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σ-ideals of Distributive p-algebras

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Abstract: The concepts of boosters and σ-ideals are introduced in distributive p-algebras. Many properties of σ-ideals are studied in terms of boosters. It is proved that the class of all boosters of a distributive p-algebra is a Boolean algebra. It is also observed that the lattice of all σ-ideals of a distributive p-algebra is isomorphic to the ideal lattice of the lattice of all boosters. Finally, some properties of σ-ideals are studied with respect to homomorphisms.

Keywords: Distributive p-algebras; boosters; σ-ideals; homomorphisms

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Introduction

The theory of pseudo-complements was introduced in semi-lattices and distributive lattices by O. Frink [6] and G. Birkhof [3]. Later pseudo-complements in Stone algebras has been studied by many authors like R. Balbes[1], O. Frink[6], G. Gratzer[4] etc.

In this paper, the notion of boosters is introduced in distributive p-algebras and then many properties of boosters are studied. It is proved that the set $B_*(L)$ of all boosters of a distributive p-algebra L forms a Boolean algebra on its own. It is also observed that a distributive p-algebra L is homomorphic to $B_*(L)$. The concept of σ-ideals is introduced in distributive p-algebras. Some properties of σ-ideals of a distributive p-algebra are studied in terms of boosters and then proved

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that the set $I_\sigma(L)$ of all σ-ideals can be made into a distributive lattice. It is proved that $I_\sigma(L)$ is isomorphic to the ideal lattice of $B_*(L)$. It is proved that every minimal prime ideal of a distributive p-algebra containing a given σ-ideal is a σ-ideal. Also, it is proved that every proper σ-ideal of a distributive p-algebra is the intersection of all prime σ-ideals containing it. Finally, some properties of σ-ideals are studied with respect to homomorphisms. If a distributive p-algebras L is homomorphic to a distributive p-algebra M, then the lattice $B_*(L)$ of boosters is homomorphic to $B_*(M)$ the lattice of Boosters of M and the ideal lattice of $B_*(L)$ is homomorphic to the ideal lattice of $B_*(M)$.

1 Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [2], [5] and [6] for the ready reference of the reader.

A (distributive)p-algebra is a universal algebra $(L, \vee, \wedge, ^*, 0, 1)$ where $(L, \vee, \wedge, 0, 1)$ is a bounded (distributive)lattice and the unary operation * is defined by

$$x \leq a^* \Leftrightarrow x \wedge a = 0$$

Here the above operation * is called pseudo-complementation on L. It is will known that the class of all p-algebras is equational(See [6]). A distributive p-algebra L in which $x^* \vee x^{**} = 1$ for all $x \in L$ holds is called a Stone algebra.

We shall frequently use the following rules of the computations in p-algebras. For any two elements a, b of a p-algebra L, we have (see [2],[5])

(1) $0^{**} = 0$ and 1^{**},
(2) $a \wedge a^* = 0$,
(3) $a \leq b$ implies $b^* \leq a^*$,
(4) $a \leq a^{**}$,
(5) $a^{***} = a^*$,
(6) $(a \vee b)^* = a^* \wedge b^*$,
(7) $(a \wedge b)^* \geq a^* \vee b^*$,
(8) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,
(9) $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^{**} \vee b^{**})^{**}$.

An element x of a p-algebra L is called closed if $x^{**} = x$ and the set of all closed elements of L is denoted by $B(L) = \{a \in L : a = a^{**}\}$. It is known that $(B(L), \triangledown, \wedge, ^*, 0, 1)$ is a Boolean algebra, where $a \triangledown b = (a^* \wedge b^*)^*$. An element a is called dense if $a^* = 0$. The set $D(L) = \{d \in L : d^* = 0\}$ is a filter of L.

2 Boosters of distributive p-algebras

In this section, the concept of boosters is introduced in a distributive p-algebra. Some properties of boosters are investigated in a distributive p-algebra. It is proved that the class of all boosters forms a Boolean algebra.

Definition 2.1. Let L be a distributive p-algebra. Then for every $a \in L$, define the booster of a as follows :

$$(a)^{\triangle} = \{x \in L : x \wedge a^* = 0\}$$

It is obvious that $(0)^{\triangle} = \{0\}$ and $(1)^{\triangle} = L$. Moreover, the class of boosters of a distributive p-algebra satisfies the following properties.

Lemma 2.2. *Let L be a distributive p-algebra. Then for any $a, b \in L$ we have*

(1) $(a)^{\triangle}$ *is an ideal of* L *containing* a*,*

(2) $(a)^{\triangle} = (a^{**})^{\triangle} = (a^{**}]$*,*

(3) $(a)^{\triangle} = (a] \Leftrightarrow a \in B(L)$*,*

(4) $(a)^{\triangle} = L \Leftrightarrow a \in D(L)$*,*

(5) $a \in (b)^{\triangle} \Rightarrow (a)^{\triangle} \subseteq (b)^{\triangle}$*.*

Proof. (1). Clearly $0 \in (a)^{\triangle}$. Let $x, y \in (a)^{\triangle}$. Then $(x \vee y) \wedge a^* = (x \wedge a^*) \vee (y \wedge a^*) = 0$. Thus $x \vee y \in (a)^{\triangle}$. Now let $x \in (a)^{\triangle}$ and $z \leq x$. Then $z \wedge a^* \leq x \wedge a^* = 0$. So, $z \in (a)^{\triangle}$. Thus $(a)^{\triangle}$ is an ideal of L. Clearly $a \in (a)^{\triangle}$.
(2). $(a)^{\triangle} = (a^{**})^{\triangle}$ follows from the fact $a^* = a^{***}$. Since $a^{**} \wedge a^* = 0$, we get that $a^{**} \in (a)^{\triangle}$. To show that a^{**} is the greatest element of $(a)^{\triangle}$, let $y \in (a)^{\triangle}$. Then $y \wedge a^* = 0$, which implies that $y \leq a^{**}$. Therefore $(a)^{\triangle} = (a^{**}]$.
(3). Let $a \in B(L)$. Then $a^{**} = a$. So by (2), we get $(a)^{\triangle} = (a]$. Conversely, let $(a)^{\triangle} = (a]$. But (2) gives $(a)^{\triangle} = (a^{**}]$. Thus $a = a^{**}$ and $a \in B(L)$.
(4). Let $a \in D(L)$. Then $x \wedge a^* = 0$ for all $x \in L$. So $(a)^{\triangle} = L$. Conversely, let $(a)^{\triangle} = L$. Then $(a)^{\triangle} = (a^{**}] = L$ and $(1)^{\triangle} = (1^{**}] = L$ imply that $a^* = 0$.
(5). Suppose $a \in (b)^{\triangle}$. Then $a \wedge b^* = 0$ and hence $b^* \leq a^*$. Let $x \in (a)^{\triangle}$.

Then $x \wedge a^* = 0$ and thus $a^* \leq x^*$. Thus it concludes $b^* \leq a^* \leq x^*$. Hence $x \wedge b^* \leq x \wedge a^* = 0$. Thus it yields $x \in (b)^\triangle$. Therefore $(a)^\triangle \subseteq (b)^\triangle$. □

Lemma 2.3. *For any two elements a, b of a distributive p-algebra L, we have*

(1) $a \leq b$ *implies* $(a)^\triangle \subseteq (b)^\triangle$,

(2) $a^* = b^* \Leftrightarrow (a)^\triangle = (b)^\triangle$,

(3) $(a)^\triangle \cap (b)^\triangle = (a \wedge b)^\triangle$,

(4) $(a)^\triangle = (b)^\triangle$ *implies* $(a \wedge c)^\triangle = (b \wedge c)^\triangle$ *for all* $c \in L$,

(5) $(a)^\triangle = (b)^\triangle$ *implies* $(a \vee c)^\triangle = (b \vee c)^\triangle$ *for all* $c \in L$,

(6) $(a)^\triangle = (0)^\triangle$ *if and only if* $a = 0$.

Proof. (1). Assume that $a \leq b$. Let $x \in (a)^\triangle$. Then we get $x \wedge b^* \leq x \wedge a^* = 0$, which implies that $x \in (b)^\triangle$. Therefore $(a)^\triangle \subseteq (b)^\triangle$.
(2). Using (3) of Lemma 2.2, we get

$$a^* = b^* \Leftrightarrow a^{**} = b^{**} \Leftrightarrow (a^{**}] = (b^{**}] \Leftrightarrow (a)^\triangle = (b)^\triangle$$

(3). It is clear that $(a \wedge b)^\triangle$ is a lower bound of both $(a)^\triangle$ and $(b)^\triangle$. Let $(c)^\triangle \subseteq (a)^\triangle$ and $(c)^\triangle \subseteq (b)^\triangle$ for some $c \in L$. Let $x \in (c)^\triangle$. Then $x \in (a)^\triangle = (a^{**}]$ and $x \in (b)^\triangle = (b^{**}]$. Then $x \leq a^{**} \wedge b^{**} = (a \wedge b)^{**}$ implies $x \in ((a \wedge b)^{**}] = (a \wedge b)^\triangle$. Therefore $(a \wedge b)^\triangle$ is the greatest lower bound of $(a)^\triangle$ and $(b)^\triangle$.
(4). Suppose $(a)^\triangle = (b)^\triangle$. Let $x \in (a \wedge c)^\triangle$, then we have

$$\begin{aligned} x \wedge (a^* \vee c^*) \leq x \wedge (a \wedge c)^* = 0 \quad &\Rightarrow \quad (x \wedge a^*) \vee (x \wedge c^*) = 0 \\ &\Rightarrow \quad x \wedge a^* = 0 \text{ and } x \wedge c^* = 0 \\ &\Rightarrow \quad x \in (a)^\triangle = (b)^\triangle \text{ and } x \in (c)^\triangle \\ &\Rightarrow \quad x \in (b)^\triangle \cap (c)^\triangle = (b \wedge c)^\triangle \text{ by (3)} \end{aligned}$$

Thus $(a \wedge c)^\triangle \subseteq (b \wedge c)^\triangle$. By similar way we can prove that $(b \wedge c)^\triangle \subseteq (a \wedge c)^\triangle$.
(5). Suppose $(a)^\triangle = (b)^\triangle$. Let $x \in (a \vee c)^\triangle$, then we have

$$\begin{aligned} x \wedge (a \vee c)^* = 0 \quad &\Rightarrow \quad x \wedge a^* \wedge c^* = 0 \\ &\Rightarrow \quad x \wedge c^* \in (a)^\triangle = (b)^\triangle \\ &\Rightarrow \quad x \wedge c^* \wedge b^* = 0 \\ &\Rightarrow \quad x \wedge (c \vee b)^* = 0 \\ &\Rightarrow \quad x \in (b \vee c)^\triangle \end{aligned}$$

Thus $(a \vee c)^{\triangle} \subseteq (b \vee c)^{\triangle}$. Similarly we can show that $(b \vee c)^{\triangle} \subseteq (a \vee c)^{\triangle}$.
(6). It is obvious. □

The following is an easy consequence of (4) and (5) of the above Lemma 2.3.

Proposition 2.4. *Let L be a distributive p-algebra. For any $x, y \in L$, define a binary relation Ψ on L as follows :*

$$\Psi = \{(x, y) : (x)^{\triangle} = (y)^{\triangle}\}$$

Then Ψ is a congruence on L.

Now, let us denote the set of all boosters of a distributive p-algebra L by $B_*(L)$. Then we get the following:

$$B_*(L) = \{(x)^{\triangle} : x \in L\} = \{(x^{**})^{\triangle} : x \in L\}$$

Theorem 2.5. *Let L be a distributive p-algebra. Then the following hold.*

(1) $B_*(L)$ is a Boolean algebra on its own,

(2) L is homomorphic of $B_*(L)$,

(3) $B(L) \cong B_*(L)$.

Proof. (1). It is easy to observe that $B_*(L)$ is a partially ordered set with respect to the set inclusion. Clearly $(0)^{\triangle} = \{0\}$ is the zero element of $B_*(L)$ and $(1)^{\triangle} = L$ is the unit element of it. Define the operations $\cap$ and $\sqcup$ on $B_*(L)$ as follows :

$$(x)^{\triangle} \cap (y)^{\triangle} = (x \wedge y)^{\triangle} \text{ and } (x)^{\triangle} \sqcup (y)^{\triangle} = (x \vee y)^{\triangle}$$

Clearly $(x \wedge y)^{\triangle}$ is the infimum of both $(x)^{\triangle}$ and $(y)^{\triangle}$ in $B_*(L)$. Since $x, y \leq x \vee y$, we get $(x)^{\triangle}, (y)^{\triangle} \subseteq (x \vee y)^{\triangle}$. So $(x \vee y)^{\triangle}$ is an upper bound for both $(x)^{\triangle}$ and $(y)^{\triangle}$. Suppose $(z)^{\triangle}$ is an upper bound of $(x)^{\triangle}$ and $(y)^{\triangle}$ for some $z \in L$. Then $(x)^{\triangle}, (y)^{\triangle} \subseteq (z)^{\triangle}$. Thus we get

$$\begin{aligned} a \in (x \vee y)^{\triangle} &\Rightarrow a \wedge (x \vee y)^* = 0 \\ &\Rightarrow a \wedge x^* \wedge y^* = 0 \\ &\Rightarrow a \wedge x^* \in (y)^{\triangle} \subseteq (z)^{\triangle} \\ &\Rightarrow a \wedge x^* \wedge z^* = 0 \\ &\Rightarrow a \wedge z^* \in (x)^{\triangle} \subseteq (z)^{\triangle} \\ &\Rightarrow a \wedge z^* = 0 \\ &\Rightarrow a \in (z)^{\triangle} \end{aligned}$$

Then $(x \vee y)^{\triangle}$ is the supremum for $(x)^{\triangle}$ and $(y)^{\triangle}$ in $B_*(L)$. Therefore $(B_*(L), \cap, \sqcup, 0, L)$ is a bounded lattice. For all $(x)^{\triangle}, (y)^{\triangle}$ and $(z)^{\triangle}$ in $B_*(L)$ we have

$$\begin{aligned}
(x)^{\triangle} \cap ((y)^{\triangle} \sqcup (z)^{\triangle}) &= (x)^{\triangle} \cap (y \vee z)^{\triangle} \\
&= (x \wedge (y \vee z)^{\triangle} \\
&= ((x \wedge y) \vee (x \wedge z))^{\triangle} \\
&= (x \wedge y)^{\triangle} \sqcup (x \wedge z)^{\triangle} \\
&= ((x)^{\triangle} \wedge (y)^{\triangle}) \sqcup ((x)^{\triangle} \wedge (z)^{\triangle})
\end{aligned}$$

Therefore it concludes that $B_*(L)$ is a distributive lattice. Define a unary operation $^-$ on $B_*(L)$ by $(x)^{\triangle *} = (x^*)^{\triangle}, \forall (x) \in B_*(L)$, so we get

$$(x)^{\triangle} \wedge (x)^{\triangle -} = (x)^{\triangle} \wedge (x^*)^{\triangle} = (x \wedge x^*)^{\triangle} = (0)^{\triangle} = \{0\},$$
$$(x)^{\triangle} \vee (x)^{\triangle -} = ((x \vee x^*)^{**})^{\triangle} = ((x^* \wedge x^{**})^*)^{\triangle} = (1)^{\triangle} = L$$

Thus it yields that $B_*(L)$ is a complemented lattice. Therefore $(B_*(L), \cap, \sqcup, ^-, 0, L)$ forms a Boolean algebra.

(2). Define $\varphi : L \to B_*(L)$ by $\varphi(x) = (x^{**})^{\triangle}$. Then by Lemma 2.2(2), we get $(x^{**})^{\triangle} = (x)^{\triangle}$. Clearly $\varphi(0) = \{0\}$ and $\varphi(1) = L$. For every $x, y \in L$ we have

$$\varphi(x \wedge y) = (x \wedge y)^{\triangle} = (x)^{\triangle} \cap (y)^{\triangle} = \varphi(x) \cap \varphi(y),$$
$$\varphi(x \vee y) = (x \vee y)^{\triangle} = (x)^{\triangle} \sqcup (y)^{\triangle} = \varphi(x) \sqcup \varphi(y),$$
$$\varphi(x^*) = (x^*)^{\triangle} = (x)^{\triangle -} = [\varphi(x)]^-$$

Obviously φ is an onto map. Therefore φ is an onto homomorphism. Moreover φ is not a one-one, because of $(a)^{\triangle} = (x)^{\triangle}$ defined by $a^* = x^*$ and $a \neq x$.

(3). Clearly the map $f : B(L) \to B_*(L)$ with $f(a) = (a)^{\triangle}$ is an isomorphism. □

Definition 2.6. Let L be a distributive p-algebra. Then define as follows:

(1) For any ideal I of L, define an operator σ as $\sigma(I) = \{(x)^{\triangle} : x \in I\}$

(2) For any ideal I of $B_*(L)$, define an operator $\overleftarrow{\sigma}$ as $\overleftarrow{\sigma}(I) = \{x \in L : (x)^{\triangle} \in I\}$

Lemma 2.7. *The following conditions hold in a distributive p-algebra L.*

(1) for any ideal I of L, $\sigma(I)$ is an ideal of $B_*(L)$,

(2) for any ideal I of $B_*(L)$, $\overleftarrow{\sigma}(I)$ is an ideal of L,

(3) $\overleftarrow{\sigma}$ and σ are isotones,

(4) $\sigma(\overleftarrow{\sigma}(I)) = I$*, for all ideal* I *of* $B_*(L)$*.*

Proof. (1). Let I be an ideal of L. Clearly $(0)^{\triangle} \in \sigma(I)$ as $0 \in I$. For any $(x)^{\triangle}, (y)^{\triangle} \in \sigma(I)$, we get $(x)^{\triangle} \sqcup (y)^{\triangle} = (x \vee y)^{\triangle} \in \sigma(I)$ as $x \vee y \in I$. Again let $(x)^{\triangle} \in \sigma(I)$ and $(z)^{\triangle} \in B_*(L)$ such that $(z)^{\triangle} \subseteq (x)^{\triangle}$, then $(z)^{\triangle} = (z)^{\triangle} \cap (x)^{\triangle} = (x \wedge z)^{\triangle} \in \sigma(I)$ as $z \wedge x \in I$. Therefore $\sigma(I)$ is an ideal of $B_*(L)$.
(2). Let I be an ideal of $B_*(L)$. Then $0 \in \overleftarrow{\sigma}(I)$ as $(0)^{\triangle} \in I$. Let $x, y \in \overleftarrow{\sigma}(I)$. Then $(x \vee y)^{\triangle} = (x)^{\triangle} \sqcup (y)^{\triangle} \in I$ implies $x \vee y \in \overleftarrow{\sigma}(I)$. Now let $x, y \in \overleftarrow{\sigma}(I)$ and $y \leq x$, for some $y \in L$. Since $(y)^{\triangle} = (y)^{\triangle} \cap (x)^{\triangle} \in I$. Then $y \in \overleftarrow{\sigma}(I)$. Therefore $\overleftarrow{\sigma}(I)$ is an ideal of L.
(3). Let I, H be two ideals of $B_*(L)$. Suppose $I \subseteq H$ and $x \in \overleftarrow{\sigma}(I)$. Then $(x)^{\triangle} \in I \subseteq H$ implies $x \in \overleftarrow{\sigma}(H)$. Therefore $\overleftarrow{\sigma}$ is an isotone operator from the lattice $I(B_*(L))$ of all ideals of $B_*(L)$ to the lattice $I(L)$ of all ideals of L. Similarly, we can also prove that σ is an isotone operator.
(4). Let I be an ideal of $B_*(L)$, then $\overleftarrow{\sigma}$ is an ideal of L (by (2)). So we have

$$(x)^{\triangle} \in I \Leftrightarrow x \in \overleftarrow{\sigma}(I) \Leftrightarrow (x)^{\triangle} \in \sigma\overleftarrow{\sigma}(I)$$

Then $\sigma\overleftarrow{\sigma}(I) = I$. So $\sigma\overleftarrow{\sigma} : I(B_*(L)) \to I(B_*(L))$ is the identity map. □

Theorem 2.8. *The map* $I \to \overleftarrow{\sigma}\sigma(I)$ *is a closure operator of a lattice of ideals of* L*, that is*

(1) $I \subseteq \overleftarrow{\sigma}\sigma(I)$*,*

(2) $I \subseteq H$ *implies* $\overleftarrow{\sigma}\sigma(I) \subseteq \overleftarrow{\sigma}\sigma(H)$*,*

(3) $\overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(I)\} = \overleftarrow{\sigma}\sigma(I)$ *for any ideals* I, H *of* L*.*

Proof. (1). Let $x \in I$. Then we get $(x)^{\triangle} \in \sigma(I)$. Since $\sigma(I)$ is an ideal of $B_*(L)$, we get that $x \in \overleftarrow{\sigma}\sigma(I)$. Therefore $I \subseteq \overleftarrow{\sigma}\sigma(I)$.
(2). Suppose $I \subseteq H$. Let $x \in \overleftarrow{\sigma}\sigma(I)$. Hence $(x)^{\triangle} \in \sigma(I)$. We have $(x)^{\triangle} = (y)^{\triangle}$ for some $y \in I \subseteq H$. Then $(x)^{\triangle} = (y)^{\triangle} \in \sigma(H)$. Since $\sigma(H)$ is an ideal of $B_*(L)$, then $x \in \overleftarrow{\sigma}\sigma(H)$. Therefore $I \subseteq \overleftarrow{\sigma}\sigma(I)$.
(3). We have $\overleftarrow{\sigma}\sigma(I) \subseteq \overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(I)\}$ as $\sigma\{\overleftarrow{\sigma}\sigma(I)\}$ is an ideal of $B_*(L)$. Conversely, let $x \in \overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(I)\}$. Then $(x)^{\triangle} \in \sigma\{\overleftarrow{\sigma}\sigma(I)\}$. Hence $(x)^{\triangle} = (y)^{\triangle}$ for some $y \in \overleftarrow{\sigma}\sigma(I)$. Thus $(x)^{\triangle} = (y)^{\triangle} \in \sigma(I)$. So $x \in \overleftarrow{\sigma}(I)$. □

Corollary 2.9. *Let* I, H *be two ideals of a distributive p-algebra* L*. Then* $\overleftarrow{\sigma}\sigma(I \cap H) = \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(H)$

Proof. Clearly $\overleftarrow{\sigma}\sigma(I \cap H) \subseteq \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(H)$. Conversely, let $x \in \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(H)$. Then we get $(x)^{\triangle} \in \sigma(I) \cap \sigma(H) = \sigma(I \cap H)$ as h is a homomorphism. Then we have $x \in \overleftarrow{\sigma}\sigma(I \cap H)$. Therefore $\overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(H) \subseteq \overleftarrow{\sigma}\sigma(I \cap H)$. □

3 σ-ideals of distributive p-algebras

In this section, the notion of σ-ideals is introduced in distributive p-algebras. The class of σ-ideals is characterized by means of boosters.

Definition 3.1. An ideal I of a distributive p-algebra L is called σ-ideal if $\overleftarrow{\sigma}\sigma(I) = I$.

Theorem 3.2. *Let I be an ideal of a distributive p-algebra L. The following conditions are equivalent.*

(1) I is a σ-ideal,

(2) for all $x, y \in L, (x)^{\triangle} = (y)^{\triangle}$ and $x \in I$ imply $y \in I$,

(3) $I = \bigcup_{x \in I}(x)^{\triangle}$,

(4) $x \in I$ implies $(x)^{\triangle} \subseteq I$.

Proof. (1) $\Rightarrow$ (2): Assume that I is a σ-ideal of L. Let $x, y \in L$ be such that $(x)^{\triangle} = (y)^{\triangle}$. Suppose $x \in I$. Then $(x)^{\triangle} = (y)^{\triangle} \in \sigma(I)$. Since $\sigma(I)$ is an ideal of $B_*(L)$, we have $y \in \overleftarrow{\sigma}\sigma(I) = I$.
(2) $\Rightarrow$ (3): For any $x \in I$, we have $(x] \subseteq (x)^{\triangle}$. Hence $I = \bigcup_{x \in I}(x] \subseteq \bigcup_{x \in I}(x)^{\triangle}$. Conversely, let $x \in I$ and $y \in (x)^{\triangle}$. Then we get $(y)^{\triangle} \subseteq (x)^{\triangle}$. Hence $(y)^{\triangle} = (y)^{\triangle} \cap (x)^{\triangle} = (y \wedge x)^{\triangle}$. Since $y \wedge x \in I$, by condition (2), we get $y \in I$. Hence $(x)^{\triangle} \subseteq I$ for all $x \in I$. This it yields $\bigcup_{x \in I}(x)^{\triangle} \subseteq I$. Therefore $I = \bigcup_{x \in I}(x)^{\triangle}$.
(3)$\Rightarrow$(4): Assume the condition (3). Let $x \in I$. Then by condition (3), we get $x \in (a)^{\triangle}$ for some $a \in I$. Let $t \in (x)^{\triangle}$. Then it concludes $t \in (x)^{\triangle} \subseteq (a)^{\triangle}$ and $a \in I$. Hence $t \in \bigcup_{a \in I}(a)^{\triangle} = I$.
(4) $\Rightarrow$ (1): Assume the condition (4). Clearly, $I \subseteq \overleftarrow{\sigma}\sigma(I)$. Conversely, let $x \in \overleftarrow{\sigma}\sigma(I)$. Then $(x)^{\triangle} \in \sigma(I)$. Hence $(x)^{\triangle} = (y)^{\triangle}$ for some $y \in I$. Since $y \in I$, by condition (4), it yields $x \in (x)^{\triangle} \subseteq (y)^{\triangle} \subseteq I$. □

Lemma 3.3. *For any distributive p-algebra L, the principal ideal $(a]$ is a σ-ideal if and only if a is a closed element of L.*

Proof. For all $a \in B(L)$, we have $(a)^{\triangle} = (a]$. Then $\sigma((a)^{\triangle}) = \sigma((a]) = \{(x)^{\triangle} : x \in (a]\} = \{(x)^{\triangle} : x \leq a\} = \{(x)^{\triangle} : (x)^{\triangle} \subseteq (a)^{\triangle}\} = ((a)^{\triangle}]\overleftarrow{\sigma}\sigma((a)^{\triangle}) = \overleftarrow{\sigma}\{((a)^{\triangle}]\} = \{x \in L : (x)^{\triangle} \in ((a)^{\triangle}]\} = \{x \in L : (x^{**})^{\triangle} = (x)^{\triangle} \subseteq (a]\} = \{x \in L : x \leq x^{**} \leq a\} = (a] = (a)^{\triangle}$. Then any principal ideal of L generated by a closed element is a σ-ideal. Conversely, let $I = (a]$ be a σ-ideal of L. Then we get $(a)^{\triangle} = (a^{**})^{\triangle}$. Then $a \in I$ implies $a^{**} \in I$. Hence $a = a^{**}$. □

Theorem 3.4. *Let L be a distributive p-algebra. If P is a minimal in the class of all prime ideals containing a given σ-ideal, then P is a σ-ideal.*

Proof. Let I be a σ-ideal of L and P minimal in the class of all prime ideals of L such that $I \subseteq P$. Suppose P is not a σ-ideal. Then there exist elements $x, y \in L$ and $y \notin P$. Consider the filter $F = (L-P) \vee [x \wedge y)$. Then $F \cap I = \emptyset$. Otherwise, choose $a \in F \cap I$. Then $a = r \wedge s$ for some $r \in L - P$ and $s \in [x \wedge y)$. Then

$$a = r \wedge s = r \wedge (s \vee (x \wedge y)) = (r \wedge s) \vee (r \wedge x \wedge y) \in I \text{ as } s \geq x \wedge y$$

Since $s \geq x \wedge y$, then $a = r \wedge s \geq r \wedge x \wedge y$. Thus $r \wedge x \wedge y \in I$. Since $(x)^{\triangle} = (y)^{\triangle}$, then we get $(r \wedge y)^{\triangle} = (r \wedge x \wedge y)^{\triangle}$. since I is a σ-ideal and $r \wedge x \wedge y \in I$, we get $r \wedge y \in I \subseteq P$. Hence $r \in P$ or $y \in P$, which is a contradiction. Thus $F \cap I = \phi$. Then there exists a prime ideal H such that $H \cap H = \phi$ and $I \subseteq H$. Since $F \cap H = \phi$, we get $H \subseteq P$. Also $x \wedge y \notin H$ and $x \wedge y \in P$. Hence $H \subset P$. Therefore P is not minimal in the class of all prime ideals containing I, which is a contradiction. Therefore P is a σ-ideal of L. □

Theorem 3.5. *Let L be a distributive p-algebra. Then every proper σ-ideal of L is the intersection of all prime σ-ideals containing it.*

Proof. Let I be a proper σ-ideal of L. Consider the following set

$$I_0 = \{P : P \text{ is a prime ideal and } I \subseteq P\}$$

Clearly $I \subseteq I_0$. Conversely, let $a \notin I$. Take $R = \{H : H$ is a σ-ideal, $I \subseteq H$, $a \notin H\}$. Clearly R satisfies the hypothesis of Zorn's Lemma. Let M be a maximal element of R. Let $b, c \in L$ be such that $b \notin M$ and $c \notin M$. Then

$$M \subseteq M \vee (b] \subseteq \overleftarrow{\sigma}\sigma\{M \vee (b]\} \text{ and } M \subseteq M \vee (c] \subseteq \overleftarrow{\sigma}\sigma\{M \vee (c]\}$$

By maximality of M, we get

$$a \in \overleftarrow{\sigma}\sigma\{M \vee (b]\} \text{ and } a \in \overleftarrow{\sigma}\sigma\{M \vee (c]\}$$

Thus we get

$$\begin{aligned} a &\in \overleftarrow{\sigma}\sigma\{M \vee (b]\} \cap \overleftarrow{\sigma}\sigma\{M \vee (c]\} \\ &= \overleftarrow{\sigma}\sigma\{\{M \vee (b]\} \cap \{M \vee (c]\}\} \\ &= \overleftarrow{\sigma}\sigma\{M \vee (b \wedge c]\} \end{aligned}$$

If $b \wedge c \in M$, then $a \in \overleftarrow{\sigma}\sigma(M) = M$, which is a contradiction. Thus M is a prime σ-ideal such that $a \notin M$. Therefore $a \in I_0$. Then $I_0 \subseteq I$. Therefore $I_0 = I$. □

Now, for any distributive p-algebra L, let $I(L)$ denotes the set of all ideals of L and $I_\sigma(L)$ denotes the set of all σ-ideals of L. It is known that $(I(L), \wedge, \vee)$ is a distributive lattice, where $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : x \geq i \vee j, i \in I, j \in J\}$. We will prove that the set $I_\sigma(L)$ of all σ-ideal of a distributive p-algebra L forms a bounded distributive lattice.

Theorem 3.6. *Let L be a distributive p-algebra. Then the set $I_\sigma(L)$ forms a bounded distributive lattice on its own.*

Proof. Define the operations $\wedge$ and $\veebar$ on $I_\sigma(L)$ as follows :

$$I \wedge J = I \cap J \text{ and } I \veebar J = \overleftarrow{\sigma}\sigma(I \vee J) \text{ for all } I, J \in I_\sigma(L)$$

where $I \vee J$ is the supremum of both I and J in the lattice $I(L)$. For every $I, J \in I_\sigma(L)$, we get

$$\overleftarrow{\sigma}\sigma(I \cap J) = \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(J) = I \cap J \Rightarrow I \cap J \in I_\sigma(L)$$

Since $I, J \subseteq I \vee J$, we get $I \subseteq \overleftarrow{\sigma}\sigma(I), J \subseteq \overleftarrow{\sigma}\sigma(J) \subseteq \overleftarrow{\sigma}\sigma(I \vee J)$. Then $\overleftarrow{\sigma}\sigma(I \vee J)$ is an upper bound of both I and J. Suppose $K \in I_\sigma(L)$ such that $I, J \subseteq K$. Then $I \vee J \subseteq K$. Thus $\overleftarrow{\sigma}\sigma(I \vee J) \subseteq \overleftarrow{\sigma}\sigma(K) = K$. Then $\overleftarrow{\sigma}\sigma(I \vee J)$ is the supremum of I and J. Clearly $\overleftarrow{\sigma}\sigma(I \vee J)$ is a σ-ideal of L. It is clear that $\{0\}, L \in I_\sigma(L)$. So $(I_\sigma(L), \veebar, \wedge, \{0\}, L)$ is a bounded lattice. Now, let $I, J, H \in I_\sigma(L)$. Then by the distributivity of $I(L)$ we get $I \wedge (J \veebar H) = I \cap \overleftarrow{\sigma}\sigma(J \vee H) = \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(J \vee H) = \overleftarrow{\sigma}\sigma(I \cap (J \vee H)) = \overleftarrow{\sigma}\sigma((I \cap J) \vee (I \cap H)) = (I \cap J) \veebar (I \cap H) = (I \wedge J) \veebar (I \wedge H)$. Therefore $(I_\sigma(L), \veebar, \wedge, \{0\}, L)$ is a distributive lattice. □

Theorem 3.7. *Let L be distributive p-algebra. Then the lattice $I_\sigma(L)$ is isomorphic to the lattice $I(B_*(L))$ of all ideals of $B_*(L)$.*

Proof. Let φ be the restriction of $\sigma : I(L) \to I(B_*(L))$ to $I_\sigma(L)$. Then $\varphi(I) = \sigma(I), I \in I_\sigma(I)$. Let $\varphi(I) = \varphi(J)$. Then $\sigma(I) = \sigma(J)$ implies $I = \overleftarrow{\sigma}\sigma(I) = \overleftarrow{\sigma}\sigma(J) = J$. So φ is an injective map. Now we prove that φ is a surjective map. For any $I \in I(B_*(L))$. Then $\overleftarrow{\sigma}(I)$ is an ideal of L and $\sigma\overleftarrow{\sigma}(I) = I$. We observe that $\overleftarrow{\sigma}(I) \in I_\sigma(L)$ because of $\overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}(I)\} = \overleftarrow{\sigma}(I)$. Then we have $\varphi(\overleftarrow{\sigma}(I)) = \sigma\overleftarrow{\sigma}(I) = I$. Therefore φ is a surjective map. Since σ is a homomorphism and $\sigma\overleftarrow{\sigma}\sigma(I \vee J)) = \sigma(I \vee J)$ we get

$$\varphi(I \wedge J) = \sigma(I \cap J) = \sigma(I) \cap \sigma(J) = \varphi(I) \cap \varphi(J)$$
$$\varphi(I \veebar J) = \sigma\overleftarrow{\sigma}\sigma(I \vee J) = \sigma(I \vee J) = \sigma(I) \sqcup \sigma(J) = \varphi(I) \sqcup \varphi(J)$$

for every $I, J \in I_\sigma(L)$. Therefore φ is an isomorphism. □

For any ideal I of a distributive p-algebra L. Consider $I^* = \{x \in L : x \geq i^*, i \in I\}$ and $I_{**} = \{x \in L : x \leq i^{**}, i \in I\}$. It is easy to check that I^* ia a filter of L and I_{**} is an ideal of L. If $I = (a]$, we observe that $(a]_{**} = (a^{**}] = (a)^\triangle$.

Theorem 3.8. *Let I be an ideal of be a distributive p-algebra L. Then*

(1) $I = I_{**}$ if and only if $(i \in I \Rightarrow i^{**} \in I)$,

(2) I is a σ-ideal of L if and only if $I = I_{**}$,

(3) I is a σ-ideal of L implies $(i)^\triangle \subseteq I$ for all $i \in I$.

Proof. (1). Let $I = I_{**}$ and $i \in I$. Then we get $i \leq a^{**} \in I_{**}$ for some $a \in I$. Then $i^{**} \leq a^{**} \in I$ implies $i^{**} \in I$. Conversely, Suppose $(i \in I \Rightarrow i^{**} \in I)$ and $x \in I_{**}$. Then $x \leq i^{**}$ for some $i \in I$. Hence $i^{**} \in I$. Therefore $x \in I$. Thus $I_{**} \subseteq I$. Obviously $I \subseteq I_{**}$. Therefore $I = I_{**}$.
(2). Suppose $I = I_{**}$. Then $I \subseteq \overleftarrow{\sigma}\sigma(I)$. Let $x \in \overleftarrow{\sigma}\sigma(I)$, then $(x)^\triangle \in \sigma(I)$. Thus $(x)^\triangle = (y)^\triangle$ for some $y \in I$. Then $x \wedge y^* = 0$ implies $x \leq y^{**} \in I$. So $x \in I$ and $\overleftarrow{\sigma}\sigma(I) \subseteq I$. Therefore I is a σ-ideal of L. Conversely, let $z \in I_{**}$. So we have $x \leq i^{**}$ for some $i \in I$ and $(i)^\triangle \in \sigma(I)$. Since $(i)^\triangle = (i^{**})^\triangle$, we get $i^{**} \in I$. Therefore $z \in I$ and $I_{**} \subseteq I$. Clearly $I \subseteq I_{**}$. Thus $I_{**} = I$.
(3). Let $x \in (i)^\triangle$, then $x \wedge i^* = 0$. Thus $x \leq i^{**} \in I_{**} = I$. Therefore $x \in I$. □

4 σ-ideals and homomorphisms

In this section, some properties of the homomorphic images and the inverse images of σ-ideals are studied. By a homomorphism on a distributive p-algebra L, we mean a lattice homomorphism h which preserves the pseudo-complementation i.e. $(h(x))^* = h(x^*)$ for all $x \in L$.

Theorem 4.1. *Let $h : L \to M$ be a homomorphism of a distributive p-algebra L onto a distributive p-algebra M. Then*

(1) $h(a)^{\triangle} = (h(a))^{\triangle}$ for all $a \in L$,

(2) $h(I) \in I_{\sigma}(M)$ for all $I \in I_{\sigma}(L)$,

(3) $I \in I_{\sigma}(L) \Rightarrow (h(i))^{\triangle} \subseteq (h(i))^{\triangle} \subseteq h(I)$ for all $i \in I$.

Proof. (1). Let $a \in L$. Then we get $h(a)^{\triangle} = h\{x \in L : x \wedge a^* = 0\} = \{h(x) \in M : h(x) \wedge h(a^*) = h(0)\} = \{h(x) \in M : h(x) \wedge (h(a))^* = 0\} = (h(a))^{\triangle}$.

(2). For every ideal I of L, it is known that $h(I)$ is an ideal of M. Hence $h(I) \subseteq \overleftarrow{\sigma}\sigma(h(I))$. So we have to prove only that $\overleftarrow{\sigma}\sigma(h(I)) \subseteq h(I)$. Let $y \in \overleftarrow{\sigma}\sigma(h(I))$. Then $(y)^{\triangle} \in \sigma(h(I))$. Hence $(y)^{\triangle} = (z)^{\triangle}$ for some $z \in h(I)$. Thus we get $y \wedge (h(a))^* = 0$ where $z = h(a), a \in I$. Hence $y \leq (h(a))^{**} = h(a^*) \in h(I_{**}) = h(I) \Rightarrow y \in h(I)$. Therefore $h(I)$ is a σ-ideal of M.

(3). It can be proved similarly. □

Theorem 4.2. *Let $h : L \to M$ be a homomorphism of a distributive p-algebra L into a distributive p-algebra M. Then*

(1) $h^{-1}(H) \in I_{\sigma}(L)$ for all $H \in I_{\sigma}(M)$,

(2) $Kerh \in I_{\sigma}(L)$.

Proof. (1). Since $h^{-1}(H)$ is an ideal of L, we get $h^{-1}(H) \subseteq \overleftarrow{\sigma}\sigma(h^{-1}(H))$. Let $x \in \overleftarrow{\sigma}\sigma(h^{-1}(H))$. Then $(x)^{\triangle} \in \sigma(h^{-1}(H))$. Hence $(y)^{\triangle} = (x)^{\triangle}$ for some $y \in h^{-1}(H)$. Thus $(h(y)^{\triangle}) = (h(x)^{\triangle}), h(y) \in H$. Hence $h(x) \in H$ which implies $x \in h^{-1}(H)$. Then $h^{-1}(H)$ is a σ-ideal of L.

(2). Clearly $Kerh \subseteq \overleftarrow{\sigma}\sigma(Kerh)$. Let $x \in \overleftarrow{\sigma}\sigma(Kerh)$. Then $(x)^{\triangle} \in \sigma(kerh)$. Hence $(x)^{\triangle} = (y)^{\triangle}$ for some $y \in Kerh$. Hence $x \wedge y^* = 0$. Then $h(x \wedge y^*) = 0$. Hence $h(x) \wedge (h(y))^* = 0$, which means $h(x) = 0$ as $(h(y))^* = 1$. Thus $x \in Kerh$. Then $\overleftarrow{\sigma}\sigma(Kerh) \subseteq Kerh$. Hence $Kerh$ is a σ-ideal of L. □

Theorem 4.3. *Let $h : L \to M$ be a homomorphism of a distributive p-algebra L onto a distributive p-algebra M. Then*

(1)$B_*(L)$ is homomorphic of $B_*(M)$,

(2)$I_{\sigma}(L)$ is homomorphic of $I_{\sigma}(M)$.

Proof. (1). Define $g : B_*(L) \to B_*(M)$ by $g(a)^{\triangle} = h(a)^{\triangle}$. For every $(a)^{\triangle}, (b)^{\triangle} \in B_*(L)$, we have $g((a)^{\triangle} \cap (b)^{\triangle}) = g(a \wedge b)^{\triangle} = h(a \wedge b)^{\triangle} = (h(a \wedge b))^{\triangle} = (h(a) \wedge h(b))^{\triangle} = h(a)^{\triangle} \cap h(b)^{\triangle} = g(a)^{\triangle} \cap g(b)^{\triangle}$. Also $g((a)^{\triangle} \sqcup (b)^{\triangle}) = g(a \vee b)^{\triangle} = h(a \vee b)^{\triangle} = (h(a \vee \vee b))^{\triangle} = (h(a) \vee h(b))^{\triangle} = h(a)^{\triangle} \sqcup h(b)^{\triangle} = g(a)^{\triangle} \sqcup g(b)^{\triangle}$. Also $g((a)^{\triangle -}) = h((a)^{\triangle -}) = h(a^*)^{\triangle} = ((h(a))^*)^{\triangle} = (h(a)^{\triangle})^-$. Clearly $g(0_L)^{\triangle} = (0_M)^{\triangle}$ and $g(1_L)^{\triangle} = (1_M)^{\triangle}$, where $0_L, 0_M$ are the smallest elements of L and M respectively and $1_L, 1_M$ are the greatest elements of L and M respectively. Therefore g is a Boolean homomorphism.

(2). Define the map $\pi : I_\sigma(L) \to I_\sigma(M)$ by $\pi(I) = h(I)$. It is clear that π is a $\{0, 1, \wedge\}$-homomorphism. So we have to prove that $\pi(I \veebar J) = \pi(I) \veebar \pi(J)$. Since $\overleftarrow{\sigma}\sigma(I \vee J)$ is a σ-ideal of L then $h\{\overleftarrow{\sigma}\sigma(I \vee J)\}$ is a σ-ideal of M. Now we have $\pi(I) \veebar \pi(J) = h(I) \veebar h(J) = \overleftarrow{\sigma}\sigma\{h(I) \vee h(J)\} = \overleftarrow{\sigma}\sigma\{h(I \vee J)\} \subseteq \overleftarrow{\sigma}\sigma\{h\{\overleftarrow{\sigma}\sigma(I \vee J)\}\} = h\{\overleftarrow{\sigma}\sigma(I \vee J)\} = \pi(I \veebar J)$.

Conversely, for $(i)^{\triangle}, (j)^{\triangle} \in I_\sigma(L)$, we have

$$(i)^{\triangle} \veebar (j)^{\triangle} = \overleftarrow{\sigma}\sigma\{(i)^{\triangle} \vee (j)^{\triangle}\} = \overleftarrow{\sigma}\sigma(i \vee j)^{\triangle} = (i \vee j)^{\triangle} = (i)^{\triangle} \sqcup (j)^{\triangle}$$

Now, we can get

$$\begin{aligned}
y \in \pi(I \veebar J) = h\{\overleftarrow{\sigma}\sigma(I \vee J)\} \quad &\Rightarrow \quad y = h(x) \text{ where } x \in \overleftarrow{\sigma}\sigma(I \vee J) \\
&\Rightarrow \quad (x)^{\triangle} \in \sigma(I \vee J) \\
&\Rightarrow \quad (x)^{\triangle} = (z)^{\triangle} \text{ for some } z \in I \vee J \\
&\Rightarrow \quad (x)^{\triangle} = (z)^{\triangle} \text{ and } z \le i \vee j, i \in I, j \in J \\
&\Rightarrow \quad (x)^{\triangle} = (z)^{\triangle} \subseteq (i \vee j)^{\triangle} \\
&\Rightarrow \quad (y)^{\triangle} = (h(x))^{\triangle} = h((x)^{\triangle}) \subseteq h(i \vee j)^{\triangle} \\
&\Rightarrow \quad y \in (y)^{\triangle} \subseteq (h(i))^{\triangle} \sqcup (h(j))^{\triangle} \\
&\Rightarrow \quad y \in h(I) \veebar h(J) \\
&\Rightarrow \quad y \in \pi(I) \veebar \pi(J)
\end{aligned}$$

Therefore π is a homomorphism and the proof is completed. □

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