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Congruence Pairs of Algebras Abstracting Double Kleene and Stone algebras

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Abstract: In this note, we extend the result of Beazer on congruence pairs of K_2-algebras to the class of double K_2-algebras. We show that any congruence α on a double K_2-algebra can be represented by a congruence pair $\langle\theta_1, \theta_2\rangle$, where θ_1 is a Kleene congruence and θ_2 is a lattice one. As an application of this result, we give a sufficient condition for a double K_2-algebra is congruence permutable (n-permutable).

Keywords: double K_2-algebra, congruence pair, congruence permutable, congruence n-permutable

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1 Introduction

Lakser [9] and Katriňák [10] independently showed that any congruence on a distributive p-algebra can be represented by a congruence pair $\langle\theta_1, \theta_2\rangle$, where θ_1 is a (Boolean) congruence of the skeleton Boolean algebra $B(L)$ of L and θ_2 is a (lattice) congruence of the dense filter $D(L)$ of L. Subsequently, Katriňák [11] showed that a congruence on a p-algebra with a modular frame can be described in exactly the same way as in the distributive p-algebra and Beazer [2] gave a different method to describe the congruence pairs of a distributive double p-algebra with non-empty core. In [3], Beazer characterized the congruence pairs for algebras abstracting Kleene and Stone algebras, that is the class of K_2-algebras which was introduced by Blyth and Varlet in [7]. The purpose of this note is to extend the result of Beazer on congruence pairs of K_2-algebras to the class of double K_2-algebras. We show that any congruence α on a double K_2-algebra can be represented by a congruence pair $\langle\theta_1, \theta_2\rangle$, where θ_1 is a (Kleene) congruence and θ_2 is a lattice one. As an application of this result, we give a sufficient condition that a double K_2-algebra is congruence permutable (n-permutable).

2 Preliminaries

An *MS-algebra* is an algebra $\langle L; \vee, \wedge, ^\circ, 0, 1\rangle$ of type $\langle 2,2,1,0,0\rangle$ whose reduct $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and such that, for any $x, y \in L$,

$$x \le x^{\circ\circ},\ \ (x\wedge y)^\circ = x^\circ \vee y^\circ,\ \ 1^\circ = 0.$$

Clearly, the class **MS** of MS-algebras is a variety. The subvariety K_2 of MS-algebra L is defined by

$$(*)\ \ x\wedge x^\circ = x^\circ \wedge x^{\circ\circ} \text{ and } x\wedge x^\circ \le y \vee y^\circ.$$

A *double MS-algebra* must be an algebra $\langle L; \vee, \wedge, ^\circ, ^+, 0, 1\rangle$ of type $\langle 2,2,1,1,0,0\rangle$ such that $(L; ^\circ)$ is an MS-algebra, $(L; ^+)$ is a dual MS-algebra, and the unary operations are linked by the properties:

$$(\forall x \in L) \quad x^{\circ +} = x^{\circ\circ} \text{ and } x^{+\circ} = x^{++}.$$

In this note, our aim is to describe the congruence pairs of the class of double K_2-algebras. A *congruence* on a double K_2-algebra $(L; ^\circ, ^+)$ is a lattice congruence θ such that

$$(x,y)\in\theta \Rightarrow (x^\circ, y^\circ)\in\theta \text{ and } (x^+, y^+)\in\theta$$

Through what follows, for a double K_2-algebra $(L; ^\circ, ^+)$, we shall denote by $ConL$ the lattice of congruences of L. If S is a subalgebra of L and α is a congruence on L then we denote by $\alpha|_S$ a restriction of α on S. The basic congruence Φ on L given by

$$(x,y)\in\Phi \iff \quad x^\circ = y^\circ \text{ and } x^+ = y^+$$

The *skeleton* of L is the Kleene algebra

$$S(L) = \{x \in L; x = x^{\circ\circ}\} = \{x \in L; x = x^{++}\}.$$

The *core* of L is the set

$$K(L) = \{x \vee x^{\circ}; x \in L\} \cap \{x \wedge x^{+}; x \in L\} = \{x \in L; x^{\circ} \leq x \leq x^{+}\}.$$

For the basic properties of MS-algebras and double MS-algebras we refer the reader to [5]-[8]. The notations and terminologies we shall use in this note are same as used in [5] and [8].

3 Congruence pairs

Let $(L;^{\circ},^{+})$ be a double K_2-algebra. We shall be concerned with the condition that $K(L) = [k, l]$ is a bounded non-empty core. We now begin with the following:

Lemma 3.1. *Let $(L;^{\circ},^{+})$ be a double K_2-algebra and let $K(L) = [k, l]$. Then every $x \in L$ is of the form*

$$(\dagger) \quad x = x^{++} \vee [x^{\circ\circ} \wedge (x \vee k) \wedge l].$$

Proof. We first show that for each $x \in L$, $x^{\circ\circ} \wedge k = x \wedge k$. Since $x \leq x^{\circ\circ}$, we have $x \wedge k \leq x^{\circ\circ} \wedge k$. Suppose now, by way of obtaining the contradiction, that $x \wedge k < x^{\circ\circ} \wedge k$. Here we must have $k \neq x \wedge x^{\circ}$; for otherwise, it follows the contradiction that $x \wedge x^{\circ} = x \wedge k < x^{\circ\circ} \wedge k = x \wedge x^{\circ}$. Let $a = x^{\circ\circ} \wedge k$, $b = x^{\circ} \wedge k$ and $c = x \wedge k$. Since, by [2, Theorem 1], that $L^{\vee} = \{x \vee x^{\circ} \mid x \in L\}$ is a filter, it follows that $k = inf L^{\vee}$ and so $k \leq x \vee x^{\circ}$. Hence we have $x \wedge x^{\circ} < k \leq x \vee x^{\circ}$. Then by $(*)$ we obtain

$$a \wedge b = x^{\circ\circ} \wedge x^{\circ} \wedge k = x \wedge x^{\circ} \wedge k = x \wedge x^{\circ} = b \wedge c$$

and

$$a \vee b = (x^{\circ\circ} \vee x^{\circ}) \wedge k = k = (x^{\circ} \vee x) \wedge k = (x^{\circ} \wedge k) \vee (x \wedge k) = b \vee c.$$

Here we must have $k \neq b$; for otherwise, it follows the contradiction that

$$a = x^{\circ\circ} \wedge k = x^{\circ\circ} \wedge b = x^{\circ\circ} \wedge x^{\circ} \wedge k = x \wedge x^{\circ} \wedge k = x \wedge b = x \wedge k = c.$$

Therefore, $\{x \wedge x^{\circ}, c, a, b, k\}$ is a five-element non-modular sublattice of L, which contradicts to the distributivity of L. Consequently, we obtain that

$$x = x \vee (x \wedge k) = x \vee (x^{\circ\circ} \wedge k) = x^{\circ\circ} \wedge (x \vee k).$$

Similarly, we can show that $x = x^{++} \vee (x \wedge l)$. Thus we obtain

$$x = x^{++} \vee [x^{\circ\circ} \wedge (x \vee k) \wedge l].$$

□

Given a double K_2-algebra L with $K(L) = [k, l]$. Every member of L contains two simpler substructures, one being a Kleene algebra $S(L)$ and the other being a distributive sublattice $K(L)$. For any $\alpha \in ConL$, define the pair

$$\langle \theta_1, \theta_2 \rangle \in Con(S(L)) \times Con(K(L))$$

where θ_1 is the restriction $\alpha|_{S(L)}$ of α to $S(L)$ and θ_2 is the restriction $\alpha|_{K(L)}$ of α to $K(L)$, where $Con(S(L))$ and $Con(K(L))$ are the congruence lattice of $S(L)$ and $K(L)$ respectively.

Definition 3.2. Let $(L;^{\circ}, ^{+})$ be a double K_2-algebra and let $K(L) = [k, l]$. We say that the pair $\langle \theta_1, \theta_2 \rangle \in Con(S(L)) \times Con(K(L))$ is a *double K_2-congruence pair* if it satisfies the following conditions:

$(CP1)$: $c \overset{\theta_2}{\equiv} d \Rightarrow c^{\circ} \overset{\theta_1}{\equiv} d^{\circ}$ and $c^{+} \overset{\theta_1}{\equiv} d^{+}$;

$(CP2)$: $a \overset{\theta_1}{\equiv} b \Rightarrow (a \vee k) \wedge l \overset{\theta_2}{\equiv} (b \vee k) \wedge l$.

In what follows, we shall denote by $Con_p(L)$ the set of all double K_2-congruence pairs of a double K_2-algebra L.

Theorem 3.3. *Let $(L;^{\circ}, ^{+})$ be a double K_2-algebra and let $K(L) = [k, l]$. Then $Con_p(L)$ is a sublattice of $Con(S(L)) \times Con(K(L))$.*

Proof. Let $\langle \theta_1, \theta_2 \rangle$, $\langle \varphi_1, \varphi_2 \rangle \in Con_p(L)$. Clearly, $\langle \theta_1 \wedge \varphi_1, \theta_2 \wedge \varphi_2 \rangle \in Con_p(L)$. To see that $\langle \theta_1 \vee \varphi_1, \theta_2 \vee \varphi_2 \rangle \in Con_p(L)$, let $a \overset{\theta_1 \vee \varphi_1}{\equiv} b$ and $c \overset{\theta_2 \vee \varphi_2}{\equiv} d$. Then there exist

$$a = x_0, x_1, \cdots, x_m = b \text{ in } S(L)$$

and

$$c = y_0, y_1, \cdots, y_n = d \text{ in } K(L)$$

such that

$$a = x_0 \equiv x_1 \equiv x_2 \equiv \cdots \equiv x_m = b$$

and

$$c = y_0 \equiv y_1 \equiv y_2 \equiv \cdots \equiv y_n = d$$

where $x_i \overset{\theta_1}{\equiv} x_{i+1}$ or $x_i \overset{\varphi_1}{\equiv} x_{i+1}$ and $y_j \overset{\theta_2}{\equiv} y_{j+1}$ or $y_j \overset{\varphi_2}{\equiv} y_{j+1}$. By properties $(CP1)$ and $(CP2)$, we see that $y_j^\circ \overset{\theta_1}{\equiv} y_{j+1}^\circ$, $y_j^+ \overset{\theta_1}{\equiv} y_{j+1}^+$ or $y_j^\circ \overset{\varphi_1}{\equiv} y_{j+1}^\circ$, $y_j^+ \overset{\varphi_1}{\equiv} y_{j+1}^+$ and $(x_i \vee k) \wedge l \overset{\theta_2}{\equiv} (x_{i+1} \vee k) \wedge l$ or $(x_i \vee k) \wedge l \overset{\varphi_2}{\equiv} (x_{i+1} \vee k) \wedge l$. Thus, $c^\circ \overset{\theta_1 \vee \varphi_1}{\equiv} d^\circ$, $c^+ \overset{\theta_1 \vee \varphi_1}{\equiv} d^+$ and $(a \vee k) \wedge l \overset{\theta_2 \vee \varphi_2}{\equiv} (b \vee k) \wedge l$. Consequently, $\langle \theta_1 \vee \varphi_1, \theta_2 \vee \varphi_2 \rangle \in Con_p(L)$ and we obtain that $Con_p(L)$ is a sublattice of $Con(S(L)) \times Con(K(L))$. □

Theorem 3.4. *Every congruence α on a double K_2-algebra $(L;^\circ,^+)$ with $K(L) = [k, l]$ determines a double K_2-congruence pair. Conversely, every double K_2-congruence pair $\langle \theta_1, \theta_2 \rangle \in Con(S(L)) \times Con(K(L))$ uniquely determines a congruence α on L satisfying $\alpha|_{S(L)} = \theta_1$ and $\alpha|_{K(L)} = \theta_2$ such that the following conditions are equivalent:*

(i) $x \overset{\alpha}{\equiv} y$;

(ii) $x^\circ \overset{\theta_1}{\equiv} y^\circ$, $x^+ \overset{\theta_1}{\equiv} y^+$ *and* $(x \vee k) \wedge l \overset{\theta_2}{\equiv} (y \vee k) \wedge l$.

Proof. Let α be a relation on L defined by (i) and (ii). Clearly, α is an equivalence relation. Let now $a \overset{\alpha}{\equiv} b$ and $c \overset{\alpha}{\equiv} d$. Then

$$a^\circ \overset{\theta_1}{\equiv} b^\circ, \;\; a^+ \overset{\theta_1}{\equiv} b^+, \;\; (a \vee k) \wedge l \overset{\theta_2}{\equiv} (b \vee k) \wedge l$$

and

$$c^\circ \overset{\theta_1}{\equiv} d^\circ, \;\; c^+ \overset{\theta_1}{\equiv} d^+, \;\; (c \vee k) \wedge l \overset{\theta_2}{\equiv} (d \vee k) \wedge l.$$

By the distributivity of L, it is not hard to see that

$$(a \vee c)^\circ \overset{\theta_1}{\equiv} (b \vee d)^\circ, \;\; (a \vee c)^+ \overset{\theta_1}{\equiv} (b \vee d)^+, \;\; (a \vee c \vee k) \wedge l \overset{\theta_2}{\equiv} (b \vee d \vee k) \wedge l$$

and

$$(a \wedge c)^\circ \overset{\theta_1}{\equiv} (b \wedge d)^\circ, \;\; (a \wedge c)^+ \overset{\theta_1}{\equiv} (b \wedge d)^+, \;\; ((a \wedge c) \vee k) \wedge l \overset{\theta_2}{\equiv} ((b \wedge d) \vee k) \wedge l.$$

Thus $(a \vee c, b \vee d) \in \alpha$ and $(a \wedge c, b \wedge d) \in \alpha$. Hence, we obtain that α is a lattice congruence on L.

We now show that α preserves the unary operations $^\circ$ and $^+$. Let $x \stackrel{\alpha}{\equiv} y$. Then

$$x^\circ \stackrel{\theta_1}{\equiv} y^\circ,\ \ x^+ \stackrel{\theta_1}{\equiv} y^+,\ \ (x \vee k) \wedge l \stackrel{\theta_2}{\equiv} (y \vee k) \wedge l.$$

Since $a^{\circ\circ} = a^{\circ +}$ and $a^{++} = a^{+\circ}$ for any $a \in L$, and by property $(CP2)$, we observe that

$$x^{\circ\circ} \stackrel{\theta_1}{\equiv} y^{\circ\circ},\ \ x^{\circ +} \stackrel{\theta_1}{\equiv} y^{\circ +},\ \ (x^\circ \vee k) \wedge l \stackrel{\theta_2}{\equiv} (y^\circ \vee k) \wedge l$$

and

$$x^{+\circ} \stackrel{\theta_1}{\equiv} y^{+\circ},\ \ x^{++} \stackrel{\theta_1}{\equiv} y^{++},\ \ (x^+ \vee k) \wedge l \stackrel{\theta_2}{\equiv} (y^+ \vee k) \wedge l.$$

Thus α preserves the unary operations $^\circ$ and $^+$ and so α is a congruence on L

Next, we show that $\alpha|_{S(L)} = \theta_1$ and $\alpha|_{K(L)} = \theta_2$. Let $x, y \in S(L)$. If $x \stackrel{\theta_1}{\equiv} y$, then $x^\circ \stackrel{\theta_1}{\equiv} y^\circ$ and $x^+ \stackrel{\theta_1}{\equiv} y^+$. By property $(CP2)$, we obtain that $(x \vee k) \wedge l \stackrel{\theta_2}{\equiv} (y \vee k) \wedge l$ and so $x \stackrel{\alpha|_{S(L)}}{\equiv} y$. Then $\theta_1 \leq \alpha|_{S(L)}$. If $x \stackrel{\alpha|_{S(L)}}{\equiv} y$ then $x^\circ \stackrel{\theta_1}{\equiv} y^\circ$ so that $x = x^{\circ\circ} \stackrel{\theta_1}{\equiv} y^{\circ\circ} = y$. Thus $\alpha|_{S(L)} \leq \theta_1$ and consequently, $\alpha|_{S(L)} = \theta_1$. Now, let $x, y \in K(L)$. If $x \stackrel{\theta_2}{\equiv} y$ then $x^\circ \stackrel{\theta_1}{\equiv} y^\circ$ and $x^+ \stackrel{\theta_1}{\equiv} y^+$ by property $(CP1)$. Since $K(L)$ is sublattice of L, we see that $(x \vee k) \wedge l \stackrel{\theta_2}{\equiv} (y \vee k) \wedge l$. Hence $x \stackrel{\alpha|_{K(L)}}{\equiv} y$ and whence $\theta_2 \leq \alpha|_{K(L)}$. If $x \stackrel{\alpha|_{K(L)}}{\equiv} y$ then $x = (x \vee k) \wedge l \stackrel{\theta_2}{\equiv} (y \vee k) \wedge l = y$. Thus $\alpha|_{K(L)} \leq \theta_2$ and consequently, $\alpha|_{K(L)} = \theta_2$.

Suppose now, by the way of obtaining the uniqueness of the theorem, that there exist $\alpha, \beta \in ConL$ such that $\alpha|_{S(L)} = \beta|_{S(L)}$ and $\alpha|_{K(L)} = \beta|_{K(L)}$. Let $x \stackrel{\alpha}{\equiv} y$. Then $x^{\circ\circ} \stackrel{\alpha|_{S(L)}}{\equiv} y^{\circ\circ}$, $x^{++} \stackrel{\alpha|_{S(L)}}{\equiv} y^{++}$ imply $x^{\circ\circ} \stackrel{\beta|_{S(L)}}{\equiv} y^{\circ\circ}$, $x^{++} \stackrel{\beta|_{S(L)}}{\equiv} y^{++}$, and $(x \vee k) \wedge l \stackrel{\alpha|_{K(L)}}{\equiv} (y \vee k) \wedge l$ implies $(x \vee k) \wedge l \stackrel{\beta|_{K(L)}}{\equiv} (y \vee k) \wedge l$. By $(\dagger)$, we obtain that

$$x = x^{++} \vee [x^{\circ\circ} \wedge (x \vee k) \wedge l] \stackrel{\beta}{\equiv} y^{++} \vee [y^{\circ\circ} \wedge (y \vee k) \wedge l] = y.$$

Similarly, we can show that $\beta \leq \alpha$. Consequently, we have $\alpha = \beta$.

□

The following result is an immediate consequence from Theorem 3.4.

Corollary 3.5. *Let $(L;^\circ,^+)$ be a double K_2-algebra and let $K(L) = [k, l]$. Then $ConL$ is lattice isomorphic to $Con_p(L)$ by the prescription $\alpha \rightarrow \langle \alpha|_{S(L)}, \alpha|_{K(L)} \rangle$.*

We recall from [2] that if $(L;^\circ,^+)$ is double Stone algebra then $S(L)$ is a boolean sublattice of L and $K(L)$ is a Φ-classes of L. In addition, a pair $\langle \theta_1, \theta_2 \rangle \in$

$Con(S(L)) \times Con(K(L))$ is called a *reduced congruence pair* of L if, for any $x, y \in K(L)$, $x \leq y$ and $z \in S(L)$, the following condition hold:

$$(\ddagger) \quad x \wedge z = y \wedge z \text{ and } z \overset{\theta_1}{\equiv} 1 \Rightarrow x \overset{\theta_2}{\equiv} y.$$

Corollary 3.6. *Let* $(L; ^\circ, ^+)$ *be a double Stone algebra with* $K(L) = [k, l]$ *and let* $\langle \theta_1, \theta_2 \rangle \in Con(S(L)) \times Con(K(L))$. *Then* $\langle \theta_1, \theta_2 \rangle \in Con_p(L)$ *if and only if it is a reduced congruence pair.*

Proof. ($\Rightarrow$:) Suppose that $\langle \theta_1, \theta_2 \rangle \in Con_p(L)$. Let $x \wedge z = y \wedge z$ and $z \overset{\theta_1}{\equiv} 1$, where $x, y \in K(L)$, $x \leq y$ and $z \in S(L)$. Then $(z \vee k) \wedge l \overset{\theta_2}{\equiv} l$ by $(CP2)$. Observe that

$$x \overset{\theta_2}{\equiv} x \wedge (z \vee k) \wedge l = (x \wedge z) \vee k = (y \wedge z) \vee k \overset{\theta_2}{\equiv} y \wedge (z \vee k) \wedge l \overset{\theta_2}{\equiv} y.$$

Hence we see that $x \overset{\theta_2}{\equiv} y$. Thus, $\langle \theta_1, \theta_2 \rangle$ is a reduced congruence pair.

($\Leftarrow$:) Suppose that $\langle \theta_1, \theta_2 \rangle$ is a reduced congruence pair. Since, for $x \in K(L)$, $x^\circ = 0$ and $x^+ = 1$, the property $(CP1)$ holds. To see that $\langle \theta_1, \theta_2 \rangle \in Con_p(L)$, let $a \overset{\theta_1}{\equiv} b$ for some $a, b \in S(L)$. Now, let $c = (a \vee b^\circ) \wedge (b \vee a^\circ)$. Then $c \in S(L)$, $a \wedge c = b \wedge c = a \wedge b$ and $\alpha \overset{\theta_1}{\equiv} 1$. Hence, by $(\ddagger)$ we have $a \overset{\theta_2}{\equiv} b$, whence $(a \vee k) \wedge l \overset{\theta_2}{\equiv} (b \vee k) \wedge l$ and we conclude that $(CP2)$ holds. Consequently, $\langle \theta_1, \theta_2 \rangle \in Con_p(L)$. □

4 Applications

If A is an algebra and α_1, α_2 are congruences on A then α_1 and α_2 are said to be *permutable* provided that, for any $a, b, c \in A$ with $(a, b) \in \alpha_1$ and $(b, c) \in \alpha_2$, there exists $d \in A$ such that $(a, d) \in \alpha_2$ and $(d, c) \in \alpha_1$, that is $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$, where $\alpha_1 \circ \alpha_2$ is their relational product. In what follows we shall denote by $\alpha_1 \circ^n \alpha_2$ the compound relational product $\alpha_1 \circ \alpha_2 \circ \alpha_1 ...$, where $n \geq 2$. If $\alpha_1 \circ^n \alpha_2 = \alpha_2 \circ^n \alpha_1$ then, we shall say that α_1 and α_2 are n*-permutable*. An algebra A is said to be *congruence permutable (congruence* n*-permutable)* if every pair of congruences on it is permutable (n-permutable). For the purpose of characterizing the congruence permutability (n-permutability) of a double K_2-algebra, we shall make use of the following useful result.

Theorem 4.1. *Let* $(L;^{\circ},^{+})$ *be a double* K_2*-algebra and* $K(L)=[k,l]$. *If* $\langle\theta_1,\theta_2\rangle,\langle\varphi_1,\varphi_2\rangle\in Con(S(L))\times Con(K(L))$ *then*

$$\langle\theta_1,\theta_2\rangle\circ\langle\varphi_1,\varphi_2\rangle=\theta_1\circ\varphi_1\vee\theta_2\circ\varphi_2.$$

Proof. Suppose that $\langle\theta_1,\theta_2\rangle,\langle\varphi_1,\varphi_2\rangle\in Con(S(L))\times Con(K(L))$. Then by Corollary 3.5 there exist unique $\alpha,\beta\in ConL$ such that $\alpha=\langle\theta_1,\theta_2\rangle$ and $\beta=\langle\varphi_1,\varphi_2\rangle$. Clearly, $\theta_1\circ\varphi_1\vee\theta_2\circ\varphi_2\leq\alpha\circ\beta$.

To see the reverse inequality, we let $x\overset{\alpha\circ\beta}{\equiv}y$. Then there exists $t\in L$ such that $x\overset{\alpha}{\equiv}t$ and $t\overset{\beta}{\equiv}y$. Then $x^{\circ\circ}\overset{\alpha}{\equiv}t^{\circ\circ}$, $x^{++}\overset{\alpha}{\equiv}t^{++}$ and $t^{\circ\circ}\overset{\beta}{\equiv}y^{\circ\circ}$, $t^{++}\overset{\beta}{\equiv}y^{++}$. Hence $x^{\circ\circ}\overset{\theta_1}{\equiv}t^{\circ\circ}$, $x^{++}\overset{\theta_1}{\equiv}t^{++}$ and $t^{\circ\circ}\overset{\varphi_1}{\equiv}y^{\circ\circ}$, $t^{++}\overset{\varphi_1}{\equiv}y^{++}$, and so $x^{\circ\circ}\overset{\theta_1\circ\varphi_1}{\equiv}y^{\circ\circ}$ and $x^{++}\overset{\theta_1\circ\varphi_1}{\equiv}y^{++}$. In addition, since $x\overset{\alpha}{\equiv}t$ and $t\overset{\beta}{\equiv}y$ imply $(x\vee k)\wedge l\overset{\alpha}{\equiv}(t\vee k)\wedge l$ and $(t\vee k)\wedge l\overset{\beta}{\equiv}(y\vee k)\wedge l$. Thus, we have $(x\vee k)\wedge l\overset{\theta_2}{\equiv}(t\vee k)\wedge l$ and $(t\vee k)\wedge l\overset{\varphi_2}{\equiv}(y\vee k)\wedge l$, and whence $(x\vee k)\wedge l\overset{\theta_2\circ\varphi_2}{\equiv}(y\vee k)\wedge l$. Since $x^{\circ\circ}\overset{\theta_1\circ\varphi_1}{\equiv}y^{\circ\circ}$, $x^{++}\overset{\theta_1\circ\varphi_1}{\equiv}y^{++}$ and $(x\vee k)\wedge l\overset{\theta_2\circ\varphi_2}{\equiv}(y\vee k)\wedge l$, there exist $a,b,c\in L$ such that $x^{\circ\circ}\overset{\theta_1}{\equiv}a$, $a\overset{\varphi_1}{\equiv}y^{\circ\circ}$, $x^{++}\overset{\theta_1}{\equiv}b$, $b\overset{\varphi_1}{\equiv}y^{++}$ and $(x\vee k)\wedge l\overset{\theta_2}{\equiv}c$, $c\overset{\varphi_2}{\equiv}(y\vee k)\wedge l$. By (†), it follows that

$$x=x^{++}\vee[x^{\circ\circ}\wedge(x\vee k)\wedge l]\equiv b\vee(a\wedge c)(\theta_1\circ\varphi_1\vee\theta_2\circ\varphi_2);$$

$$y=y^{++}\vee[y^{\circ\circ}\wedge(y\vee k)\wedge l]\equiv b\vee(a\wedge c)(\theta_1\circ\varphi_1\vee\theta_2\circ\varphi_2).$$

Therefore, $x\equiv y(\theta_1\circ\varphi_1\vee\theta_2\circ\varphi_2)$. Thus, we have $\alpha\circ\beta\leq\theta_1\circ\varphi_1\vee\theta_2\circ\varphi_2$.

Consequently, we have from the above observations that $\alpha\circ\beta=\langle\theta_1,\theta_2\rangle\circ\langle\varphi_1,\varphi_2\rangle=\theta_1\circ\varphi_1\vee\theta_2\circ\varphi_2$.

□

Corollary 4.2. *Let* $(L;^{\circ},^{+})$ *be a double* K_2*-algebra and let* $K(L)=[k,l]$. *If* $S(L)$ *and* $K(L)$ *are congruence permutable (congruence* n*-permutable), then so* L.

Since every boolean algebra is congruence permutable and congruence n-permutable, the following result is an immediate consequence from Corollary 4.2, [1, Lemma 3.2] and [2, Theorem 2.5].

Corollary 4.3. *Let* $(L;^{\circ},^{+})$ *be a double Stone algebra with* $K(L)=[k,l]$. *Then* L *is congruence permutable (congruence* n*-permutable) if and only if* $K(L)$ *is congruence permutable (congruence* n*-permutable).*

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References

[1] M. E. Adams and R. Beazer: Congruence properties of distributive double p-algebras, *Czech. Math. J.*, **41(2)**(1991), 216–231.

[2] R. Beazer: Congruence pairs for distributive double p-algebras with non-empty core, *Houston Math. J.*, **6**(1980), 443–454.

[3] R. Beazer: Congruence pairs for algebras abstracting Kleene and Stone algebras, *Czech. Math. J.*, **35(2)**(1985), 260–268.

[4] R. Beazer: Distributive p-algebras and double p-algebras having n-permutable congruences, *Proc. Edin. Math. Soc.*, **35**(1992), 301–307.

[5] T. S. Blyth and J. C. Varlet: *Ockham Algebras*, Oxford University Press, Oxford, 1994.

[6] T. S. Blyth and J. C. Varlet: On a common abstraction of de Morgan and Stone algebras, *Proc. Roy. Soc. Edin. Sect*, **A 94A**(1983), 301–308.

[7] T. S. Blyth and J. C. Varlet: Subvarieties of the class of MS-algebras, *Proc. Roy. Soc. Edin. Sect*, **A 95A**(1983), 157–169.

[8] T. S. Blyth and J. C. Varlet: Double MS-algebras, *Proc. Roy. Soc. Edin. Sect*, **A 98A**(198)4, 37–47.

[9] H. Lakser: The structure of pseudocomplemented distributive lattice. I. Subdirect decomposition, *Trans. Amer. Math. Soc.*, **156**(1971), 335–342.

[10] T. Katriňák: Über eine Konstruktion der distributiven pseudokomplementaren Verbände, *Math. Nachr.*, **53**(1972), 85–99.

[11] T. Katriňák: Congruence pairs on p-algebras with a modular frame, *ALgebra Univ.*, **8**(1978), 205–220.

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