Chamchuri Journal of Mathematics
Volume 2(2010) Number 1, 93–100
http://www.math.sc.chula.ac.th/cjm

Reciprocal Sums of Elements Satisfying Second Order Linear Recurrences*

Kantaphon Kuhapatanakul† and Vichian Laohakosol

Received 30 Apr 2010
Revised 6 Aug 2010
Accepted 26 Aug 2010

Abstract: Let $\{U_n\}_{n=0}^{\infty}$ and $\{W_n\}_{n=0}^{\infty}$ be two sequences defined by $U_0 = 0, U_1 = 1$, $U_{n+2} = pU_{n+1} + qU_n$ and $W_{n+2} = pW_{n+1} + qW_n$ (W_0, W_1 arbitrary) with $p, q \in \mathbb{R}; p^2 + 4q > 0$. The aim of this paper is to prove

$$\sum_{n=1}^{N} \frac{(-q)^{at-1} U_{at^n(t-1)}}{W_{at^n} W_{at^{n+1}}} = \sum_{n=1}^{\frac{at^{N+1}-at}{2}} \frac{(-q)^{at-1} p}{W_{at+2(n-1)} W_{at+2n}} = \frac{1}{W_0 W_2 - W_1^2} \Big(\frac{W_{at-1}}{W_{at}} - \frac{W_{at^{N+1}-1}}{W_{at^{N+1}}} \Big),$$

where $a, t \in \mathbb{N}$ and $t \geq 2$. This identity generalizes a number of known identities such as $\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$, where $\{F_n\}$ is the Fibonacci sequence.

Keywords: reciprocal sum, linear recurrence, Fibonacci numbers

2000 Mathematics Subject Classification: 11B37, 11B39

1 Introduction

Let

$$\{F_n\}_{n\geq 0} = \{0, 1, \dots, F_{n+1} = F_n + F_{n-1}, \dots\}$$

Supported by the Commission on Higher Education and the Thailand Research Fund RTA5180005 and by Kasetsart University Research and Development Institute

†*Corresponding author*

and

$$\{L_n\}_{n\geq 0} = \{2, 1, \ldots, L_{n+1} = L_n + L_{n-1}, \ldots\}$$

denote the sequences of Fibonacci, respectively, Lucas numbers. In 1974, Millin [9] posed the problem of showing that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}. \tag{1}$$

A proof of (1) by Good is given in [2], while in [4], Hoggatt and Bicknell demonstrated eleven different methods of finding the same sum. The identity (1) was further extended by Hoggatt and Bicknell in [5], where they showed that

$$\sum_{n=1}^{\infty} \frac{1}{F_{a2^{n+1}}} = \begin{cases} \dfrac{5F_a^2 + 2L_a - F_{2a}\sqrt{5}}{2F_{2a}} & \text{if } a \text{ is odd} \\ \dfrac{L_a + 2 - F_a\sqrt{5}}{2F_a} & \text{otherwise.} \end{cases}$$

In [7], this last sum was also found to equal

$$\sum_{n=1}^{\infty} \frac{1}{F_{a2^{n+1}}} = \frac{F_{2a+1}}{F_{2a}} - \frac{1+\sqrt{5}}{2}, \tag{2}$$

while a finite version of this sum was shown by Greig [3] to be

$$\sum_{i=0}^{n} \frac{1}{F_{a2^i}} = \begin{cases} \dfrac{1+F_{a-1}}{F_a} - \dfrac{F_{a2^n-1}}{F_{a2^n}} & ; \quad a \text{ even} \\ \dfrac{1+F_{a-1}}{F_a} + \dfrac{2}{F_{2a}} - \dfrac{F_{a2^n-1}}{F_{a2^n}} & ; \quad a \text{ odd.} \end{cases}$$

In 1976, using Lambert series expansions, Bruckman and Good [1] evaluated several reciprocal sums including

$$\sum_{n=0}^{\infty} \frac{L_{k\cdot 3^n}}{F_{k\cdot 3^{n+1}}} = \frac{(\sqrt{5}-1)^k}{2^k F_k} \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{F_{k\cdot 3^n}}{L_{k\cdot 3^{n+1}}} = \frac{(\sqrt{5}-1)^k}{2^k\sqrt{5}\, L_k}. \tag{4}$$

Since the Fibonacci and the Lucas numbers are elements satisfying the same second order linear recurrence relation but with different initial values, it is natural to ask whether the above-mentioned identities continue to hold for elements satisfying a general second order linear recurrence relation. We answer this question affirmatively here.

Define three second order linear recurrences $\{U_n\}_{n\geq 0}, \{V_n\}_{n\geq 0}$ and $\{W_n\}_{n\geq 0}$ by

$$U_{n+2} = pU_{n+1} + qU_n, \quad U_0 = 0, \quad U_1 = 1$$
$$V_{n+2} = pV_{n+1} + qV_n, \quad V_0 = 2, \quad V_1 = p$$

and

$$W_{n+2} = pW_{n+1} + qW_n, \qquad W_0, W_1 \text{ arbitrary},$$

where $p, q \in \mathbb{R}$ are subject to $p^2 + 4q > 0$. Let α, β be the two roots of its characteristic equation $x^2 - px - q = 0$ with $|\beta| < |\alpha|$. It is well-known, [6], that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \quad V_n = \alpha^n + \beta^n, \quad W_n = r_1\alpha^n + r_2\beta^n,$$

where $r_1 = \frac{W_1 - W_0\beta}{\alpha - \beta}$ and $r_2 = \frac{W_0\alpha - W_1}{\alpha - \beta}$. If $p = 1, q = 1$, then $U_n = F_n$ and $V_n = L_n$ are the Fibonacci and Lucas numbers, respectively. The following identities are easily verified

$$F_{2n} = F_n L_n \tag{5}$$
$$\alpha^n = \alpha F_n + F_{n-1} \quad \text{or} \quad \beta^n = \beta F_n + F_{n-1} \tag{6}$$
$$\sqrt{5}\alpha^n = \alpha L_n + L_{n-1} \quad \text{or} \quad \sqrt{5}\beta^n = -\beta L_n - L_{n-1}. \tag{7}$$

Certain reciprocal sums of elements in the sequence $\{U_n\}$ have previously appeared such as in 1995, Melham and Shanon [8] found, when $q = 1$, that

$$\sum_{n=0}^{\infty} \frac{1}{U_{k\cdot 2^n}} = \begin{cases} \frac{1-U_{k-1}}{U_k} + \frac{1}{\alpha} & \text{if } p > 2 \\ \frac{1-U_{k-1}}{U_k} + \frac{1}{\beta} & \text{if } p < -2 \end{cases}. \tag{8}$$

2 Results

The notation of Section 1 will be kept standard throughout. Observe that the Fibonacci and Lucas numbers satisfy the following identities

$$F_m F_{n-1} - F_{m-1} F_n = (-1)^n F_{m-n} \tag{9}$$
$$L_m L_{n-1} - L_{m-1} L_n = (-1)^{n-1} 5F_{m-n}, \tag{10}$$

In (9), taking $m = n + 1$, we get

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n,$$

which is the result found by Cassini (Theorem 5.3 in Chapter 5 of [6]). These identities are special cases of (11) in the next lemma.

Lemma 2.1. *If* m, n *are two positive integers with* $m \geq n$, *then*

$$W_mW_{n-1} - W_{m-1}W_n = (-q)^{n-1}(W_0W_2 - W_1^2)U_{m-n}. \tag{11}$$

Proof. Both sides of (11) are zero when $m = n$. Observe from the recurrence relation that

$$\begin{aligned}\begin{pmatrix} W_m & W_n \\ W_{m-1} & W_{n-1}\end{pmatrix} &= \begin{pmatrix} p & q \\ 1 & 0\end{pmatrix}\begin{pmatrix} W_{m-1} & W_{n-1} \\ W_{m-2} & W_{n-2}\end{pmatrix} \\ &= \begin{pmatrix} p & q \\ 1 & 0\end{pmatrix}\begin{pmatrix} p & q \\ 1 & 0\end{pmatrix}\begin{pmatrix} W_{m-2} & W_{n-2} \\ W_{m-3} & W_{n-3}\end{pmatrix} \\ &\vdots \\ &= \underbrace{\begin{pmatrix} p & q \\ 1 & 0\end{pmatrix}\begin{pmatrix} p & q \\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} p & q \\ 1 & 0\end{pmatrix}}_{n-1 \text{ terms}}\begin{pmatrix} W_{m-n+1} & W_1 \\ W_{m-n} & W_0\end{pmatrix}.\end{aligned}$$

Evaluating the determinants on both sides, we get

$$W_mW_{n-1} - W_{m-1}W_n = (-q)^{n-1}(W_{m-n+1}W_0 - W_{m-n}W_1).$$

It remains to show that

$$W_{m-n+1}W_0 - W_{m-n}W_1 = (W_0W_2 - W_1^2)U_{m-n},$$

i.e.

$$W_{k+1}W_0 - W_kW_1 = (W_0W_2 - W_1^2)U_k, \tag{12}$$

where $m = n+k$. Clearly, (12) holds for $k = 1$. Assume it is true for an arbitrary positive integer i. By definition and the induction hypothesis, we get

$$\begin{aligned} W_{i+2}W_0 - W_{i+1}W_1 &= (pW_{i+1} + qW_i)W_0 - (pW_i + qW_{i-1})W_1 \\ &= p(W_{i+1}W_0 - W_iW_1) + q(W_iW_0 - W_{i-1}W_1) \\ &= p(W_0W_2 - W_1^2)U_i + q(W_0W_2 - W_1^2)U_{i-1} \\ &= (W_0W_2 - W_1^2)U_{i+1}, \end{aligned}$$

i.e., (12) also holds for $i + 1$. □

We next state and prove our main theorem.

Theorem 2.2. *Let $N, a, t \in \mathbb{N}$ with $t \geq 2$. If $W_0W_2 \neq W_1^2$ and $W_n \neq 0$ for all n, then*

1) $\sum_{i=1}^{N} \frac{(-q)^{at-1}p}{W_{at+2(i-1)}W_{at+2i}} = \frac{1}{W_0W_2-W_1^2}\left(\frac{W_{at-1}}{W_{at}} - \frac{W_{at+2N-1}}{W_{at+2N}}\right)$;

2) $\sum_{i=1}^{N} \frac{(-q)^{at-1}U_{at^i(t-1)}}{W_{at^i}W_{at^{i+1}}} = \frac{1}{W_0W_2-W_1^2}\left(\frac{W_{at-1}}{W_{at}} - \frac{W_{at^{N+1}-1}}{W_{at^{N+1}}}\right)$;

3) $\sum_{n=1}^{N} \frac{U_{at^n(t-1)}}{W_{at^n}W_{at^{n+1}}} = \sum_{n=1}^{\frac{at^{N+1}-at}{2}} \frac{p}{W_{at+2(n-1)}W_{at+2n}}$.

Proof. Rewrite the equation (11) of Lemma 2.1 as

$$\frac{(-q)^{n-1}U_{m-n}}{W_nW_m} = \frac{1}{W_0W_2 - W_1^2}\Big(\frac{W_{n-1}}{W_n} - \frac{W_{m-1}}{W_m}\Big). \tag{13}$$

Putting $m = at + 2i, n = at + 2(i-1)$, we get

$$\frac{(-q)^{at-1}p}{W_{at+2(i-1)}W_{at+2i}} = \frac{1}{W_0W_2 - W_1^2}\Big(\frac{W_{at+2i-3}}{W_{at+2i-2}} - \frac{W_{at+2i-1}}{W_{at+2i}}\Big).$$

Summing over i from 1 to N, we get the result of Part 1).

For Part 2), putting $m = at^{i+1}, n = at^i$, we get

$$\frac{(-q)^{at-1}U_{at^i(t-1)}}{W_{at^i}W_{at^{i+1}}} = \frac{1}{W_0W_2 - W_1^2}\Big(\frac{W_{at^i-1}}{W_{at^i}} - \frac{W_{at^{i+1}}}{W_{at^{i+1}}}\Big).$$

The identity in Part 2) follows by summing over i.

Part 3) follows by taking $N = \frac{at^{N+1}-at}{2}$ in Part 1). □

3 Applications

Theorem 2.2 is a host of a good deal of identities as we now show.

Corollary 3.1. *If $N, a, t \in \mathbb{N}$ with $t \geq 2$, then*

1) $\sum_{i=1}^{N} \frac{(-q)^{at-1}p}{U_{at+2(i-1)}U_{at+2i}} = \frac{U_{at+2N-1}}{U_{at+2N}} - \frac{U_{at-1}}{U_{at}}$;

2) $\sum_{i=1}^{N} \frac{(-q)^{at-1}p}{V_{at+2(i-1)}V_{at+2i}} = \frac{1}{p^2+4q}\left(\frac{V_{at-1}}{V_{at}} - \frac{V_{at+2N-1}}{V_{at+2N}}\right)$;

3) $\sum_{i=1}^{N} \frac{(-q)^{at-1}U_{at^i(t-1)}}{U_{at^i}U_{at^{i+1}}} = \frac{U_{at^{N+1}-1}}{U_{at^{N+1}}} - \frac{U_{at-1}}{U_{at}}$;

4) $\sum_{i=1}^{N} \frac{(-q)^{at-1}U_{at^i(t-1)}}{V_{at^i}V_{at^{i+1}}} = \frac{1}{p^2+4q}\left(\frac{V_{at-1}}{V_{at}} - \frac{V_{at^{N+1}-1}}{V_{at^{N+1}}}\right)$.

Proof. For Parts 1) and 3), put $W_n = U_n$ in Theorem 2.2 Part 1) and 2), respectively.
For Parts 2) and 4), put $W_n = V_n$ in Theorem 2.2 Part 1) and 2), respectively. □

Letting $N \to \infty$ in Corrolary 3.1, and using

$$\lim_{N\to\infty} \frac{U_{N-1}}{U_N} = \lim_{N\to\infty} \frac{V_{N-1}}{V_N} = \frac{1}{\alpha},$$

we obtain:

Corollary 3.2. *If $a, t \in \mathbb{N}$ with $t \geq 2$, then*

1) $\sum_{i=1}^{\infty} \frac{(-q)^{at-1}}{p} U_{at+2(i-1)} U_{at+2i} = \frac{1}{\alpha} - \frac{U_{at-1}}{U_{at}}$;

2) $\sum_{i=1}^{\infty} \frac{(-q)^{at-1} p}{V_{at+2(i-1)} V_{at+2i}} = \frac{1}{p^2+4q} \left(\frac{V_{at-1}}{V_{at}} - \frac{1}{\alpha} \right)$;

3) $\sum_{i=1}^{\infty} \frac{(-q)^{at-1} U_{at^i(t-1)}}{U_{at^i} U_{at^{i+1}}} = \frac{1}{\alpha} - \frac{U_{at-1}}{U_{at}}$;

4) $\sum_{i=1}^{\infty} \frac{(-q)^{at-1} U_{at^i(t-1)}}{V_{at^i} V_{at^{i+1}}} = \frac{1}{p^2+4q} \left(\frac{V_{at-1}}{V_{at}} - \frac{1}{\alpha} \right)$.

Specializing certain parameters in Corrolary 3.2, several identities, mentioned in Section 1, follow easily as we illustrate now.

1) Putting $t = 2$ and $q = 1$ in Part 3), we get

$$\sum_{i=1}^{\infty} \frac{1}{U_{a2^{i+1}}} = \frac{U_{2a-1}}{U_{2a}} - \frac{1}{\alpha}$$

which, after a little more computation, is (8).

2) When $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$ are the Fibonacci and Lucas sequences, we have $\alpha = (1+\sqrt{5})/2$. Part 3) of Corrolary 3.2 gives, when $t = 2$, the identity

$$\sum_{i=1}^{\infty} \frac{1}{F_{a2^{i+1}}} = \frac{F_{2a-1}}{F_{2a}} - \frac{2}{1+\sqrt{5}} = \frac{F_{2a+1} - F_{2a}}{F_{2a}} + 1 - \frac{1+\sqrt{5}}{2} = \frac{F_{2a+1}}{F_{2a}} - \frac{1+\sqrt{5}}{2},$$

which is (2), and when $t = 3$ and $q = 1$, the identity

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{F_{2a3^i}}{F_{a3^i}F_{a3^{i+1}}} = \sum_{i=1}^{\infty} \frac{L_{a3^i}}{F_{a3^{i+1}}} &= (-1)^{a-1}\Big(\frac{2}{1+\sqrt{5}} - \frac{F_{3a-1}}{F_{3a}}\Big) && \text{(using (5))} \\
&= (-1)^a \cdot \frac{1-\sqrt{5}}{2} + \frac{(-1)^a F_a F_{3a-1} + F_{2a}}{F_a F_{3a}} - \frac{F_{2a}}{F_a F_{3a}} \\
&= (-1)^a \cdot \frac{1-\sqrt{5}}{2} + \frac{(-1)^a F_{a-1} F_{3a}}{F_a F_{3a}} - \frac{F_{2a}}{F_a F_{3a}} && \text{(using (9))} \\
&= \frac{(-1)^a}{F_a}\Big(\frac{1-\sqrt{5}}{2} F_a + F_{a-1}\Big) - \frac{F_{2a}}{F_a F_{3a}} \\
&= \frac{(\sqrt{5}-1)^a}{2^a F_a} - \frac{F_{2a}}{F_a F_{3a}}, && \text{(using (6))}
\end{aligned}$$

which is (3). Part 4) of Corrolary 3.2 gives, when $t = 3$, the identity

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{F_{2a3^i}}{L_{a3^i}L_{a3^{i+1}}} = \sum_{i=1}^{\infty} \frac{F_{a3^i}}{L_{a3^{i+1}}} &= \frac{(-1)^a}{5}\Big(\frac{2}{1+\sqrt{5}} - \frac{L_{3a-1}}{L_{3a}}\Big) && \text{(using (5))} \\
&= \frac{(-1)^a}{5 L_a L_{3a}}\Big(\frac{\sqrt{5}-1}{2} L_a L_{3a} - L_{3a-1} L_a\Big) \\
&= \frac{(-1)^a}{5 L_a L_{3a}}\Big(\frac{\sqrt{5}-1}{2} L_a L_{3a} - L_{3a} L_{a-1} - (-1)^a 5 F_{2a}\Big) && \text{(using (10))} \\
&= \frac{(\sqrt{5}-1)^a}{2^a \sqrt{5} L_a} - \frac{F_{2a}}{L_a L_{3a}}, && \text{(using (7))}
\end{aligned}$$

which is (4).

References

[1] P.S. Bruckman and I.J. Good, A generalization of series of De Morgan, with applications of Fibonacci type, *Fibonacci Quart.*, **14**(1976), 193–196.

[2] I.J. Good, A reciprocal series of Fibonacci numbers, *Fibonacci Quart.*, **12**(4)(1974), 346.

[3] W.E. Greig, Sum of Fibonacci reciprocals, *Fibonacci Quart.*, **15**(1)(1977), 46–48.

[4] V.E. Hoggart, Jr., and M. Bicknell, A primer for the Fibonacci Numbers, part XV: variations on summing a series of reciprocals of Fibonacci Numbers, *Fibonacci Quart.*, **14**(3)(1976), 272–276.

[5] V.E. Hoggart, Jr., and M. Bicknell, A reciprocal series of Fibonacci Numbers with subscripts $2^n k$, *Fibonacci Quart.*, **14**(5)(1976), 453–455.

[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, 2001.

[7] V. Laohakosol and K. Kuhapatanakul, The reverse irrationality criteria of Brun and Badea, Contributions in General Algebra II, *Proc. Intern. Conf. Discrete Math. Appl., Bangkok, March 5-7, 2008; a special volume published by East-West J. Math.*, 217–234.

[8] R.S. Melham and A.G. Shannon, On reciprocal sums of Chebyshev related sequences, *Fibonacci Quart.*, **33**(1995), 194–202.

[9] D.A. Millin, Problem H-237, *Fibonacci Quart.*, **12**(3)(1974), 309.

Kantaphon Kuhapatanakul
Department of Mathematics
Kasetsart University
Bangkok 10900, Thailand
Email: `fscikpkk@ku.ac.th`

Vichian Laohakosol
Department of Mathematics
Kasetsart University
Bangkok 10900, Thailand
Email: `fscivil@ku.ac.th`