) &= z + |b_1|\overline{z} + \frac{1}{2^n}\left[\frac{1-\gamma}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)}\right. \\ &\quad \left. - \frac{2-\lambda(1+\gamma) - (-1)^{m-n}(1+\gamma)(1-\lambda)}{2^{m-n}(2-\lambda(1+\gamma)) - (1-\lambda)(1+\gamma)}|b_1|\right]\overline{z}^2, \\ |b_1| &\leq \frac{1-\gamma}{(2-\lambda(1+\gamma)) - (-1)^{m-n}(1-\lambda)(1+\gamma)} \end{aligned}$$

□

The following covering result follows from the second inequality in Theorem 2.4.

Corollary 2.5. *Let* f_m *of the form (4) be so that* $f_m \in TS_H^*(m,n,\lambda,\gamma)$. *Then*

$$\begin{aligned} &\left\{ w : |w| < \frac{2^{m+1} - (1+\gamma)[2^m\lambda + 2^n(1-\lambda)] - (1-\gamma)}{2^{m+1} - 2^m\lambda(1+\gamma) - 2^n(1-\lambda)(1+\gamma)} \right. \\ &\qquad \left. - \frac{2^{m+1} - (1+\gamma)[2^m\lambda + 2^n(1-\lambda)] + \lambda - (-1)^{m-n}(1-\lambda) + 2}{2^{m+1} - 2^m\lambda(1+\gamma) - 2^n(1-\lambda)(1+\gamma)}\, b_1 \right\} \\ &\qquad \subset f_m(U) \end{aligned}$$

We now consider the convolution of two harmonic functions, $f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k \overline{z}^k$ and $F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k \overline{z}^k$ as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k B_k \overline{z}^k \tag{12}$$

Using this definition, we show that the class $TS_H^*(m,n,\lambda,\gamma)$ is closed under convolution.

Theorem 2.6. *For* $0 \leq \beta \leq \gamma < 1$, *let* $f_m \in TS_H^*(m,n,\lambda,\gamma)$ *and* $F_m \in TS_H^*(m,n,\lambda,\beta)$. *Then* $f_m * F_m \in TS_H^*(m,n,\lambda,\gamma) \subset TS_H^*(m,n,\lambda,\beta)$.

Proof. Let $f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k \overline{z}^k$ be in $TS_H^*(m,n,\lambda,\gamma)$ and $F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k \overline{z}^k$, be in $TS_H^*(m,n,\lambda,\beta)$. Then the convolution $f_m * F_m$ is given by (12). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2.2. For

$F_m \in TS_H^*(m,n,\lambda,\beta)$ we note that $A_k < 1$ and $B_k < 1$. Now, for the convolution function $f_m * F_m$ we obtain

$$\sum_{k=2}^{\infty} \frac{(2-\lambda(1+\beta))k^m - (1-\lambda)(1+\beta)k^n}{1-\beta} a_k A_k + \sum_{k=1}^{\infty} \frac{(2-\lambda(1+\beta))k^m - (-1)^{m-n}(1-\lambda)(1+\beta)k^n}{1-\beta} b_k B_k$$

$$\leq \sum_{k=2}^{\infty} \frac{(2-\lambda(1+\beta))k^m - (1-\lambda)(1+\beta)k^n}{1-\beta} a_k + \sum_{k=1}^{\infty} \frac{(2-\lambda(1+\beta))k^m - (-1)^{m-n}(1-\lambda)(1+\beta)k^n}{1-\beta} b_k$$
$$\leq \sum_{k=2}^{\infty} \frac{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}{1-\gamma} a_k + \sum_{k=1}^{\infty} \frac{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma} b_k \leq 1$$

since $0 \leq \beta \leq \gamma < 1$ and $f_m \in TS_H^*(m,n,\lambda,\gamma)$. Hence $f_m * F_m \in TS_H^*(m,n,\lambda,\gamma) \subset TS_H^*(m,n,\lambda,\beta)$. □

Now we show that $TS_H^*(m,n,\lambda,\gamma)$ is closed under convex combination of its members.

Theorem 2.7. *The class $TS_H^*(m,n,\lambda,\gamma)$ is closed under convex combination.*

Proof. For $i = 1,2,3,\ldots$. Let $f_{m_i} \in TS_H^*(m,n,\lambda,\gamma)$, where f_{m_i} is given by $f_{m_i}(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k,i} \overline{z}^k$. Then by (9)

$$\sum_{k=1}^{\infty} \frac{(2-\lambda(1+\gamma))k^m - (1-\lambda)(1+\gamma)k^n}{1-\gamma} a_{k,i} + \frac{(2-\lambda(1+\gamma))k^m - (-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma} b_{k,i} \leq 2 \tag{13}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k,i} \right) z^k + (-1)^{m-1} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k,i} \right) \overline{z}^k \tag{14}$$

Then by (13)

$$\sum_{k=1}^{\infty}\left[\frac{(2-\lambda(1+\gamma))k^m-(1-\lambda)(1+\gamma)k^n}{1-\gamma}\sum_{i=1}^{\infty}t_i a_{k,i}\right.$$
$$\left.+\frac{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma}\sum_{i=1}^{\infty}t_i b_{k,i}\right]$$
$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{k=1}^{\infty}\frac{(2-\lambda(1+\gamma))k^m-(1-\lambda)(1+\gamma)k^n}{1-\gamma}a_{k,i}\right.$$
$$\left.+\frac{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma}b_{k,i}\right\}$$
$$\leq 2\sum_{i=1}^{\infty}t_i=2,\ \text{which is the required coefficient condition.}$$
□

Theorem 2.8. *If $f_m \in TS_H^*(m,n,\lambda,\gamma)$ then f_m is convex in the disc*

$$|z|\leq \min_k\left\{\frac{(1-\gamma)(1-b_1)}{k[1-\gamma-(2-\lambda(1+\gamma)-(-1)^{m-n}(1+\gamma)(1-\lambda))]b_1}\right\}^{\frac{1}{k-1}},$$

$k=2,3,\ldots.$

Proof. Let $f_m \in TS_H^*(m,n,\lambda,\gamma)$ and let r, $0<r<1$ be fixed. Then $r^{-1}f_m(rz)\in TS_H^*(m,n,\lambda,\gamma)$ and we have

$$\sum_{k=2}^{\infty}k^2(a_k+b_k)r^{k-1}=\sum_{k=2}^{\infty}k(a_k+b_k)(kr^{k-1})$$
$$\leq\sum_{k=2}^{\infty}\left(\frac{(2-\lambda(1+\gamma))k^m-(1-\lambda)(1+\gamma)k^n}{1-\gamma}a_k\right.$$
$$\left.+\frac{(2-\lambda(1+\gamma))k^m-(-1)^{m-n}(1-\lambda)(1+\gamma)k^n}{1-\gamma}b_k\right)kr^{k-1}$$
$$\leq 1-b_1$$

if

$$kr^{k-1}\leq\frac{1-b_1}{1-\frac{(2-\lambda(1+\gamma)-(-1)^{m-n}(1-\lambda)(1+\gamma))}{1-\gamma}b_1}$$

or

$$r\leq\min_k\left\{\frac{(1-\gamma)(1-b_1)}{k[1-\gamma-(2-\lambda+\lambda\gamma-(-1)^{m-n}(1+\gamma)(1-\lambda))]b_1}\right\}^{\frac{1}{k-1}},k=2,3,\ldots.$$

□

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