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Some Inclusion Relationships of Certain Subclasses of p-valent Mermorphic Functions Associated with Certain Integral Operator

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Abstract: In this paper we investigate a family of integral operators defined on the space of p-valent mermorphic functions. By making use of these novel integral operators, we introduce and investigate several new subclasses of p-valent mermorphic functions. Also we establish some inclusion relationships associated with the aforementioned integral operators. Several interesting integral preserving properties also considered.

Keywords: Analytic functions, p-valent mermorphic starlikefunctions, p-valent mermorphic convex functions, p-valent mermorphic close-to-convex functions, p-valent mermorphic quasi-convex functions, integral operator, Hadmard product, subordination

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1 Introduction

Let $\mathcal{M}_p$ denote the class of p-valent functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \qquad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and mermorphic in the punctured disc

$$U^{*}=\{z: z\in\mathbb{C} \text{ and } |z|<1\}=U\backslash\{0\}.$$

If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω, analytic in U with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in U)$, such that $f(z)=g(\omega(z))$ $(z\in U)$. In particular, if the function g is univalent in U, $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

For $0\le\alpha,\beta<p$, we denote by $\mathcal{MS}_p(\alpha), \mathcal{MC}_p(\alpha), \mathcal{MK}_p(\alpha,\beta)$ and $\mathcal{MQ}_p(\alpha,\beta)$ the subclasses of $\mathcal{M}_p$ consisting of all analytic functions which are, respectively, p-valent mermorphic starlike of order α, p-valent mermorphic convex of order α, p-valent mermorphic close-to-convex of order α, and type β and p-valent mermorphic quasi-convex of order α, and type β in U (see [1] and [4]).

Let P be the class of all functions ϕ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0)=1$ and $\Re\{\phi(z)\}>0$ for $z\in U$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}_p(\alpha;\phi)$, $\mathcal{MC}_p(\alpha;\phi)$, $\mathcal{MK}_p(\alpha,\beta;\phi,\psi)$ and $\mathcal{MQ}_p(\alpha,\beta;\phi,\psi)$ of the class $\mathcal{M}_p$ for $0\le\alpha,\beta<p$, and $\phi,\psi\in P$, which are defined by

$$\mathcal{MS}_p(\alpha;\phi)=\left\{f\in\mathcal{M}_p: -\frac{1}{p-\alpha}\left(\frac{zf'(z)}{f(z)}+\alpha\right)\prec\phi(z) \text{ in } U\right\}, \tag{1.2}$$

$$\mathcal{MC}_p(\alpha;\phi)=\left\{f\in\mathcal{M}_p: -\frac{1}{p-\alpha}\left(\frac{(zf'(z))'}{f'(z)}+\alpha\right)\prec\phi(z) \text{ in } U\right\}, \tag{1.3}$$

$$\mathcal{MK}_p(\alpha,\beta;\phi,\psi) = \left\{f\in\mathcal{M}_p: \exists\, g\in\mathcal{MS}_p(\alpha;\phi), -\frac{1}{p-\beta}\left(\frac{zf'(z)}{g(z)}+\beta\right)\prec\psi(z) \text{ in } U\right\}, \tag{1.4}$$

$$\mathcal{MQ}_p(\alpha,\beta;\phi,\psi) = \left\{f\in\mathcal{M}_p: \exists\, g\in\mathcal{MC}_p(\alpha;\phi), -\frac{1}{p-\beta}\left(\frac{(zf'(z))'}{g'(z)}+\beta\right)\prec\psi(z) \text{ in } U\right\}. \tag{1.5}$$

Also let the Hadamard product (or convolution) $f*g$ of two analytic functions $f(z)$, is defined by (1.1), and $g(z)$ is defined by

$$g(z)=z^{-p}+\sum_{k=1}^{\infty}b_k z^{k-p} \qquad (p\in\mathbb{N}), \tag{1.6}$$

be given (as usual) by

$$(f*g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g*f)(z). \tag{1.7}$$

For complex parameters $a_1, \dots, a_q;\ b_1, \dots, b_s$ ($b_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$; $j = 1, \dots, s$), the generalized hypergeometric function ${}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z)$ is given by (see [6])

$$ {}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!} \tag{1.8}$$

$$(q \le s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(x)_k$ is the Pochhammer symbol defined (in terms of the Gamma function) by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & (k=0) \\ x(x+1)\cdots(x+k-1) & (k \in \mathbb{N}). \end{cases} \tag{1.9}$$

Corresponding to a function $F_{\mu,p}(a_1, \dots, a_q; b_1, \dots, b_s; z)$, defined by

$$\begin{aligned} F_{\mu,p}(a_1, \dots, a_q; b_1, \dots, b_s; z) = {} & \frac{1-\mu}{z^p}\ {}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z) \\ & + \frac{\mu}{z^p}\left(z^{p+1}\ {}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z)\right)', \end{aligned} \tag{1.10}$$

we introduce a function $F_{\mu,p}^{\lambda}(a_1, \dots, a_q;\ b_1, \dots, b_s; z)$ given by

$$F_{\mu,p}(a_1, \dots, a_q; b_1, \dots, b_s; z) * F_{\mu,p}^{\lambda}(a_1, \dots, a_q; b_1, \dots, b_s; z) = \frac{1}{z^p(1-z)^{\lambda+p}} \quad (\lambda > -p). \tag{1.11}$$

Corresponding to the function $F_{\mu,p}^{\lambda}(a_1, \dots, a_q;\ b_1, \dots, b_s; z)$, we introduce the linear operator

$$J_{\mu,p}^{\lambda}(a_1, \dots, a_q; b_1, \dots, b_s) : \mathcal{M}_p \to \mathcal{M}_p$$

which is defined the following convolution

$$J_{\mu,p}^{\lambda}(a_1, \dots, a_q; b_1, \dots, b_s) f(z) = F_{\mu,p}^{\lambda}(a_1, \dots, a_q;\ b_1, \dots, b_s; z) * f(z). \tag{1.12}$$

For $f \in \mathcal{M}_p$ given by (1.1), then from (1.12), we have

$$J_{\mu,p}^{\lambda}(a_1, \dots, a_q; b_1, \dots, b_s) f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (b_1)_k \cdots (b_s)_k}{(1+\mu k)(a_1)_k \cdots (a_q)_k} a_k z^{k-p} \tag{1.13}$$

$$\left(a_i, b_j \in \mathbb{C}\backslash\mathbb{Z}_0^-, i=1,\ldots,s, j=1,\ldots,q; \lambda > -p; p \in \mathbb{N}; \mu \geq 0; z \in U^*\right).$$

For convenience, we write

$$J_{\mu,p,q,s}^{\lambda}\left(a_1\right) = J_{\mu,p}^{\lambda}\left(a_1,\ldots,a_q; b_1,\ldots,b_s\right). \tag{1.14}$$

It is easily verified from (1.13) that

$$z\left(J_{\mu,p,q,s}^{\lambda}(a_1+1)f(z)\right)' = a_1 J_{\mu,p,q,s}^{\lambda}(a_1)f(z) - (a_1+p)J_{\mu,p,q,s}^{\lambda}(a_1+1)f(z), \tag{1.15}$$

and

$$z\left(J_{\mu,p,q,s}^{\lambda}\left(a_1\right)f\left(z\right)\right)' = (\lambda+p)J_{\mu,p,q,s}^{\lambda}(a_1)f(z) - (\lambda+2p)J_{\mu,p,q,s}^{\lambda}(a_1)f(z). \tag{1.16}$$

We note that

(i) for $p=1, \mu=0, \lambda=\sigma-1\,(\sigma>0)$, we have $J_{0,1,q,s}^{\sigma-1}\left(a_1\right) = H_{\sigma,q,s}\left(a_1\right)$, where $H_{\sigma,q,s}\left(a_1\right)$ was defined by Cho and Kim [2];

(ii) for $p=s=1, q=2, a_1=n+1(n>-1), a_2=b_1, \mu=0, \lambda=\sigma-1(\sigma>0)$, we have $J_{0,1,2,1}^{\sigma-1}\left(n+1\right) = I_{n,\sigma}$, where $I_{n,\sigma}$ was defined by Yuan et al. [7].

Next, by using the operator $J_{\mu,p,q,s}^{\lambda}\left(a_1\right)$, we introduce the following classes of analytic functions for $\phi,\psi \in P$ and $0 \leq \alpha,\beta < p$

$$\mathcal{MS}_{\mu,p,q,s}^{\lambda}\left(a_1;\alpha;\phi\right) = \left\{f \in \mathcal{M}_p : J_{\mu,p,q,s}^{\lambda}\left(a_1\right)f \in \mathcal{MS}_p\left(\alpha;\phi\right)\right\}, \tag{1.17}$$

$$\mathcal{MC}_{\mu,p,q,s}^{\lambda}\left(a_1;\alpha;\phi\right) = \left\{f \in \mathcal{M}_p : J_{\mu,p,q,s}^{\lambda}\left(a_1\right)f \in \mathcal{MC}_p\left(\alpha;\phi\right)\right\}, \tag{1.18}$$

$$\mathcal{MK}_{\mu,p,q,s}^{\lambda}(a_1;\alpha,\beta;\phi,\psi) = \{f \in \mathcal{M}_p : J_{\mu,p,q,s}^{\lambda}(a_1)f \in \mathcal{MK}_p(\alpha,\beta;\phi,\psi)\}, \tag{1.19}$$

$$\mathcal{MQ}_{\mu,p,q,s}^{\lambda}(a_1;\alpha,\beta;\phi,\psi) = \{f \in \mathcal{M}_p : J_{\mu,p,q,s}^{\lambda}(a_1)f \in \mathcal{MQ}_p(\alpha,\beta;\phi,\psi)\}. \tag{1.20}$$

We also note that

$$f \in \mathcal{MC}_{\mu,p,q,s}^{\lambda}\left(a_1;\alpha;\phi\right) \Leftrightarrow -\frac{zf'}{p}\mathcal{MS}_{\mu,p,q,s}^{\lambda}\left(a_1;\alpha;\phi\right) \tag{1.21}$$

and

$$f \in \mathcal{MQ}_{\mu,p,q,s}^{\lambda}\left(a_1;\alpha,\beta;\phi,\psi\right) \Leftrightarrow -\frac{zf'}{p} \in \mathcal{MK}_{\mu,p,q,s}^{\lambda}\left(a_1;\alpha,\beta;\phi,\psi\right). \tag{1.22}$$

In particular, we set

$$\mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\frac{1+Az}{1+Bz}) = \mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;A,B) \qquad (-1<B<A\leq 1) \tag{1.23}$$

and

$$\mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\frac{1+Az}{1+Bz}) = \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;A,B) \qquad (-1 < B < A \leq 1). \quad (1.24)$$

In this paper, we investigate several inclusion properties of the classes $\mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$, $\mathcal{MC}^{\lambda}_{\mu,p,q,s}(\mu,a_1;\alpha;\phi)$, $\mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$ and $\mathcal{MQ}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$ associated with the operator $J^{\lambda}_{\mu,p,q,s}(a_1)$. Some applications involving integral operators are also considered.

2 Inclusion properties involving the operator $J^{\lambda}_{\mu,p,q,s}(a_1)$

The following lemmas will be required in our investigation.

Lemma 1 [3]. *Let ϕ be convex univalent in U with $\phi(0)=1$ and $\Re\{\eta\phi(z)+\gamma\}>0$ $(\eta,\gamma\in\mathbb{C})$. If q is analytic in U with $q(0)=1$, then*

$$q(z) + \frac{zq'(z)}{\eta q(z)+\gamma} \prec \phi(z) \tag{2.1}$$

implies $q(z)\prec\phi(z)$.

Lemma 2 [5]. *Let ϕ be convex univalent in U and let w be analytic in U with $\Re\{w(z)\}\geq 0$. If q is analytic in U and $q(0)=\phi(0)$, then*

$$q(z) + w(z)\,zq'(z) \prec \phi(z) \tag{2.2}$$

implies $q(z)\prec\phi(z)$.

Theorem 1. *Let $\phi\in P$ with*

$$\max_{z\in U}(\Re\{\phi(z)\}) < \min\left(\frac{\lambda+2p-\alpha}{p-\alpha}, \frac{a_1+p-\alpha}{p-\alpha}\right) \qquad (\mu>0; 0\leq\alpha<p). \tag{2.3}$$

Then,

$$\mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi) \subset \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) \subset \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha;\phi). \tag{2.4}$$

Proof. We begin by showing the first inclusion relationship

$$\mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi) \subset \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi). \tag{2.5}$$

Let $f \in \mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi)$ and set

$$q(z) = -\frac{1}{p-\alpha}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1)f(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1)f(z)}+\alpha\right), \tag{2.6}$$

where the function q is analytic in U with $q(0)=1$. Using (1.16) and (2.6), we have

$$-\frac{1}{p-\alpha}\left(\frac{z(J^{\lambda+1}_{\mu,p,q,s}(a_1)f(z))'}{J^{\lambda+1}_{\mu,p,q,s}(a_1)f(z)}+\alpha\right) = q(z) + \frac{zq'(z)}{\lambda+2p-\alpha-(p-\alpha)q(z)} \qquad (z\in U). \tag{2.7}$$

From (2.3), we see that

$$\Re\{\lambda+2p-\alpha-(p-\alpha)\phi(z)\}>0 \quad (z\in U). \tag{2.8}$$

Applying Lemma 1 to (2.7), it follows that $q \prec \phi$, that is, $f \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$.

To prove the second part, let $f \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$ and put

$$s(z) = -\frac{1}{p-\alpha}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1+1)f(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1+1)f(z)}+\alpha\right) \quad (z\in U), \tag{2.9}$$

where the function s is an analytic function with $s(0)=1$. Then, by using the arguments similar to those detailed above with (1.15), it follows that $s\prec\phi$ in U, which implies that $f\in\mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha;\phi)$. Therefore, we complete the proof of Theorem 1. □

Theorem 2. *Let* $\phi\in P$ *with* (2.3) holds. *Then,*

$$\mathcal{MC}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi)\subset\mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)\subset\mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha;\phi). \tag{2.10}$$

Proof. Applying (1.24) and Theorem 1, we observe that

$$\begin{aligned} f\in\mathcal{MC}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi) &\iff -\frac{zf'}{p}\in\mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi) && \text{(from (1.21))}\\ &\Longrightarrow -\frac{zf'}{p}\in\mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) && \text{(by Theorem 1)}\\ &\iff f\in\mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi), \end{aligned}$$

and

$$\begin{aligned} f\in\mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) &\iff -\frac{zf'}{p}\in\mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) && \text{(from (1.21))}\\ &\Longrightarrow -\frac{zf'}{p}\in\mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha;\phi) && \text{(by Theorem 1)}\\ &\iff f\in\mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha;\phi), \end{aligned}$$

which evidently proves Theorem 2. □

Taking $\phi(z) = \dfrac{1+Az}{1+Bz} \quad (-1 < B < A \leq 1; z \in U)$ in Theorems 1 and 2, respectively, we have the following corollary.

Corollary 1. *Let* $\phi \in P$ *with*

$$\frac{1+A}{1+B} < \min\left(\frac{\lambda+2p-\alpha}{p-\alpha}, \frac{a_1+p-\alpha}{p-\alpha}\right) \qquad (\mu > 0; 0 \leq \alpha < p; -1 < B < A \leq 1).$$

Then,

$$\mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;A,B) \subset \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;A,B) \subset \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha;A,B), \tag{2.11}$$

and

$$\mathcal{MC}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;A,B) \subset \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;A,B) \subset \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha;A,B). \tag{2.12}$$

Next, by using Lemma 2, we obtain the following inclusion relation for $\mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$.

Theorem 3. *Let* $\phi, \psi \in P$ *with* (2.3) *holds. Then,*

$$\mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha,\beta;\phi,\psi). \tag{2.13}$$

Proof. We begin by proving that

$$\mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi).$$

Let $f \in \mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$. Then, from the definition of $\mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$, there exists a function $r \in \mathcal{MS}_p(\alpha;\phi)$ such that

$$-\frac{1}{p-\beta}\left(\frac{z\left(J^{\lambda+1}_{\mu,p,q,s}(a_1)f(z)\right)'}{r(z)}+\beta\right) \prec \psi(z) \quad (z \in U). \tag{2.14}$$

Choose the function g such that $J^{\lambda+1}_{\mu,p,q,s}(a_1)g(z) = r(z)$. Then, $g \in \mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi)$ and

$$-\frac{1}{p-\beta}\left(\frac{z\left(J^{\lambda+1}_{\mu,p,q,s}(a_1)f(z)\right)'}{J^{\lambda+1}_{\mu,p,q,s}(a_1)g(z)}+\beta\right) \prec \psi(z) \quad (z \in U). \tag{2.15}$$

Now let

$$q(z) = -\frac{1}{p-\beta}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1)f(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1)g(z)}+\beta\right) \quad (z \in U), \tag{2.16}$$

where the function q is analytic in U with $q(0)=1$. Using (1.16), we have

$$\begin{aligned}(p-\beta)\,zq'(z)\; & J^{\lambda}_{\mu,p,q,s}(a_1)g(z) + [(p-\beta)q(z)+\beta]\; z\left(J^{\lambda}_{\mu,p,q,s}(a_1)g(z)\right)' \\ &= -(\lambda+p)\,z\left(J^{\lambda}_{\mu,p,q,s}(a_1)f(z)\right)' + (\lambda+2p)\,z\left(J^{\lambda}_{\mu,p,q,s}(a_1)f(z)\right)'. \end{aligned} \tag{2.17}$$

Since $g \in \mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi)$, by Theorem 1, we know that $g \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$. Let

$$t(z) = -\frac{1}{p-\alpha}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1)g(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1)g(z)}+\alpha\right). \tag{2.18}$$

Then, using (1.16) once again, we have

$$(\lambda+p)\frac{J^{\lambda+1}_{\mu,p,q,s}(a_1)g(z)}{J^{\lambda}_{\mu,p,q,s}(a_1)g(z)} = \lambda+2p-\alpha-(p-\alpha)\,t(z). \tag{2.19}$$

From (2.17) and (2.19), we obtain

$$-\frac{1}{p-\beta}\left(\frac{z(J^{\lambda+1}_{\mu,p,q,s}(a_1)f(z))'}{J^{\lambda+1}_{\mu,p,q,s}(a_1)g(z)}+\beta\right) = q(z) + \frac{zq'(z)}{\lambda+2p-\alpha-(p-\alpha)t(z)} \prec \psi(z). \tag{2.20}$$

Since $\lambda > -p$ and $t \prec \phi$ in U with (2.3) holds, we obtain

$$\Re\{\lambda+2p-\alpha-(p-\alpha)\,t(z)\} > 0 \qquad (z \in U).$$

Hence, by taking

$$w(z) = \frac{1}{\lambda+2p-\alpha-(p-\alpha)\,t(z)}$$

in equation (2.20), and then applying Lemma 2, we can show that $q \prec \psi$, so that $f \in \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$. For the second part, by using the arguments similar to those detailed above with (1.15), we obtain

$$\mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha,\beta;\phi,\psi).$$

Therefore, we complete the proof of Theorem 3. □

Theorem 4. *Let* $\phi,\psi \in P$ *with* (2.3) *holds. Then,*

$$\mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1+1;\alpha,\beta;\phi,\psi).$$

Proof. Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (1.21), we can also prove Theorem 4 by using Theorem 3 the equivalence (1.22). □

3 Inclusion properties involving the integral operator $F_{p,c}(f)$

In this section, we present several integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator $F_{p,c}(f)$ (see [4]) defined by

$$F_{p,c}(f)(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)\, dt \quad (f \in \mathcal{M}_p; c > 0),$$

which satisfies the following relationship:

$$z(J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(f)(z))' = cJ^{\lambda}_{\mu,p,q,s}(a_1)f(zt) - (c+p)J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(f)(z). \quad (3.1)$$

We first prove the following inclusion relationship for the integral operator $F_{p,c}(f)$.

Theorem 5. *Let* $\phi \in P$ *with*

$$\max_{z\in U}(\Re\{\phi(z)\}) < \frac{c+p-\alpha}{p-\alpha} \quad (c > -p\ ; 0 \le \alpha < p). \quad (3.2)$$

If $f \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1; \alpha; \phi)$, *then* $F_{p,c}(f) \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1; \alpha; \phi)$.

Proof. Let $f \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1; \alpha; \phi)$ and set

$$q(z) = -\frac{1}{p-\alpha}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(f)(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(f)(z)} + \alpha\right) \quad (z \in U), \quad (3.3)$$

where the function q is analytic in U with $q(0) = 1$. From the identity (3.1), we obtain

$$c\frac{J^{\lambda}_{\mu,p,q,s}(a_1)f(z)}{J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(f)(z)} = c+p-\alpha-(p-\alpha)q(z). \quad (3.4)$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$-\frac{1}{p-\alpha}\left(\frac{z(J^{\lambda}_{\mu,p,q,s}(a_1)f(z))'}{J^{\lambda}_{\mu,p,q,s}(a_1)f(z)} + \alpha\right) = q(z) + \frac{z'(z)}{c+p-\alpha-(p-\alpha)q(z)} \prec \phi(z) \quad (z \in U). \quad (3.5)$$

Hence, by virtue of Lemma 1, we conclude that $q \prec \phi$ in U, which implies that $F_{p,c}(f) \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$. □

Theorem 6. *Let* $\phi \in P$ *with* (3.2) holds. *If* $f \in \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$, *then* $F_{p,c}(f) \in \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$.

Proof. Applying Theorem 5, it follows that

$$\begin{aligned} f \in \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) &\Longleftrightarrow -\frac{zf'}{p} \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) \\ &\Longrightarrow F_{p,c}\left(-\frac{zf'}{p}\right) \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) \qquad \text{(by Theorem 5)} \\ &\Longleftrightarrow -\frac{z\left(F_{p,c}(f)\right)'}{p} \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi) \\ &\Longleftrightarrow F_{p,c}(f) \in \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi), \end{aligned}$$

which proves Theorem 6. □

From Theorems 5 and 6, respectively, we have the following corollary.

Corollary 2. *Let* $\phi \in P$ *with*

$$\frac{1+A}{1+B} < \frac{c+p-\alpha}{p-\alpha} \qquad (c > 0; 0 \le \alpha < p; -1 < B < A \le 1).$$

Then, for the function classes defined by (1.23) *and* (1.24), *the following inclusion relationships hold ture*

$$f \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;A,B) \Longrightarrow F_{p,c}(f) \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;A,B),$$

and

$$f \in \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;A,B) \Longrightarrow F_{p,c}(f) \in \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;A,B).$$

Theorem 7. *Let* $\phi, \psi \in P$ *with (3.2) holds. If* $f \in \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$, *then* $F_{p,c}(f) \in \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$.

Proof. Let $f \in \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$. Then, in view of the definition of $\mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$, there exists a function $g \in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$ such that

$$-\frac{1}{p-\beta}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1) f(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1) g(z)} + \beta\right) \prec \psi(z). \qquad (3.6)$$

Thus, we set

$$q(z) = -\frac{1}{p-\beta}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(f)(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(g)(z)}+\beta\right) \quad (z\in U), \tag{3.7}$$

where the function q is analytic in U with $q(0)=1$. Using (3.1) in (3.7), we have

$$(p-\beta)zq'(z)J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(g)(z)+[(p-\beta)q(z)+\beta]z\left(J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(g)(z)\right)'$$
$$= (c+p)z\left(J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(f)(z)\right)' - cz\left(J^{\lambda}_{\mu,p,q,s}(a_1)f(z)\right)'. \tag{3.8}$$

Since $g\in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$, by using Theorem 5, we obtain that $F_{p,c}(g)\in \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1;\alpha;\phi)$. Let

$$t(z) = -\frac{1}{p-\alpha}\left(\frac{z\left(J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(g)(z)\right)'}{J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(g)(z)}+\alpha\right) \quad (z\in U). \tag{3.9}$$

Then, using (3.1) once again, we have

$$c\frac{zJ^{\lambda}_{\mu,p,q,s}(a_1)g(z)}{J^{\lambda}_{\mu,p,q,s}(a_1)F_{p,c}(g)(z)} = c+p-\alpha-(p-\alpha)t(z). \tag{3.10}$$

From (3.8) and (3.10) Hence, we have

$$-\frac{1}{p-\beta}\left(\frac{z(J^{\lambda}_{\mu,p,q,s}(a_1)f(z))'}{J^{\lambda}_{\mu,p,q,s}(a_1)g(z)}+\beta\right) = q(z)+\frac{zq'(z)}{c+p-\alpha-(p-\alpha)t(z)} \prec \psi(z). \tag{3.11}$$

Since $c>0$ and $t\prec\phi$ in U with (3.2) holds, we have

$$\Re\{c+p-\alpha-(p-\alpha)t(z)\}>0.$$

Hence, by taking

$$w(z) = \frac{1}{c+p-\alpha-(p-\alpha)t(z)}$$

in (3.11) and then applying Lemma 2, we find that $q\prec\psi$ in U, so that have $F_{p,c}(f)\in \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$. The proof of Theorem 7 is evidently completed. □

Theorem 8. *Let $\phi,\psi\in P$ with (3.2) holds. If $f\in \mathcal{MQ}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$, then $F_{p,c}(f)\in \mathcal{MQ}^{\lambda}_{\mu,p,q,s}(a_1;\alpha,\beta;\phi,\psi)$.*

Proof. Just as we derived Theorem 6 as consequence of Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7. □

Remark. Putting $p = 1, \mu = 0, \lambda = \sigma - 1\,(\sigma > 0)$ in the above results, we obtain the results of Cho and Kim [2].

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