

The Normed Ordered Cone of Operator Connections

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Abstract: A connection in Kubo-Ando sense is a binary operation for positive operators on a Hilbert space satisfying the monotonicity, the transformer inequality and continuity from above. A mean is a connection σ such that $A\sigma A = A$ for all positive operators A . In this paper, we consider the interplay between the cone of connections, the cone of operator monotone functions on the nonnegative reals \mathbb{R}^+ and the cone of finite Borel measures on $[0, \infty]$. The set of operator connections is shown to be isometrically order-isomorphic, as normed ordered cones, to the set of operator monotone functions on \mathbb{R}^+ . This set is isometrically isomorphic, as normed cones, to the set of finite Borel measures on $[0, \infty]$. It follows that the convergences of the sequence of connections, the sequence of their representing functions and the sequence of their representing measures are equivalent. In addition, we obtain characterizations for a connection to be a mean. In fact, a connection is a mean if and only if it has norm 1.

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1 Introduction

A general theory of operator means was given by Kubo and Ando [9]. Denote by $B(\mathcal{H})$ the von Neumann algebra of bounded linear operators on a complex Hilbert

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space \mathcal{H} and $B(\mathcal{H})^+$ its positive cone. A *connection* is a binary operation σ assigned to each pair of positive operators such that for all $A, B, C, D \geq 0$:

$$(M1) \text{ monotonicity: } A \leq C, B \leq D \implies A \sigma B \leq C \sigma D$$

$$(M2) \text{ transformer inequality: } C(A \sigma B)C \leq (CAC) \sigma (CBC)$$

(M3) *continuity from above*: for $A_n, B_n \in B(\mathcal{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A \sigma B$. Here, $A_n \downarrow A$ indicates that A_n is a decreasing sequence converging strongly to A .

This definition is modeled from the notion of the parallel sum introduced in [1] for analyzing multiport electrical networks. A *mean* is a connection σ such that $A \sigma A = A$ for all $A \geq 0$ or, equivalently, $I \sigma I = I$. Here are examples of means in practical usage:

- arithmetic mean: $A \nabla B = (A + B)/2$
- geometric mean [2, 3]: $A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$
- harmonic mean: $A ! B = 2(A^{-1} + B^{-1})^{-1}$
- logarithmic mean: $(A, B) \mapsto A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$ where f is given by $f(x) = (x - 1)/\log x$ and $f(1) := 1$ is defined by the continuity.

For connections σ and η on $B(\mathcal{H})^+$, we define

$$\begin{aligned} \sigma + \eta &: B(\mathcal{H})^+ \times B(\mathcal{H})^+ \rightarrow B(\mathcal{H})^+, (A, B) \mapsto (A \sigma B) + (A \eta B), \\ k\sigma &: B(\mathcal{H})^+ \times B(\mathcal{H})^+ \rightarrow B(\mathcal{H})^+, (A, B) \mapsto k(A \sigma B), \quad k \in \mathbb{R}^+. \end{aligned}$$

Denote by $C(B(\mathcal{H})^+)$ the set of connections on $B(\mathcal{H})^+$. Define a partial order \leq for connections on $B(\mathcal{H})^+$ by $\sigma_1 \leq \sigma_2$ if $A \sigma_1 B \leq A \sigma_2 B$ for all $A, B \in B(\mathcal{H})^+$. It is straightforward to show that the set $C(B(\mathcal{H})^+)$ is an ordered cone in which the neutral element is the zero connection $0 : (A, B) \mapsto 0$. This cone is pointed (i.e. $\sigma \geq 0$ for all σ) and order cancellative (i.e. $\sigma_1 + \eta \leq \sigma_2 + \eta$ implies $\sigma_1 \leq \sigma_2$).

A major tool in Kubo-Ando theory of connections and means is the class of operator monotone functions. This concept was introduced in [10]; see more information in [5, 6, 8]. Recall that a continuous real-valued function f on an interval I is called an *operator monotone function* if for all Hilbert spaces \mathcal{H} and

for all Hermitian operators $A, B \in B(\mathcal{H})$ whose spectra are contained in I , we have

$$A \leq B \implies f(A) \leq f(B).$$

Denote by $OM(\mathbb{R}^+)$ the set of operator monotone functions from $\mathbb{R}^+ = [0, \infty)$ to itself. This set is a cone under usual addition and scalar multiplication in which the zero function $0 : x \mapsto 0$ is the neutral element. The partial order on $OM(\mathbb{R}^+)$ is defined pointwise. This cone becomes an ordered cone which is pointed and order cancellative.

A major result in Kubo-Ando theory is a one-to-one correspondence between connections on $B(\mathcal{H})^+$ and operator monotone functions on \mathbb{R}^+ as follows:

Theorem 1.1 ([9]). *Given a connection σ , there is a unique operator monotone function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying*

$$f(x)I = I\sigma(xI), \quad x \in \mathbb{R}^+.$$

Moreover, the map $\sigma \mapsto f$ is an affine order isomorphism. In addition, σ is a mean if and only if $f(1) = 1$.

We call f the *representing function* of σ . There is also a one-to-one correspondence between connections and finite Borel measures on $[0, \infty]$ given by the following integral representation:

Theorem 1.2 ([9]). *Given a connection σ , there is a unique finite Borel (Radon) measure μ on $[0, \infty]$ such that*

$$A\sigma B = \int_{[0, \infty]} \frac{\lambda + 1}{2\lambda} (\lambda A \sharp B) d\mu(\lambda), \quad A, B \geq 0. \quad (1)$$

Moreover, the map $\sigma \mapsto \mu$ is an affine isomorphism. In addition, σ is a mean if and only if $\mu([0, \infty]) = 1$.

Note that any finite Borel measure on $[0, \infty]$ is always a Radon measure. The measure μ in this theorem is called the *representing measure* of σ . Here, the cone of finite Borel measures on $[0, \infty]$, denoted by $BM([0, \infty])$, is equipped with the usual algebraic operations and pointwise order, i.e.

$$\mu_1 \leq \mu_2 \quad \text{if and only if} \quad \mu_1(E) \leq \mu_2(E) \quad \text{for all Borel sets } E \subseteq [0, \infty].$$

Then $BM([0, \infty])$ is an ordered cone where the zero measure is the neutral element.

In this paper, we investigate structures of the cone of connections in relation with the cone of operator monotone functions on \mathbb{R}^+ and the cone of finite Borel measures on $[0, \infty]$. We define a norm for a connection such that the set of operator connections becomes a normed ordered cone. The cone of operator monotone functions on \mathbb{R}^+ and the cone of finite Borel measures on $[0, \infty]$ are also equipped with suitable norms. The set $C(B(\mathcal{H})^+)$ is shown to be isometrically order-isomorphic, as normed ordered cones, to the set $OM(\mathbb{R}^+)$ via the map sending a connection to its representing function. This set is isometrically isomorphic, as normed cones, to the set $BM([0, \infty])$ via the map sending a connection to its representing measure. Hence the convergences of the sequence of connections, the sequence of their representing functions and the sequence of their representing measures are equivalent. In addition, we obtain characterizations for a connection to be a mean. In fact, a connection is a mean if and only if it has norm 1.

2 Norms for connections, operator monotone functions and Borel measures

In this section, we consider topological structures of the cone of connections, the cone of operator monotone functions on \mathbb{R}^+ and the cone of finite Borel measures on $[0, \infty]$. In fact, there are norms equipped naturally on these cones.

Recall that a *normed cone* is a cone C equipped with a function $\|\cdot\| : C \rightarrow \mathbb{R}^+$ such that for each $x, y \in C$ and $k \in \mathbb{R}^+$,

$$(i) \quad \|x\| = 0 \implies x = 0,$$

$$(ii) \quad \|kx\| = k\|x\|,$$

$$(iii) \quad \|x + y\| \leq \|x\| + \|y\|.$$

A *normed ordered cone* is an ordered cone (C, \leq) which is also a normed cone such that for each $x, y \in C$, $x \leq y \implies \|x\| \leq \|y\|$.

We define a function $\|\cdot\| : C(B(\mathcal{H})^+) \rightarrow \mathbb{R}^+$ by

$$\|\sigma\| = \sup \{ \|A \sigma B\| : A, B \geq 0, \|A\| = \|B\| = 1 \}$$

for each connection σ .

Recall that each connection σ on $B(\mathcal{H})^+$ induces a unique connection $\tilde{\sigma}$ on \mathbb{R}^+ such that

$$(xI)\sigma(yI) = (x\tilde{\sigma}y)I, \quad x, y \in \mathbb{R}^+.$$

A connection and its induced connection may be written by the same notation.

Lemma 2.1. ([4]) *For each connection σ , we have $\|A\sigma B\| \leq \|A\|\sigma\|B\|$ for all $A, B \geq 0$.*

Proposition 2.2. *For each connection σ , we have*

$$\begin{aligned} \|\sigma\| &= \sup \left\{ \frac{\|A\sigma A\|}{\|A\|} : A > 0 \right\} \\ &= \sup \{ \|A\sigma A\| : A \geq 0, \|A\| = 1 \} \\ &= \frac{\|A\sigma A\|}{\|A\|} \quad \text{for any } A > 0 \\ &= \|I\sigma I\|. \end{aligned}$$

Proof. Clearly, $\|\sigma\| \geq \|I\sigma I\|$. For each $A, B \geq 0$ with $\|A\| = \|B\| = 1$, it follows from Lemma 2.1 that

$$\|A\sigma B\| \leq \|A\|\sigma\|B\| = 1\sigma 1 = \|(1\sigma 1)I\| = \|I\sigma I\|.$$

Hence, $\|\sigma\| \leq \|I\sigma I\|$. For each $A > 0$, we have $\|\frac{1}{\|A\|}A\| = 1$ and hence

$$\|\sigma\| \geq \left\| \frac{1}{\|A\|}A\sigma \frac{1}{\|A\|}A \right\| = \left\| \frac{1}{\|A\|}(A\sigma A) \right\| = \frac{\|A\sigma A\|}{\|A\|}$$

On the other hand, for $A > 0$ we have

$$\begin{aligned} \|I\sigma I\| &= \|A^{-1/2}(A\sigma A)A^{-1/2}\| \\ &\leq \|A^{-1/2}\| \|A\sigma A\| \|A^{-1/2}\| \\ &= \|A\|^{-1/2} \|A\sigma A\| \|A\|^{-1/2}. \end{aligned}$$

Thus, $\|\sigma\| = \|A\sigma A\|/\|A\|$ for any $A > 0$. □

Proposition 2.3. *The pair $(C(B(\mathcal{H})^+), \|\cdot\|)$ is a normed ordered cone.*

Proof. For each $\sigma, \eta \in C(B(\mathcal{H})^+)$ and $k \in \mathbb{R}^+$, by Proposition 2.2 we have

$$\begin{aligned}\|k\sigma\| &= \|I(k\sigma)I\| = \|k(I\sigma I)\| = k\|I\sigma I\| = k\|\sigma\|, \\ \|\sigma + \eta\| &= \|I(\sigma + \eta)I\| = \|(I\sigma I) + (I\eta I)\| \leq \|I\sigma I\| + \|I\eta I\| = \|\sigma\| + \|\eta\|.\end{aligned}$$

Suppose now that $\|\sigma\| = 0$, i.e. $I\sigma I = 0$. For each $x \in [0, 1]$, since $xI \leq I$, we have $(xI)\sigma I \leq I\sigma I = 0$, i.e. $(xI)\sigma I = 0$. Then for each $x > 1$,

$$I\sigma(xI) = x \left(\frac{1}{x} I\sigma I \right) = 0.$$

Consider $A \in B(\mathcal{H})^+$ in the form $A = \sum_{i=1}^m \lambda_i P_i$ where $\lambda_i > 0$ and P_i 's are projections such that $P_i P_j = 0$ for $i \neq j$ and $\sum_{i=1}^m P_i = I$. We have

$$I\sigma A = \sum (I\sigma A)P_i = \sum P_i \sigma A P_i = \sum P_i \sigma \lambda_i P_i = \sum P_i (I\sigma \lambda_i I) = 0.$$

For general $A \in B(\mathcal{H})^+$, let $\{A_n\}$ be a sequence of invertible positive operators such that $A_n \downarrow A$. Then $I\sigma A = \lim_{n \rightarrow \infty} I\sigma A_n = 0$ for all $A \geq 0$. Hence, for $A, B \in B(\mathcal{H})^+$,

$$A\sigma B = \lim_{\epsilon \downarrow 0} A_\epsilon \sigma B = \lim_{\epsilon \downarrow 0} A_\epsilon^{1/2} (I\sigma A_\epsilon^{-1/2} B A_\epsilon^{-1/2}) A_\epsilon^{1/2} = 0,$$

here $A_\epsilon \equiv A + \epsilon I$. Thus $\sigma = 0$.

If $\sigma \leq \eta$, then $\|\sigma\| = \|I\sigma I\| \leq \|I\eta I\| = \|\eta\|$ since $I\sigma I \leq I\eta I$. \square

Recall that a function f from a cone C into a cone D is called *linear* or *affine* if $f(rx + sy) = rf(x) + sf(y)$ for each $x, y \in C$ and $r, s \in \mathbb{R}^+$.

Define a function $\|\cdot\| : OM(\mathbb{R}^+) \rightarrow \mathbb{R}^+$ by $\|f\| = f(1)$ for each $f \in OM(\mathbb{R}^+)$.

Proposition 2.4. *The pair $(OM(\mathbb{R}^+), \|\cdot\|)$ is a normed ordered cone. Moreover, the function $\|\cdot\|$ is linear.*

Proof. The only non-trivial part is to show that $\|f\| = 0$ implies $f = 0$. Consider $f \in OM(\mathbb{R}^+)$ such that $f(1) = 0$. Suppose that there is an $a > 0$ such that $f(a) = 0$. Then $f(x) = 0$ for $0 \leq x \leq a$. Since $f \in OM(\mathbb{R}^+)$, f is a concave function by [7]. The concavity of f implies that $f = 0$. \square

Assign to each measure $\mu \in BM([0, \infty])$ its total variation:

$$\|\mu\| = \mu([0, \infty]) < \infty.$$

Proposition 2.5. *The pair $(BM([0, \infty]), \|\cdot\|)$ is a normed ordered cone. Moreover, the function $\|\cdot\|$ is linear.*

Given any normed cone, we can equip it with a topology as follows.

Proposition 2.6. *Let $(C, \|\cdot\|)$ be a normed cone. We have the followings:*

- (1) *the function $d : C \times C \rightarrow \mathbb{R}^+$, $d(x, y) = \|\|x\| - \|y\|\|$ is a pseudo metric; in particular, C is a 1st-countable topological space with respect to the topology induced by d .*
- (2) *the functions $\|\cdot\|$ and d are continuous, where the topology on $C \times C$ is given by the product topology.*
- (3) *if $\|\cdot\|$ is linear, then C becomes a topological cone in the sense that the addition and the scalar multiplication are continuous.*

Proof. The nonnegative function d satisfies $d(x, y) = d(y, x)$ for all $x, y \in C$ and the triangle inequality. This shows that the pair (C, d) is a pseudo metric space and hence a 1st-countable space. Recall that in any 1st-countable topology, a function is continuous if and only if it is sequentially continuous. Let (x_n) be a sequence in C such that $x_n \rightarrow x \in C$. Then $\|\|x_n\| - \|x\|\| = d(x_n, x) \rightarrow 0$, i.e. $\|x_n\| \rightarrow \|x\|$. Let (x_n) and (y_n) be sequences in C such that $x_n \rightarrow x \in C$ and $y_n \rightarrow y \in C$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, y_n) - d(x, y) &\leq d(x_n, x) + d(x, y) + d(y, y_n) - d(x, y) \\ &= d(x_n, x) + d(y_n, y) \end{aligned}$$

and, similarly, $d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y)$. Hence

$$0 \leq |d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \quad \text{for all } n.$$

This implies $d(x_n, y_n) \rightarrow d(x, y)$.

It is easy to see that $kx_n \rightarrow kx$ for each $k \in \mathbb{R}^+$. The linearity of the norm yields the continuity of the addition as follows. We have

$$\|x_n + y_n\| - \|x + y\| = \|x_n\| + \|y_n\| - \|x\| - \|y\| \leq d(x_n, x) + d(y_n, y).$$

and $\|x + y\| - \|x_n + y_n\| \leq d(x_n, x) + d(y_n, y)$. Thus $x_n + y_n \rightarrow x + y$. \square

Hence the cones $OM(\mathbb{R}^+)$ and $BM([0, \infty])$ are topological cones since the norm for operator monotone functions and the norm for finite Borel measures are linear.

3 The isomorphism theorem

In this section, we establish isomorphisms between the cone of connections, the cones of operator monotone functions on \mathbb{R}^+ and the cone of finite Borel measures on $[0, \infty]$.

Recall the following terminology. A function $\varphi : C \rightarrow D$ between normed cones is called an *isomorphism* if it is a continuous linear bijection whose inverse is continuous. By an *isometry*, we mean a linear function $\phi : C \rightarrow D$ such that $\|\phi(c)\| = \|c\|$ for all $c \in C$. Note that every isometry between normed cones is continuous and injective. The inverse of an isometry is an isometry. If $\varphi : C \rightarrow D$ is an isomorphism which is also an isometry, we say that φ is an *isometric isomorphism* and C is said to be *isometrically isomorphic* to D .

Let C and D be normed ordered cones. A function $\varphi : C \rightarrow D$ is called an *order isomorphism* if it is an isomorphism (between normed cones) such that φ and φ^{-1} are order-preserving. If, in addition, φ is an isometry, we say that φ is an *isometric order-isomorphism* and C is said to be *isometrically order-isomorphic* to D .

Theorem 3.1. (1) *The normed ordered cones $C(B(\mathcal{H})^+)$ and $OM(\mathbb{R}^+)$ are isometrically order-isomorphic via the isometric order-isomorphism $\sigma \mapsto f_\sigma$, where f_σ is the representing function of σ .*

(2) *The normed cones $C(B(\mathcal{H})^+)$ and $BM([0, \infty])$ are isometrically isomorphic via the isometric isomorphism $\sigma \mapsto \mu_\sigma$, where μ_σ is the representing measure of σ .*

Proof. The function $\Phi : \sigma \mapsto f_\sigma$ is an order isomorphism by Theorem 1.1. For each connection σ , since $f_\sigma(1)I = I\sigma I$, we have

$$\|\Phi(\sigma)\| = \|f_\sigma\| = f_\sigma(1) = \|I\sigma I\| = \|\sigma\|.$$

The function $\Psi : \sigma \mapsto \mu_\sigma$ is an isomorphism by Theorem 1.2. For each connection σ , we have

$$\|\Psi(\sigma)\| = \|\mu_\sigma\| = \mu([0, \infty]) = \|I\sigma I\| = \|\sigma\|$$

since $I\sigma I = \int_{[0, \infty]} \frac{\lambda+1}{2\lambda} (\lambda I! I) d\mu(\lambda) = \mu([0, \infty])I$. □

Remark 3.2. Even though the map $\mu \mapsto \sigma$, sending finite Borel measures to their associated connections, is order-preserving, the inverse map $\sigma \mapsto \mu$ is not order-preserving in general. For example, the representing measures of the harmonic mean $!$ and the arithmetic mean ∇ are given by δ_1 and $(\delta_0 + \delta_\infty)/2$, respectively. Here, δ_x is the Dirac measure at x . We have $! \leq \nabla$ but it is not true that $\delta_1 \leq (\delta_0 + \delta_\infty)/2$.

Corollary 3.3. *The function $\|\cdot\|$ on $C(B(\mathcal{H})^+)$ is linear.*

Proof. It follows from the fact that the map $\sigma \mapsto f_\sigma$ is an isometric isomorphism and the norm on $OM(\mathbb{R}^+)$ is linear. \square

Hence the cone $C(B(\mathcal{H})^+)$ is a topological cone. We also obtain the following characterizations of a mean as follows.

Corollary 3.4. *The followings are equivalent for a connection σ :*

- (i) σ is a mean;
- (ii) $\|\sigma\| = 1$;
- (iii) $\|A \sigma A\| = \|A\|$ for all $A \geq 0$;
- (iv) $\|A \sigma A\| = \|A\|$ for some $A > 0$.

Proof. It follows from Proposition 2.2, Theorem 3.1 and the fact that a connection is a mean if and only if its representing function (measure) is normalized. \square

From the equivalence (i)-(ii) in this corollary, a mean is a normalized connection. Every mean arises as a normalization of a nonzero connection. The convex set of means is the unit sphere in the cone of connections.

Corollary 3.5. *The limit of a sequence of means is a mean.*

Proof. Use the fact that the norm for connections is continuous by Proposition 2.6 and the norm of a mean is 1 by Corollary 3.4. \square

The topologies of the cones $C(B(\mathcal{H})^+)$, $OM(\mathbb{R}^+)$ and $BM([0, \infty])$ are compatible with the isometric isomorphisms $\sigma \mapsto f_\sigma$ and $\sigma \mapsto \mu_\sigma$ in Theorem 3.1 as follows.

Corollary 3.6. For each $n \in \mathbb{N}$, let σ_n be a connection with representing function f_n and representing measure μ_n . Then the followings are equivalent for a connection σ with representing function f and representing measure μ :

- (i) $\sigma_n \rightarrow \sigma$;
- (ii) $f_n \rightarrow f$;
- (iii) $\mu_n \rightarrow \mu$.

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