

# The Gray Images of Skew-Constacyclic Codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$

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**Abstract:** We study the Gray images of three types of skew constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$ , where  $u^e = 0$ . For a given automorphism  $\Theta$  of  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$  induced by an automorphism  $\theta$  of  $\mathbb{F}_{p^m}$ , the Gray images of  $\Theta$ - $(1-u^{e-1})$ -constacyclic codes are shown to be  $\theta$ -permutation invariant codes whose algebraic structures are generalization of quasi-cyclic codes over finite fields. In addition, if the length of codes is not divisible by  $p$ , the Gray images of  $\Theta$ -cyclic and  $\Theta$ - $(1+u^{e-1})$ -constacyclic codes are permutatively equivalent to  $\theta$ -permutation invariant codes. Moreover, our works generalize known results concerning the Gray images of classical cyclic,  $(1-u^{e-1})$ -constacyclic and  $(1+u^{e-1})$ -constacyclic codes over this ring.

**Keywords:** finite chain ring, Gray map, permutation invariant code, skew constacyclic code

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## 1 Introduction

The discovery of good codes from codes over  $\mathbb{Z}_4$ , via the Gray map [10] motivated the study of the Gray images of codes over finite rings in general. Analogs of the Gray map have also been defined for codes over other finite chain rings [9], linking these codes to codes over finite fields. Cyclic codes and constacyclic codes form important classes of codes due to their rich algebraic structure. Qian, Zhang and Zhu have characterized the Gray images of  $(1 + u)$ -constacyclic and cyclic codes over the ring  $\mathbb{F}_2 + u\mathbb{F}_2$  in [13] and investigated some constacyclic codes over  $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$  in [14]. In [7], Congellenmis have introduced  $(1 - u^{e-1})$ -constacyclic codes over  $\mathbb{F}_2 + u\mathbb{F}_2 + \cdots + u^{e-1}\mathbb{F}_2$  and generalized the results of [13] and [14]. In [16], Udomkavanich and Jitman generalized these results to the ring  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$ . The Gray images of  $(1 - u^{e-1})$ -constacyclic and cyclic codes were studied in the mentioned paper.

Recently, skew (consta) cyclic codes, generalizations of cyclic codes that their (consta) cyclicity is defined with respect to an automorphism of an alphabet ring, have been established in [4], [6], [5] and [11]. Especially, in [11], an algebraic structure of skew constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$  has been studied.

Motivated by these works, we study the Gray images of three families of skew constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$ , namely,  $\Theta$ - $(1 - u^{e-1})$ -constacyclic,  $\Theta$ -cyclic and  $\Theta$ - $(1 + u^{e-1})$ -constacyclic codes. In the case where  $\Theta$  is the identity automorphism, the results on  $(1 - u^{e-1})$ -constacyclic, cyclic and  $(1 + u^{e-1})$ -constacyclic codes in [16] can be viewed as a special case of our works.

The paper is organized as follows. Some basic notion and concepts concerning rings, codes and Gray maps are given in Section 2. In Section 3, the structures of the Gray images of  $\Theta$ - $(1 - u^{e-1})$ -constacyclic codes are determined. In the case where the length of codes is not divisible by  $p$ , the Gray images of  $\Theta$ -cyclic and  $\Theta$ - $(1 + u^{e-1})$ -constacyclic codes are studied in Section 4. The conclusion and remarks are given in Section 5.

## 2 Preliminaries

### 2.1 Rings

Let  $\mathbb{F}_{p^m}$  denote the finite field of cardinality  $p^m$ , where  $p$  is a prime number and  $m$  is a positive integer. For  $e \geq 1$ , the set

$$\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m} = \left\{ \sum_{i=0}^{e-1} u^i a_i \mid a_i \in \mathbb{F}_{p^m} \right\} \quad (1)$$

forms a ring, where the addition and multiplication are the usual addition and multiplication of polynomials over  $\mathbb{F}_{p^m}$  with indeterminate  $u$  together with the rule  $u^e = 0$ . Since this ring is uniquely determined by  $p, m$  and  $e$ , for simplicity, we denoted by  $\mathcal{R}_{(p^m, e)}$  the ring  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$ .

In other words, an element  $r \in \mathcal{R}_{(p^m, e)}$  can be uniquely expressed as

$$r = a_0 + ua_1 + \cdots + u^{e-1}a_{e-1}, \quad (2)$$

where  $a_i \in \mathbb{F}_{p^m}$ , for all  $0 \leq i \leq e-1$ .

The ideals of  $\mathcal{R}_{(p^m, e)}$  are principal and form the following chain

$$\mathcal{R}_{(p^m, e)} \supseteq \langle u \rangle \supseteq \langle u^2 \rangle \cdots \supseteq \langle u^{e-1} \rangle \supseteq \langle u^e \rangle = \langle 0 \rangle,$$

where  $\langle u^i \rangle = u^i \mathcal{R}_{(p^m, e)} = u^i \mathbb{F}_{p^m} + u^{i+1} \mathbb{F}_{p^m} + \cdots + u^{e-1} \mathbb{F}_{p^m}$  for all  $0 \leq i \leq e-1$ .

According to [8, Lemma 1], the ring  $\mathcal{R}_{(p^m, e)}$  may refer as the only finite chain ring of characteristic  $p$ , nilpotency index  $e$ , and residue field  $\mathbb{F}_{p^m}$ .

Note that when  $e = 1$ , this ring becomes the finite field  $\mathbb{F}_{p^m}$ .

In [2], the structure of the automorphism group  $\text{Aut}(\mathcal{R}_{(p^m, e)})$  of  $\mathcal{R}_{(p^m, e)}$  has been characterized. For  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ ,  $\beta \in \mathbb{F}_{p^m}$  and  $w \in \mathcal{R}_{(p^m, e)}$ , let

$$\Theta_{\theta, \beta, w} : \mathcal{R}_{(p^m, e)} \rightarrow \mathcal{R}_{(p^m, e)}$$

be defined by

$$\Theta_{\theta, \beta, w} \left( \sum_{i=0}^{e-1} a_i u^i \right) = \sum_{i=0}^{e-1} u^i \beta^i w^i \theta(a_i).$$

**Proposition 2.1** ([2, Proposition 1]).  $\text{Aut}(\mathcal{R}_{(p^m, e)}) = \{ \Theta_{\theta, \beta, w} \mid \theta \in \text{Aut}(\mathbb{F}_{p^m}), \beta \in \mathbb{F}_{p^m}^* \text{ and } w \in 1 + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m} \}$ , where  $\mathbb{F}_{p^m}^*$  denotes the group of units in  $\mathbb{F}_{p^m}$ .

**Corollary 2.2** ([11]).  $\text{Aut}(\mathcal{R}_{(p^m, e)})$  is non-trivial if and only if  $m \geq 2$  or  $p$  is odd or  $e \geq 3$ .

## 2.2 Codes

A code  $C$  of length  $n$  over the finite field (resp., a finite ring)  $\mathcal{A}$  is a subspace (resp., submodule) of the  $\mathcal{A}$ -vector space (resp., module)  $\mathcal{A}^n$ . The *Hamming distance*  $d_{Ham}(\mathbf{u}, \mathbf{v})$  between  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathcal{A}^n$  is defined to be the number of entries which  $\mathbf{u}$  and  $\mathbf{v}$  differ. The *minimum Hamming distance* of a code  $C$ , denoted by  $d_{Ham}(C)$ , is defined by

$$d_{Ham}(C) = \min\{d_{Ham}(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in C, \mathbf{u} \neq \mathbf{v}\}.$$

Given an automorphism  $\Theta$  of  $\mathcal{R}_{(p^m, e)}$  and a unit  $\lambda$  in  $\mathcal{R}_{(p^m, e)}$ , a map

$$\rho_{\Theta, \lambda} : \mathcal{R}_{(p^m, e)}^n \rightarrow \mathcal{R}_{(p^m, e)}^n$$

defined by

$$\rho_{\Theta, \lambda}((a_0, a_1, \dots, a_{n-1})) = (\Theta(\lambda a_{n-1}), \Theta(a_0), \dots, \Theta(a_{n-2})) \quad (3)$$

is called the  $\Theta$ - $\lambda$ -constacyclic shift on  $\mathcal{R}_{(p^m, e)}^n$ . A code  $C$  of length  $n$  over  $\mathcal{R}_{(p^m, e)}$  is said to be *skew-constacyclic*, or specifically,  $\Theta$ - $\lambda$ -constacyclic if it is closed under the  $\Theta$ - $\lambda$ -constacyclic shift.

When  $\Theta$  is the identity automorphism, the subscript  $\Theta$  will be dropped and denoted by  $\rho_\lambda$  the map  $\rho_{\Theta, \lambda}$ . A code closed under  $\rho_\lambda$  is called a *constacyclic code*, or specifically,  $\lambda$ -constacyclic code. Once,  $\lambda = 1$ , the word “ $\lambda$ -constacyclic” will be replaced by “cyclic”.

A characterization of skew constacyclic codes over  $\mathcal{R}_{(p^m, e)}$  is given in [11] in terms of skew polynomials for every automorphism in  $\mathcal{R}_{(p^m, e)}$  and unit  $\lambda$  in  $\mathcal{R}_{(p^m, e)}$ .

In this work, we focus on the gray images of skew constacyclic codes over  $\mathcal{R}_{(p^m, e)}$  in the case where  $\lambda \in \{1, 1 - u^{e-1}, 1 + u^{e-1}\}$ . In addition, an automorphism  $\Theta$  of  $\mathcal{R}_{(p^m, e)}$  is defined by

$$\Theta(a_0 + ua_1 + \dots + u^{e-1}a_{e-1}) = \theta(a_0) + u\theta(a_1) + \dots + u^{e-1}\theta(a_{e-1}), \quad (4)$$

where  $\theta$  is an automorphism of  $\mathbb{F}_{p^m}$ , i.e.,  $\Theta$  is  $\Theta_{\theta, 1, 1}$  in Proposition 2.1. The automorphisms in this families are called *automorphisms induced by automorphism of  $\mathbb{F}_{p^m}$*  and they form a subgroup of  $\text{Aut}(\mathcal{R}_{(p^m, e)})$ . It is clear that  $\text{ord}(\Theta) = \text{ord}(\theta)$  and  $\Theta$  fixes  $1$ ,  $1 - u^{e-1}$  and  $1 + u^{e-1}$ .

Let  $\sigma : \mathbb{F}_{p^m}^{p^{m(e-1)-1}} \rightarrow \mathbb{F}_{p^m}^{p^{m(e-1)-1}}$  be defined by

$$(a^{(0)} \mid a^{(1)} \mid \dots \mid a^{(p^{m(e-1)-1}-1)}) \mapsto (\varphi(a^{(0)}) \mid \varphi(a^{(1)}) \mid \dots \mid \varphi(a^{(p^{m(e-1)-1}-1)})),$$

where  $a^{(i)} \in \mathbb{F}_{p^m}^{pn}$ ,  $|$  a vector concatenation and  $\varphi : \mathbb{F}_{p^m}^{pn} \rightarrow \mathbb{F}_{p^m}^{pn}$  denotes the cyclic shift on  $\mathbb{F}_{p^m}^{pn}$ :

$$\varphi((c_0, c_1, \dots, c_{pn-1})) = (c_{pn-1}, c_0, \dots, c_{pn-2}).$$

A code  $\tilde{C}$  of length  $p^{m(e-1)}n$  over  $\mathbb{F}_{p^m}$  satisfying  $\sigma^{\otimes p^{m(e-1)-1}}(\tilde{C}) = \tilde{C}$  is called a *quasi-cyclic code of index  $p^{m(e-1)-1}$* . In general,  $\tilde{C}$  is called a  *$\theta$ -permutation invariant code* (induced by a permutation  $T : \mathbb{F}_{p^m}^{p^{m(e-1)}n} \rightarrow \mathbb{F}_{p^m}^{p^{m(e-1)}n}$ ), a generalization of permutation invariant code [15], if it is invariant under a composition of the permutation  $T$  and the map defined by taking  $\theta$  coordinatewise.

Codes  $\tilde{C}_1$  and  $\tilde{C}_2$  are said to be *permutatively equivalent* if one can be obtained from the other by permuting the coordinates.

### 2.3 Homogenous Weights and Gray Maps

A homogeneous distance has been introduced for arbitrary finite chain ring in [9]. In light of this, the homogeneous distance on  $\mathcal{R}_{(p^m, e)}^n$  is defined in [16] in terms of the weight function  $w_{hom}(\mathbf{r})$  as follows:

$$w_{hom}(\mathbf{r}) = \sum_{i=0}^{n-1} w_{hom}(r_i)$$

for all  $\mathbf{r} = (r_0, r_1, \dots, r_{n-1}) \in \mathcal{R}_{(p^m, e)}^n$ , where

$$w_{hom}(r_i) = \begin{cases} p^{m(e-2)}(p^m - 1) & \text{if } r_i \in \mathcal{R}_{(p^m, e)} \setminus u^{e-1}\mathcal{R}_{(p^m, e)}, \\ p^{m(e-1)} & \text{if } r_i \in u^{e-1}\mathcal{R}_{(p^m, e)} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

The *homogeneous distance*  $d_{hom}(\mathbf{r}, \mathbf{s})$  between vectors  $\mathbf{r}, \mathbf{s}$  in  $\mathcal{R}_{(p^m, e)}^n$  is defined to be  $w_{hom}(\mathbf{r} - \mathbf{s})$ . The minimum homogeneous distance  $d_{hom}(C)$  of a code  $C$  over  $\mathcal{R}_{(p^m, e)}$  is defined by

$$d_{hom}(C) := \min\{d_{hom}(\mathbf{r}, \mathbf{s}) \mid \mathbf{r}, \mathbf{s} \in C, \mathbf{r} \neq \mathbf{s}\}.$$

In order to recall the Gray map for  $\mathcal{R}_{(p^m, e)}$  [16], an element  $\epsilon \in \mathbb{Z}_{p^m}$  is viewed uniquely as the  $p$ -adic representation

$$\epsilon = \xi_0(\epsilon) + \xi_1(\epsilon)p + \cdots + \xi_{m-1}(\epsilon)p^{m-1},$$

where  $\xi_i(\epsilon) \in \{0, 1, \dots, p-1\}$ , for every  $0 \leq i \leq m-1$ . Let  $\alpha$  be a primitive element of  $\mathbb{F}_{p^m}$ . For each  $\epsilon \in \mathbb{Z}_{p^m}$ , the element  $\alpha_\epsilon \in \mathbb{F}_{p^m}$  corresponding to  $\epsilon$  is given by

$$\alpha_\epsilon := \xi_0(\epsilon) + \xi_1(\epsilon)\alpha + \dots + \xi_{m-1}(\epsilon)\alpha^{m-1}.$$

Similarly, an element  $\omega \in \mathbb{Z}_{p^{m(e-1)}}$  is viewed uniquely as the  $p^m$ -adic representation

$$\omega = \bar{\xi}_0(\omega) + \bar{\xi}_1(\omega)p^m + \dots + \bar{\xi}_{e-2}(\omega)p^{m(e-2)},$$

where  $\bar{\xi}_i(\omega) \in \{0, 1, \dots, p^m-1\}$ , for every  $0 \leq i \leq e-2$ .

The Gray map linked codes over  $\mathcal{R}_{(p^m, e)}$  and codes over  $\mathbb{F}_{p^m}$ . In [16], the Gray map  $\Phi: \mathcal{R}_{(p^m, e)}^n \rightarrow \mathbb{F}_{p^m}^{m(e-1)n}$  is defined by

$$\Phi(\mathbf{r}) = (\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{p^{m(e-1)}-1}),$$

for all  $\mathbf{r} = a_0(\mathbf{r}) + ua_1(\mathbf{r}) + \dots + u^{e-1}a_{e-1}(\mathbf{r}) \in \mathcal{R}_{(p^m, e)}^n$ , where

$$\mathbf{b}_{\omega p^m + \epsilon} = \alpha_\epsilon a_0(\mathbf{r}) + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} a_l(\mathbf{r}) + a_{e-1}(\mathbf{r}), \quad (5)$$

for all  $0 \leq \omega \leq p^{m(e-2)}-1$  and  $0 \leq \epsilon \leq p^m-1$ .

The defined Gray map is a distance preserving  $\mathbb{F}_{p^m}$ -linear isomorphism.

**Theorem 2.3 ([16]).** *The Gray map  $\Phi$  is a  $\mathbb{F}_{p^m}$ -linear isometry from  $(\mathcal{R}_{(p^m, e)}^n, d_{hom})$  to  $(\mathbb{F}_{p^m}^{m(e-1)n}, d_{Ham})$ , where  $d_{Ham}$  denotes the Hamming distance on  $\mathbb{F}_{p^m}^{m(e-1)n}$ .*

In (1), an element  $r \in \mathcal{R}_{(p^m, e)}$  is uniquely written as

$$r = a_0 + ua_1 + \dots + u^{e-1}a_{e-1},$$

where  $a_i \in \mathbb{F}_{p^m}$ . Hence, an element  $\mathbf{r} \in \mathcal{R}_{(p^m, e)}^n$  can be viewed as

$$\mathbf{r} = a_0(\mathbf{r}) + ua_1(\mathbf{r}) + \dots + u^{e-1}a_{e-1}(\mathbf{r}), \quad (6)$$

where  $a_i(\mathbf{r}) = (r_{i,0}, r_{i,1}, \dots, r_{i,n-1})$  is a vector in  $\mathbb{F}_{p^m}^n$ , for every  $0 \leq i \leq e-1$ , or

$$\mathbf{r} = (r_0, r_1, \dots, r_{n-1}), \quad (7)$$

where  $r_i = r_{0,i} + ur_{1,i} + \dots + u^{e-1}r_{e-1,i} \in \mathcal{R}_{(p^m, e)}$ , for every  $0 \leq i \leq n-1$ .

In order to establish the onward results, an element  $\mathbf{r} \in \mathcal{R}_{(p^m, e)}^n$  is viewed as in (7), i.e.,  $\mathbf{r} = (r_0, r_1, \dots, r_{n-1})$ , where  $r_i = r_{0,i} + ur_{1,i} + \dots + u^{e-1}r_{e-1,i} \in \mathcal{R}_{(p^m, e)}$

for every  $0 \leq i \leq n-1$ . Corresponding to this representation of  $\mathbf{r}$ ,  $\Phi(\mathbf{r})$  is written as

$$\Phi(\mathbf{r}) = (b_0, b_1, \dots, b_{p^{m(e-1)}n-1}),$$

where

$$b_{(\omega p^m + \epsilon)n + j} = \alpha_\epsilon r_{0,j} + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} r_{l,j} + r_{e-1,j}, \quad (8)$$

for all  $0 \leq \omega \leq p^{m(e-2)} - 1$ ,  $0 \leq \epsilon \leq p^m - 1$  and  $0 \leq j \leq n - 1$ .

It follows from equations (5) and (8) that for each  $0 \leq \omega \leq p^{m(e-2)} - 1$  and  $0 \leq \epsilon \leq p^m - 1$ ,  $\mathbf{b}_{\omega p^m + \epsilon} = (b_{(\omega p^m + \epsilon)n}, b_{(\omega p^m + \epsilon)n+1}, \dots, b_{(\omega p^m + \epsilon)n+n-1})$ .

### 3 Gray Images of $\Theta$ -( $1 - u^{e-1}$ )-Constacyclic Codes

In this section, we give a characterization of the Gray images of  $\Theta$ -( $1 - u^{e-1}$ )-constacyclic codes over  $\mathcal{R}_{(p^m, e)}$ , where  $\Theta := \Theta_{\theta, 1, 1}$  is specified in (4) and  $\theta$  is an automorphism of  $\mathbb{F}_{p^m}$ .

For each  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ , let  $T_\theta : \mathbb{F}_{p^m}^{p^{m(e-1)}n} \rightarrow \mathbb{F}_{p^m}^{p^{m(e-1)}n}$  be a linear transformation given by

$$T_\theta((a_0, a_1, \dots, a_{p^{m(e-1)}n-1})) = (\theta(a_0), \theta(a_1), \dots, \theta(a_{p^{m(e-1)}n-1})).$$

Let  $\nu$  be a permutation on  $\{0, 1, \dots, p^{m(e-1)}n - 1\}$  defined by

$$\nu((\omega p^m + \epsilon)n + j) = (\varpi p^m + \epsilon)n + j$$

if  $\theta(\alpha_\epsilon) = \alpha_\epsilon$  and  $\theta(\alpha_{\bar{\xi}_l(\varpi)}) = \alpha_{\bar{\xi}_l(\omega)}$ , for all  $1 \leq l \leq e - 2$ . The linear transformation  $T_\nu : \mathbb{F}_{p^m}^{p^{m(e-1)}n} \rightarrow \mathbb{F}_{p^m}^{p^{m(e-1)}n}$  induced by  $\nu$  is given by

$$T_\nu((a_0, a_1, \dots, a_{p^{m(e-1)}n-1})) = (a_{\nu(0)}, a_{\nu(1)}, \dots, a_{\nu(p^{m(e-1)}n-1)}).$$

**Theorem 3.1.**  $\Phi \circ \rho_{\Theta, 1 - u^{e-1}} = T_\theta \circ T_\nu \circ \sigma^{\otimes p^{m(e-1)} - 1} \circ \Phi$ .

*Proof.* First, we observe that

$$\begin{aligned} \rho_{\Theta, 1 - u^{e-1}}(\mathbf{r}) &= (\theta(r_{0,n-1}) + u\theta(r_{1,n-1}) + \cdots + u^{e-1}(\theta(r_{e-1,n-1}) - \theta(r_{0,n-1})), \\ &\quad \theta(r_{0,0}) + u\theta(r_{1,0}) + \cdots + u^{e-1}\theta(r_{e-1,0}), \dots, \\ &\quad \theta(r_{0,n-2}) + u\theta(r_{1,n-2}) + \cdots + u^{e-1}\theta(r_{e-1,n-2})). \end{aligned}$$

Hence,  $\Phi \circ \rho_{\Theta, 1-u^{e-1}}(\mathbf{r}) = (d_0, d_1, \dots, d_{p^{m(e-1)}n-1})$ , where

$$d_{(\omega p^m + \epsilon)n+j} = \begin{cases} \alpha_\epsilon \theta(r_{0,j-1}) + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} \theta(r_{l,j-1}) + \theta(r_{e-1,j-1}) & \text{if } j \neq 0, \\ (\alpha_\epsilon - 1)\theta(r_{0,n-1}) + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} \theta(r_{l,n-1}) + \theta(r_{e-1,n-1}) & \text{if } j = 0, \end{cases}$$

$$= \begin{cases} \alpha_\epsilon \theta(r_{0,j-1}) + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} \theta(r_{l,j-1}) + \theta(r_{e-1,j-1}) & \text{if } j \neq 0, \\ \left( \sum_{i=0}^{m-1} \xi_i(\epsilon) \alpha^i - 1 \right) \theta(r_{0,n-1}) + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} \theta(r_{l,n-1}) \\ \quad + \theta(r_{e-1,n-1}) & \text{if } j = 0 \text{ and } \xi_0(\epsilon) \neq 0, \\ \left( \sum_{i=0}^{m-1} \xi_i(\epsilon) \alpha^i + p - 1 \right) \theta(r_{0,n-1}) + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} \theta(r_{l,n-1}) \\ \quad + \theta(r_{e-1,n-1}) & \text{if } j = 0 \text{ and } \xi_0(\epsilon) = 0, \end{cases} \quad (9)$$

for all  $0 \leq \omega \leq p^{m(e-2)} - 1$ ,  $0 \leq \epsilon \leq p^m - 1$  and  $0 \leq j \leq n - 1$ .

For the other direction, by equation (8), we have

$$\Phi(\mathbf{r}) = (b_0, b_1, \dots, b_{p^{m(e-1)}n-1}),$$

where  $b_{(\omega p^m + \epsilon)n+j} = \alpha_\epsilon r_{0,j} + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} r_{l,j} + r_{e-1,j}$ , for all  $0 \leq \omega \leq p^{m(e-2)} - 1$ ,  $0 \leq \epsilon \leq p^m - 1$  and  $0 \leq j \leq n - 1$ .

Let  $(c_0, c_1, \dots, c_{p^{m(e-1)}n-1}) = \sigma^{\otimes p^{m(e-1)-1}} \circ \Phi(\mathbf{r})$ . Then for each  $0 \leq \omega \leq p^{m(e-2)} - 1$ ,  $0 \leq \epsilon \leq p^m - 1$  and  $0 \leq j \leq n - 1$ ,

$$c_{(\omega p^m + \epsilon)n+j} = \begin{cases} \alpha_\epsilon r_{0,j-1} + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} r_{l,j-1} + r_{e-1,j-1} & \text{if } j \neq 0, \\ \left( \sum_{i=0}^{m-1} \xi_i(\epsilon) \alpha^i - 1 \right) r_{0,n-1} + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} r_{l,n-1} \\ \quad + r_{e-1,n-1} & \text{if } j = 0 \text{ and } \xi_0(\epsilon) \neq 0, \\ \left( \sum_{i=0}^{m-1} \xi_i(\epsilon) \alpha^i + p - 1 \right) r_{0,n-1} + \sum_{l=1}^{e-2} \alpha_{\bar{\xi}_{l-1}(\omega)} r_{l,n-1} \\ \quad + r_{e-1,n-1} & \text{if } j = 0 \text{ and } \xi_0(\epsilon) = 0. \end{cases}$$

Then  $T_\nu \circ \sigma^{\otimes p^{m(e-1)-1}} \circ \Phi(\mathbf{r}) = (f_0, f_1, \dots, f_{p^{m(e-1)}n-1})$ , where

$$f_{(\omega p^m + \epsilon)n+j} = \begin{cases} \theta^{-1}(\alpha_\epsilon)r_{0,j-1} + \sum_{l=1}^{e-2} \theta^{-1}(\alpha_{\bar{\xi}_{l-1}(\omega)})r_{l,j-1} + r_{e-1,j-1} & \text{if } j \neq 0, \\ \theta^{-1}\left(\sum_{i=0}^{m-1} \xi_i(\epsilon)\alpha^i - 1\right)r_{0,n-1} + \sum_{l=1}^{e-2} \theta^{-1}(\alpha_{\bar{\xi}_{l-1}(\omega)})r_{l,n-1} \\ \quad + r_{e-1,n-1} & \text{if } j = 0 \text{ and } \xi_0(\epsilon) \neq 0, \\ \theta^{-1}\left(\sum_{i=0}^{m-1} \xi_i(\epsilon)\alpha^i + p - 1\right)r_{0,n-1} + \sum_{l=1}^{e-2} \theta^{-1}(\alpha_{\bar{\xi}_{l-1}(\omega)})r_{l,n-1} \\ \quad + r_{e-1,n-1} & \text{if } j = 0 \text{ and } \xi_0(\epsilon) = 0, \end{cases} \quad (10)$$

for all  $0 \leq \omega \leq p^{m(e-2)} - 1$ ,  $0 \leq \epsilon \leq p^m - 1$  and  $0 \leq j \leq n - 1$ .

Applying  $T_\theta$  to (10), we have  $\Phi \circ \rho_{\Theta,1-u^{e-1}} = T_\theta \circ T_\nu \circ \sigma^{\otimes p^{m(e-1)-1}} \circ \Phi$  as desired.  $\square$

**Theorem 3.2.** *Let  $C$  be a code of length  $n$  over  $\mathcal{R}_{(p^m, \epsilon)}$  and  $\Theta := \Theta_{\theta,1,1}$ , where  $\theta$  is an automorphism of  $\mathbb{F}_{p^m}$ . Then  $C$  is a  $\Theta$ - $(1 - u^{e-1})$ -constacyclic if and only if  $\Phi(C)$  is a  $\theta$ -permutation invariant code of length  $p^{m(e-1)}n$  over  $\mathbb{F}_{p^m}$  (with respect to  $T_\nu \circ \sigma^{\otimes p^{m(e-1)-1}}$ ).*

*Proof.* The necessary part follows from Theorem 3.1, i.e.,

$$T_\theta \circ T_\nu \circ \sigma^{\otimes p^{m(e-1)-1}} \circ \Phi(C) = \Phi \circ \rho_{\Theta,1-u^{e-1}}(C) = \Phi(C).$$

For the sufficient part, assume that  $\Phi(C)$  is  $\theta$ -permutation invariant with respect to  $T_\nu \circ \sigma^{\otimes p^{m(e-1)-1}}$ . Then

$$\Phi(C) = T_\theta \circ T_\nu \circ \sigma^{\otimes p^{m(e-1)-1}} \circ \Phi(C) = \Phi \circ \rho_{\Theta,1-u^{e-1}}(C).$$

The injectivity of  $\Phi$  implies  $\rho_{\Theta,1-u^{e-1}}(C) = C$ , i.e.,  $C$  is  $\Theta$ - $(1 - u^{e-1})$ -constacyclic.  $\square$

## 4 Gray Images of $\Theta$ -Cyclic and $\Theta$ - $(1 + u^{e-1})$ -Constacyclic Codes

Throughout this section, we assume that  $p$  does not divide the length  $n$  of codes and  $\Theta := \Theta_{\theta,1,1}$  is specified in (4), where  $\theta$  is an automorphism of  $\mathbb{F}_{p^m}$ . Then

$\gcd(n, p) = 1$ , and hence there exists  $n' \in \{0, 1, \dots, p - 1\}$  such that  $nn' \equiv 1 \pmod{p}$ . Let  $\beta = 1 + n'u^{e-1}$ . Then  $\beta^j = (1 + n'u^{e-1})^j = 1 + jn'u^{e-1} \in \mathcal{R}_{(p^m, e)}$ , for all  $j \in \mathbb{Z}$ . In particular,  $\beta^n = 1 + u^{e-1}$  and  $\beta^{-n} = 1 - u^{e-1}$ . Moreover,  $1, 1 + u^{e-1}$  and  $\beta$  are fixed by  $\Theta$ .

Let  $\mu : \mathcal{R}_{(p^m, e)}^n \rightarrow \mathcal{R}_{(p^m, e)}^n$  be defined by

$$(r_0, r_1, \dots, r_{n-1}) \mapsto (r_0, \beta r_1, \dots, \beta^{n-1} r_{n-1}). \tag{11}$$

Then both  $\mu$  and  $\mu^2 = \mu \circ \mu$  are  $\mathcal{R}_{(p^m, e)}$ -module automorphisms on  $\mathcal{R}_{(p^m, e)}^n$ .

**Proposition 4.1.** *Let  $C$  be a non-empty subset of  $\mathcal{R}_{(p^m, e)}^n$ . Then  $C$  is a  $\Theta$ -cyclic code if and only if  $\mu(C)$  is a  $\Theta$ - $(1 - u^{e-1})$ -constacyclic code.*

*Proof.* Assume that  $C$  is a linear cyclic code. Let  $(r_0, \beta r_1, \dots, \beta^{n-1} r_{n-1}) \in \mu(C)$ . Since  $\mu$  is injective,  $(r_0, r_1, \dots, r_{n-1}) \in C$ . By the linearity and  $\Theta$ -cyclicity of  $C$ , we have  $\beta^{-1}(r_{n-1}, r_0, r_1, \dots, r_{n-2}) \in C$ . Thus

$$\begin{aligned} \rho_{\Theta, 1-u^{e-1}}((r_0, \beta r_1, \dots, \beta^{n-1} r_{n-1})) &= ((1 - u^{e-1})\beta^{n-1}\Theta(r_{n-1}), \Theta(r_0), \beta\Theta(r_1), \dots, \beta^{n-2}\Theta(r_{n-2})) \\ &= (\beta^{-n}\beta^{n-1}\Theta(r_{n-1}), \Theta(r_0), \beta\Theta(r_1), \dots, \beta^{n-2}\Theta(r_{n-2})) \\ &= ((\beta^{-1}\Theta(r_{n-1})), \beta(\beta^{-1}\Theta(r_0)), \beta^2(\beta^{-1}\Theta(r_1)), \dots, \beta^{n-1}(\beta^{-1}\Theta(r_{n-2}))) \\ &= \mu((\beta^{-1}\Theta(r_{n-1}), \beta^{-1}\Theta(r_0), \beta^{-1}\Theta(r_1), \dots, \beta^{-1}\Theta(r_{n-2}))) \\ &= \mu(\beta^{-1}(\Theta(r_{n-1}), \Theta(r_0), \Theta(r_1), \dots, \Theta(r_{n-2}))) \in \mu(C) \end{aligned}$$

since  $\beta^{-1}(\Theta(r_{n-1}), \Theta(r_0), \Theta(r_1), \dots, \Theta(r_{n-2})) \in C$ . Hence,  $\mu(C)$  is  $\Theta$ - $(1 - u^{e-1})$ -constacyclic as desired.

For the other direction, assume that  $\mu(C)$  is a  $\Theta$ - $(1 - u^{e-1})$ -constacyclic code. Let  $(r_0, r_1, \dots, r_{n-1}) \in C$ . Then  $(r_0, \beta r_1, \dots, \beta^{n-1} r_{n-1}) \in \mu(C)$ . By linearity and  $\Theta$ - $(1 - u^{e-1})$ -constacyclicity of  $\mu(C)$ , we have

$$\beta((1 - u^{e-1})\beta^{n-1}\Theta(r_{n-1}), \Theta(r_0), \beta\Theta(r_1), \dots, \beta^{n-2}\Theta(r_{n-2})) \in \mu(C).$$

Since  $\mu$  is bijective, we have

$$\begin{aligned} \rho_{\Theta}((r_0, r_1, \dots, r_{n-1})) &= (\Theta(r_{n-1}), \Theta(r_0), \Theta(r_1), \dots, \Theta(r_{n-2})) \\ &= \mu^{-1}((\Theta(r_{n-1}), \beta\Theta(r_0), \beta^2\Theta(r_1), \dots, \beta^{n-1}\Theta(r_{n-2}))) \\ &= \mu^{-1}((\beta^{-n}\beta^n\Theta(r_{n-1}), \beta\Theta(r_0), \beta^2\Theta(r_1), \dots, \beta^{n-1}\Theta(r_{n-2}))) \\ &= \mu^{-1}(\beta((1 - u^{e-1})\beta^{n-1}\Theta(r_{n-1}), \Theta(r_0), \beta\Theta(r_1), \dots, \beta^{n-2}\Theta(r_{n-2}))) \in C \end{aligned}$$

since  $\beta((1 - u^{e-1})\Theta(r_{n-1}), \Theta(r_0), \beta\Theta(r_1), \dots, \beta^{n-2}\Theta(r_{n-2})) \in \mu(C)$ . Therefore,  $C$  is  $\Theta$ -cyclic.  $\square$

**Proposition 4.2.** *Let  $C$  be a non-empty subset of  $\mathcal{R}_{(p^m, e)}^n$ . Then  $C$  is a  $\Theta$ - $(1 + u^{e-1})$ -constacyclic code if and only if  $\mu^2(C)$  is a  $\Theta$ - $(1 - u^{e-1})$ -constacyclic code.*

*Proof.* Using the definition of  $\nu^2$ ,  $\Theta$ - $(1 + u^{e-1})$ -constacyclicity and  $\Theta$ - $(1 - u^{e-1})$ -constacyclicity, the proof can be obtained similar to those in Proposition 4.1.  $\square$

The Nechaev permutation in [12] is extended in [16] to be the permutation  $\tau$  on  $\{0, 1, \dots, pn - 1\}$  defined by

$$\tau(sn + j) = (s + jn')_p n + j,$$

where  $0 \leq s \leq p - 1$ ,  $0 \leq j \leq n - 1$ , and  $(s + jn')_p$  is the least residue of  $s + jn'$  modulo  $p$ . The permutation  $\tau$  induces  $\pi : \mathbb{F}_{p^m}^{pn} \rightarrow \mathbb{F}_{p^m}^{pn}$  as follows:

$$\pi((c_0, c_1, \dots, c_{pn-1})) = (c_{\tau(0)}, c_{\tau(1)}, \dots, c_{\tau(pn-1)}).$$

The map  $\pi$  is then extended to  $\pi^{\otimes p^{m(e-1)-1}} : \mathbb{F}_{p^m}^{p^{m(e-1)}n} \rightarrow \mathbb{F}_{p^m}^{p^{m(e-1)}n}$  by

$$(a^{(p^0)} \mid a^{(p^1)} \mid \cdots \mid a^{(p^{m(e-1)-1})}) \mapsto (\pi(a^{(p^0)}) \mid \pi(a^{(p^1)}) \mid \cdots \mid \pi(a^{(p^{m(e-1)-1})})),$$

where  $a^{(i)} \in \mathbb{F}_{p^m}^{pn}$ ,  $\mid$  is a vector concatenation.

**Proposition 4.3 ([16]).**  $\Phi \circ \mu = \pi^{\otimes p^{m(e-1)-1}} \circ \Phi$ .

Next corollary follows immediately from Propositions 4.1, 4.3 and Theorem 3.1.

**Corollary 4.4.** *The Gray image of a  $\Theta$ -cyclic code of length  $n$  over  $\mathcal{R}_{(p^m, e)}$  is permutatively equivalent to a  $\theta$ -permutation invariant code of length  $p^{m(e-1)}n$  over  $\mathbb{F}_{p^m}$ .*

Finally, we establish the structure of the Gray image of a  $\Theta$ - $(1 + u^{e-1})$ -constacyclic code.

**Proposition 4.5 ([16]).**  $\Phi \circ \mu^2 = \pi^{\otimes p^{m(e-1)-1}} \circ \pi^{\otimes p^{m(e-1)-1}} \circ \Phi$ .

Next corollary follows immediately from Propositions 4.1, 4.5 and Theorem 3.1.

**Corollary 4.6.** *The Gray image of a  $\Theta$ - $(1 + u^{e-1})$ -constacyclic code of length  $n$  over  $\mathcal{R}_{(p^m, e)}$  is permutatively equivalent to a  $\theta$ -permutation invariant code of length  $p^{m(e-1)}n$  over  $\mathbb{F}_{p^m}$ .*

### 5 Conclusion and Remarks

For a given automorphism  $\Theta := \Theta_{\theta,1,1}$  of a ring  $\mathcal{R}_{(p^m,e)}$ , the Gray images of  $\Theta$ - $(1 - u^{e-1})$ -constacyclic codes are determined. In addition, if the length of codes is not divisible by  $p$ , the Gray images of  $\Theta$ -cyclic and  $\Theta$ - $(1 + u^{e-1})$ -constacyclic codes are also studied. Moreover, we have the following commuting diagram.

$$\begin{array}{ccccccc}
 \mathcal{R}_{(p^m,e)}^n & \xrightarrow{\mu} & \mathcal{R}_{(p^m,e)}^n & \xrightarrow{\mu} & \mathcal{R}_{(p^m,e)}^n & \xrightarrow{\rho_{\Theta,1-\mu^{e-1}}} & \mathcal{R}_{(p^m,e)}^n \\
 \downarrow \Phi & \circlearrowleft & \downarrow \Phi & \circlearrowleft & \downarrow \Phi & \circlearrowleft & \downarrow \Phi \\
 \mathbb{F}_p^{p^{m(e-1)}n} & \xrightarrow{\pi^{\otimes p^{m(e-1)-1}}} & \mathbb{F}_p^{p^{m(e-1)}n} & \xrightarrow{\pi^{\otimes p^{m(e-1)-1}}} & \mathbb{F}_p^{p^{m(e-1)}n} & & \mathbb{F}_p^{p^{m(e-1)}n} \\
 & & & & \downarrow \sigma^{\otimes p^{m(e-1)-1}} & & \uparrow T_{\Theta} \\
 & & & & \mathbb{F}_p^{p^{m(e-1)}n} & \xrightarrow{T_{\nu}} & \mathbb{F}_p^{p^{m(e-1)}n}
 \end{array}$$

For an arbitrary  $n$ , it follows from Theorem 3.1 that the third loop in the diagram commutes. As a consequence of Theorem 3.1, Propositions 4.3 and 4.5, all loops in the diagram commute whenever  $\gcd(n,p) = 1$ .

If  $\Theta$  is the identity automorphism, then the maps  $T_{\nu}$  and  $T_{\theta}$  are identity permutations. Hence, the diagram can be viewed as in the following.

$$\begin{array}{ccccccc}
 \mathcal{R}_{(p^m,e)}^n & \xrightarrow{\mu} & \mathcal{R}_{(p^m,e)}^n & \xrightarrow{\mu} & \mathcal{R}_{(p^m,e)}^n & \xrightarrow{\rho_{1-\mu^{e-1}}} & \mathcal{R}_{(p^m,e)}^n \\
 \downarrow \Phi & \circlearrowleft & \downarrow \Phi & \circlearrowleft & \downarrow \Phi & \circlearrowleft & \downarrow \Phi \\
 \mathbb{F}_p^{p^{m(e-1)}n} & \xrightarrow{\pi^{\otimes p^{m(e-1)-1}}} & \mathbb{F}_p^{p^{m(e-1)}n} & \xrightarrow{\pi^{\otimes p^{m(e-1)-1}}} & \mathbb{F}_p^{p^{m(e-1)}n} & \xrightarrow{\sigma^{\otimes p^{m(e-1)-1}}} & \mathbb{F}_p^{p^{m(e-1)}n}
 \end{array}$$

For an arbitrary  $n$ , the third loop in the diagram commutes. All the diagrams commute whenever  $\gcd(n,p) = 1$ .

Using the latter diagram, the following results can be obtained as corollaries of Theorem 3.2, Corollaries 4.4 and 4.6, respectively.

**Corollary 5.1 ([16]).** *Let  $C$  be a code of length  $n$  over  $\mathcal{R}_{(p^m,e)}$ . Then  $C$  is a  $(1-u^{e-1})$ -constacyclic if and only if  $\Phi(C)$  is a quasi-cyclic code of index  $p^{m(e-1)-1}$  and length  $p^{m(e-1)}n$  over  $\mathbb{F}_p$ .*

**Corollary 5.2 ([16]).** *The Gray image of a linear cyclic code of length  $n$  over  $\mathcal{R}_{(p^m,e)}$  is permutatively equivalent to a quasi-cyclic code of index  $p^{m(e-1)-1}$  and length  $p^{m(e-1)}n$  over  $\mathbb{F}_p$ .*

**Corollary 5.3.** *The Gray image of a linear  $(1+u^{e-1})$ -constacyclic code of length  $n$  over  $\mathcal{R}_{(p^m,e)}$  is permutatively equivalent to a quasi-cyclic code of index  $p^{m(e-1)-1}$  and length  $p^{m(e-1)}n$  over  $\mathbb{F}_p$ .*

Due to the automorphisms of finite chain rings established in [1], it would be interesting to study possible generalizations to the Gray images of skew constacyclic codes over finite chain rings in general.

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