Chamchuri Journal of Mathematics
Volume 6(2014), 31–46
http://www.math.sc.chula.ac.th/cjm

Closure Ideals of MS-Algebras

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Received 7 Jan 2014
Revised 21 Apr 2014
Accepted 11 June 2014

Abstract: The concepts of dominator ideals and closure ideals are introduced in MS-algebras and many properties of these ideals are studied. Closure ideals are characterized in terms of principal dominator ideals. It is then proved that the lattice of all closure ideals is isomorphic to the ideal lattice of the lattice of all principal dominator ideals. A set of equivalent conditions is obtained to characterize closure ideals of MS-algebras. Finally some properties of closure ideals are studied with respect to homomorphisms.

Keywords: MS-algebras, ideals, dominator ideals, principal dominator ideals, closure ideals, homomorphisms

2000 Mathematics Subject Classification: 06D05, 06D30

Introduction

An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contains the well-known classes for examples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras [4]. T. S. Blyth and J. C. Varlet [2] defined a subclass of Ockham algebras so called MS-algebras which generalizes both de Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by J. Berman [1]. The class of all MS-algebras form an equational class. T. S. Blyth and J. C. Varlet characterized the subvarieties of MS-algebras in [3]. Recently, Luo and Zeng

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[6] characterized the MS-algebras on which all congruences are in a one-to-one correspondence with the kernel ideals. In [7], M. Sambasiva Rao, introduced the concepts of boosters and β-filters of MS-algebras.

In this paper, we defined dominators, dominator ideals and principal dominator ideals in MS-algebras and some basic properties of dominators, dominator ideals and principal dominator ideals are studied. It is proved that the set of all principal dominator ideals of an MS-algebra can be made into a de Morgan algebra. The concept of closure ideals is introduced in MS-algebras. Many properties of closure ideals of an MS-algebra are observed. It is proved that the class $I_c(L)$ of all closure ideals of an MS-algebra L is a bounded distributive lattice. It is proved that $I_c(L)$ is isomorphic to the ideal lattice of $M_{\circ\circ}(L)$. A set of equivalent conditions is obtained to characterize closure ideals of MS-algebras by means of principal dominator ideals. Finally, some properties of closure ideals are studied with respect to homomorphisms. The concept of dominator ideal preserving homomorphism from an MS-algebra L into another MS-algebra L_1 is introduced as a homomorphism h satisfying the condition $h(I_{\circ\circ}) = \{h(I)\}_{\circ\circ}$, for any ideal I of L. It is proved that the images and the inverse images, under this homomorphism, of a closure ideal are again closure ideals. If an MS-algebras L is homomorphic to an MS-algebra L_1, then the lattice $M_{\circ\circ}(L)$ of all principal dominators of L is homomorphic to $M_{\circ\circ}(L_1)$ the lattice $M_{\circ\circ}(L_1)$ of all principal dominators of L_1 and the lattice of all closure ideals of L is homomorphic to the lattice of all closure ideals of L_1.

1 Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [2], [3] and [4] for the ready reference of the reader.

Definition 1.1. A de Morgan algebra is an algebra $(L, \vee, \wedge, ^-, 0, 1)$ of type (2,2,1,0,0) where $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $^-$ the unary operation of involution satisfies :

$$\overline{\overline{x}} = x, \overline{(x \vee y)} = \overline{x} \wedge \overline{y}, \overline{(x \wedge y)} = \overline{x} \vee \overline{y}.$$

Definition 1.2. An MS-algebra is an algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type (2,2,1,0,0) where $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $^\circ$ the unary operation of involution satisfies :

$$x \leq x^{\circ\circ}, (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ}, 1^{\circ} = 0.$$

We recall some of the basic properties of MS-algebras which were proved in [2].

Theorem 1.3. *For any two elements a, b of an MS-algebra L, we have*

(1) $0^{\circ} = 1$

(2) $a \leq b \Rightarrow b^{\circ} \leq a^{\circ}$

(3) $a^{\circ\circ\circ} = a^{\circ}$

(4) $(a \vee b)^{\circ} = a^{\circ} \wedge b^{\circ}$

(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$

(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

For any MS-algebra L we can define the set of skeleton elements $L^{\circ\circ} = \{a \in L : a = a^{\circ\circ}\}$. It is known that $(L^{\circ\circ}, \vee, \wedge, ^{\circ}, 0, 1)$ is a de Morgan subalgebra of L. An element $a \in L$ is called a dense element if $a^{\circ} = 0$. Then the set $D(L)$ of all dense elements of L forms a filter in L.

2 Dominators, dominator ideals and principal dominator ideals

In this section, the concepts of dominators, dominator ideals and principal dominator ideals are introduced in MS-algebras. Many properties of dominators, dominator ideals and principal dominator ideals are investigated.

Definition 2.1. For any non-empty subset A of an MS-algebra L, define the dominator of A as follows :

$$A_{\circ\circ} = \{x \in L : x \leq a^{\circ\circ} \text{ for some } a \in A\}.$$

Obviously $\{0\}_{\circ\circ} = \{0\}$ and $L_{\circ\circ} = L$.

Lemma 2.2. *For any two subsets A, B of an MS-algebra L, We have the following :*

(1) $A \subseteq A_{\circ\circ}$,

(2) $A \subseteq B$ *implies* $A_{\circ\circ} \subseteq B_{\circ\circ}$,

(3) $\{A_{\circ\circ}\}_{\circ\circ} = A_{\circ\circ}$.

Proof. (1). Since $a \leq a^{\circ\circ}$, it is clear that $A \subseteq A_{\circ\circ}$.
(2). It is obvious
(3). Clearly $A_{\circ\circ} \subseteq \{A_{\circ\circ}\}_{\circ\circ}$. Let $x \in \{A_{\circ\circ}\}_{\circ\circ}$. Then $x \leq a^{\circ\circ}$ for some $a \in A_{\circ\circ}$. Hence $a \leq c^{\circ\circ}$ for some $c \in A$. Thus we have $x \leq c^{\circ\circ}$. Then $x \in A_{\circ\circ}$ and $\{A_{\circ\circ}\}_{\circ\circ} \subseteq \{A_{\circ\circ}\}$. Consequently, $\{A_{\circ\circ}\}_{\circ\circ} = A_{\circ\circ}$. □

Lemma 2.3. *For any two ideals I and J of an MS-algebra L, the following hold:*

(1) $I_{\circ\circ}$ *is an ideal of* L*,*

(2) $(I \cap J)_{\circ\circ} = I_{\circ\circ} \cap J_{\circ\circ}$*,*

(3) $(I \vee J)_{\circ\circ} = I_{\circ\circ} \vee J_{\circ\circ}$.

Proof. (1) . Clearly $0 \in I_{\circ\circ}$. Let $x, y \in I_{\circ\circ}$. Then $x \leq i^{\circ\circ}, y \leq j^{\circ\circ}$ for some $i, j \in I$. Hence $x \vee y \leq i^{\circ\circ} \vee j^{\circ\circ} = (i \vee j)^{\circ\circ}$ implies $x \vee y \in I_{\circ\circ}$. Again, let $x \in I_{\circ\circ}$ and $z \leq x$. Then $x \leq i^{\circ\circ}$ for some $i \in I$. So, $z \leq i^{\circ\circ}$ implies $z \in I_{\circ\circ}$. Therefore $I_{\circ\circ}$ is an ideal of L.
(2). Clearly $(I \cap J)_{\circ\circ} \subseteq I_{\circ\circ} \cap J_{\circ\circ}$. Let $x \in I_{\circ\circ} \cap J_{\circ\circ}$. Then $x \leq i^{\circ\circ}$ and $x \leq j^{\circ\circ}$ for some $i \in I$ and $j \in J$. So $x \leq (i \wedge j)^{\circ\circ}$. Thus $x \in (I \cap J)_{\circ\circ}$ as $i \wedge j \in I \cap J$. Then $I_{\circ\circ} \cap J_{\circ\circ} \subseteq (I \cap J)_{\circ\circ}$. Therefore $(I \cap J)_{\circ\circ} = I_{\circ\circ} \cap J_{\circ\circ}$.
(3). Clearly $I_{\circ\circ} \vee J_{\circ\circ} \subseteq (I \vee J)_{\circ\circ}$. Conversely, let $x \in (I \vee J)_{\circ\circ}$. Then $x \leq a^{\circ\circ}$ for some $a \in I \vee J$. Hence $a = i \vee j$ for some $i \in I$ and $j \in J$. So, we have $x \leq a^{\circ\circ} = (i \vee j)^{\circ\circ} = i^{\circ\circ} \vee j^{\circ\circ} \in I_{\circ\circ} \vee J_{\circ\circ}$. Therefore $(I \vee J)_{\circ\circ} \subseteq I_{\circ\circ} \vee J_{\circ\circ}$. □

The result (2) of the above Lemma 3.3 can be generalized as follows:

Corollary 2.4. *If $\{I_i : i \in \Delta\}$ is a family of ideals of L, then* $\{\bigcap_{i \in \Delta} I_i\}_{\circ\circ} = \bigcap_{i \in \Delta} (I_i)_{\circ\circ}$.

Definition 2.5. An ideal I of an MS-algebra L is called a dominator ideal if $I = I_{\circ\circ}$.

We denote the set of all dominator ideals of L by $I_{\circ\circ}(L)$. Then by above lemma, it is obvious that $I_{\circ\circ}(L)$ is a bounded distributive lattice.

Definition 2.6. For any element a of an MS-algebra L, the dominator $\{a\}_{\circ\circ}$ is called a principal dominator ideal.

Then it can be easily observed that $\{0\}_{\circ\circ} = \{0\}$ and $\{1\}_{\circ\circ} = L$. The following Lemma is a direct consequence of the above definition.

Lemma 2.7. *For any two elements a, b of an MS-algebra L, we have*

(1) $\{a\}_{\circ\circ} = (a]_{\circ\circ} = (a^{\circ\circ}]$,

(2) $(a]_{\circ\circ} = (a^{\circ\circ}]_{\circ\circ}$

(3) $a \in (b]_{\circ\circ} \Leftrightarrow (a]_{\circ\circ} \subseteq (b]_{\circ\circ}$

(4) $a \leq b \Rightarrow (a]_{\circ\circ} \subseteq (b]_{\circ\circ}$,

(5) a is fixed point of L implies $(a]_{\circ\circ} = (a^{\circ}]_{\circ\circ}$,

(6) $(a]_{\circ\circ} = (a] \Leftrightarrow a \in L^{\circ\circ}$,

(7) $(a]_{\circ\circ} = L \Leftrightarrow a \in D(L)$,

(8) $(a]_{\circ\circ} = \{0\} \Leftrightarrow a = 0$.

Let us denote the set of all principal dominator ideals of L by $M_{\circ\circ}(L)$. Then some of the basic properties of $M_{\circ\circ}(L)$ are investigated in the following Theorem.

Theorem 2.8. *Let L be an MS-algebra. Then we have the following conditions*

(1) $M_{\circ\circ}(L)$ is a bounded sublattice of the lattice $I_{\circ\circ}(L)$,

(2) L is homomorphic to $M_{\circ\circ}(L)$,

(3) $M_{\circ\circ}(L)$ is a de Morgan algebra,

(4) $L^{\circ\circ}$ is isomorphic to $M_{\circ\circ}(L)$.

Proof. (1). Clearly $\{0\}, L \in M_{\circ\circ}(L)$. For every $(a]_{\circ\circ}, (b]_{\circ\circ}$ of $M_{\circ\circ}(L)$, by Lemma 3.3(2),(3), we have

$$\begin{aligned} (a]_{\circ\circ} \vee (b]_{\circ\circ} &= ((a] \vee (b])_{\circ\circ} = (a \vee b]_{\circ\circ} \\ (a]_{\circ\circ} \cap (b]_{\circ\circ} &= ((a] \cap (b])_{\circ\circ} = (a \wedge b]_{\circ\circ}. \end{aligned}$$

Then $(M_{\circ\circ}(L), \vee, \cap, \{0\}, L)$ is a bounded sublattice of $I_{\circ\circ}(L)$.

(2). Define the map $f : L \to M_{\circ\circ}(L)$ by $f(a) = (a]_{\circ\circ}$. Clearly $f(0) = \{0\}, f(1) = L$. Using Lemma 3.3(2),(3), it can be easily seen that f is a homomorphism.

(3). Define the unary operation $^{-}$ on $M_{\circ\circ}(L)$ by $\overline{(a]}_{\circ\circ} = (a^{\circ}]_{\circ\circ}$. Then we get

$$\begin{aligned} \overline{\overline{(a]}}_{\circ\circ} &= \overline{(a^{\circ}]}_{\circ\circ} = (a^{\circ\circ}]_{\circ\circ} = (a]_{\circ\circ}, \\ \overline{((a]_{\circ\circ} \cap (b]_{\circ\circ})} &= \overline{(a \wedge b]}_{\circ\circ} = ((a \wedge b)^{\circ}]_{\circ\circ} = (a^{\circ} \vee b^{\circ}]_{\circ\circ} = (a^{\circ}]_{\circ\circ} \vee (b^{\circ}]_{\circ\circ} = \overline{(a]}_{\circ\circ} \vee \overline{(b]}_{\circ\circ}, \\ \overline{((a]_{\circ\circ} \vee (b]_{\circ\circ})} &= \overline{(a \vee b]}_{\circ\circ} = ((a \vee b)^{\circ}]_{\circ\circ} = (a^{\circ} \wedge b^{\circ}]_{\circ\circ} = (a^{\circ}]_{\circ\circ} \cap (b^{\circ}]_{\circ\circ} = \overline{(a]}_{\circ\circ} \cap \overline{(b]}_{\circ\circ}, \\ \overline{(0]}_{\circ\circ} &= (0^{\circ}]_{\circ\circ} = L. \end{aligned}$$

Therefore $(M_{\circ\circ}(L), \vee, \cap, ^{-}, \{0\}, L)$ is a de Morgan algebra.

(4). It is easy to check that the map $g : L^{\circ\circ} \to M_{\circ\circ}(L)$ defined by $g(a) = (a^{\circ\circ}]_{\circ\circ}$ is an isomorphism. Then $L^{\circ\circ} \cong M_{\circ\circ}(L)$. □

Theorem 2.9. *Let I be an ideal of an MS-algebra L. Then $I_{\circ\circ} = \bigcup_{i\in I}(i]_{\circ\circ}$.*

Proof. Let $x \in \bigcup_{i\in I}(i]_{\circ\circ}$. Then $x \in (a]_{\circ\circ}$ for some $a \in I$. Then $x \le a^{\circ\circ}$ implies $x \in I_{\circ\circ}$. Therefore $\bigcup_{i\in I}(i]_{\circ\circ} \subseteq I_{\circ\circ}$. Conversely, let $x \in I_{\circ\circ}$. Then $x \le i^{\circ\circ}$ for some $i \in I$. So, $x \in (i^{\circ\circ}]_{\circ\circ} = (i]_{\circ\circ} \subseteq \bigcup_{i\in I}(i]_{\circ\circ}$. Thus $I_{\circ\circ} \subseteq \bigcup_{i\in I}(i]_{\circ\circ}$. Therefore $I_{\circ\circ} = \bigcup_{i\in I}(i]_{\circ\circ}$. □

3 Closure ideals of MS-algebras

In this section, the notion of closure ideals is introduced in MS-algebras. The class of all closure ideals is characterized by means of principal dominator ideals.

Definition 3.1. Let L be an MS-algebra. For any ideal I of L, define an operator $\sigma : I(L) \to I(M_{\circ\circ}(L))$ as follows :

$$\sigma(I) = \{(i]_{\circ\circ} : i \in I\}$$

Definition 3.2. Let L be an MS-algebra. For any ideal $\tilde{I}$ of $M_{\circ\circ}(L)$, define an operator $\overleftarrow{\sigma} : M_{\circ\circ}(L) \to I(L)$ as follows:

$$\overleftarrow{\sigma}(\tilde{I}) = \{x \in L : (x]_{\circ\circ} \in \tilde{I}\}$$

Lemma 3.3. *Let L be an MS-algebra. Then we have the following :*

(1) for any ideal I of L, $\sigma(I)$ is an ideal of $M_{\circ\circ}(L)$,

(2) for any ideal $\tilde{I}$ of $M_{\circ\circ}(L)$, $\overleftarrow{\sigma}(\tilde{I})$ is an ideal of L,

(3) $\overleftarrow{\sigma}$ and σ are isotones,

(4) $\sigma\overleftarrow{\sigma}(\tilde{I}) = \tilde{I}$, for all ideal $\tilde{I}$ of $M_{\circ\circ}(L)$,

(5) σ is a homomorphism.

Proof. (1). Let I be an ideal of L. Clearly $\{0\} \in \sigma(I)$ as $0 \in I$. For any $(x]_{\circ\circ}, (y]_{\circ\circ} \in \sigma(I)$, we get $(x]_{\circ\circ} \vee (y]_{\circ\circ} = (x \vee y]_{\circ\circ} \in \sigma(I)$ as $x \vee y \in I$. Again let $(x]_{\circ\circ} \in \sigma(I)$ and $(z]_{\circ\circ} \in M_{\circ\circ}(L)$ such that $(z]_{\circ\circ} \subseteq (x]_{\circ\circ}$. So, $(z]_{\circ\circ} = (z]_{\circ\circ} \cap (x]_{\circ\circ} = (z \wedge x]_{\circ\circ} \in \sigma(I)$ as $z \wedge x \in I$. Thus $(z]_{\circ\circ} \in \sigma(I)$. Therefore $\sigma(I)$ is an ideal of $M_{\circ\circ}(L)$.

(2). Let $\tilde{I}$ be an ideal of $M_{\circ\circ}(L)$. Then $0 \in \overleftarrow{\sigma}(\tilde{I})$ as $(0]_{\circ\circ} \in \tilde{I}$. Let $x, y \in \overleftarrow{\sigma}(\tilde{I})$. Then $(x \vee y]_{\circ\circ} = (x]_{\circ\circ} \vee (y]_{\circ\circ} \in \tilde{I}$ implies $x \vee y \in \overleftarrow{\sigma}(\tilde{I})$. Now let $x \in \overleftarrow{\sigma}(\tilde{I})$ and $y \leq x$. Since $(y]_{\circ\circ} = (y]_{\circ\circ} \cap (x]_{\circ\circ} \in \tilde{I}$, then $y \in \overleftarrow{\sigma}(\tilde{I})$. Therefore $\overleftarrow{\sigma}(\tilde{I})$ is an ideal of L.

(3). Let $\tilde{I}, \tilde{H}$ be two ideals of $M_{\circ\circ}(L)$. Suppose $\tilde{I} \subseteq \tilde{H}$ and $x \in \overleftarrow{\sigma}(\tilde{I})$. Then $(x]_{\circ\circ} \in \tilde{I} \subseteq \tilde{H}$ implies $x \in \overleftarrow{\sigma}(\tilde{H})$. Therefore $\overleftarrow{\sigma}$ is an isotone operator from the lattice $I(M_{\circ\circ}(L))$ of all ideals of $M_{\circ\circ}(L)$ to the lattice $I(L)$ of all ideals of L. Similarly, we can also prove that σ is an isotone operator.

(4). Let $\tilde{I}$ be an ideal of $M_{\circ\circ}(L)$. Then $\overleftarrow{\sigma}(\tilde{I})$ is an ideal of L (by (2)). So we have

$$(x]_{\circ\circ} \in \tilde{I} \Leftrightarrow x \in \overleftarrow{\sigma}(\tilde{I}) \Leftrightarrow (x]_{\circ\circ} \in \sigma\overleftarrow{\sigma}(\tilde{I})$$

Then $\sigma\overleftarrow{\sigma}(\tilde{I}) = \tilde{I}$. So $\sigma\overleftarrow{\sigma} : I(M_{\circ\circ}(L)) \to I(M_{\circ\circ}(L))$ is the identity map.

(5). Let $I, J \in I(L)$. Since σ is isotone, we get that $\sigma(I \cap J) \subseteq \sigma(I) \cap \sigma(J)$. Conversely, let $(x]_{\circ\circ} \in \sigma(I) \cap \sigma(J)$. Then $(x]_{\circ\circ} = (i]_{\circ\circ}$ and $(x]_{\circ\circ} = (j]_{\circ\circ}$ for some $i \in I$ and $j \in J$. Now $(x]_{\circ\circ} = (i]_{\circ\circ} \cap (j]_{\circ\circ} = (i \wedge j]_{\circ\circ} \in \sigma(I \cap J)$. Therefore $\sigma(I) \cap \sigma(J) \subseteq \sigma(I \cap J)$. Since σ is an isotone, we get $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$. Conversely, let $x \in \sigma(I \vee J)$. Then $(x]_{\circ\circ} = (y]_{\circ\circ}$ for some $y \in I \vee J$. Hence $y = i \vee j$ for some $i \in I$ and $j \in J$. Thus $(y]_{\circ\circ} = (i \vee j]_{\circ\circ} = (i]_{\circ\circ} \vee (j]_{\circ\circ} \in \sigma(I) \vee \sigma(J)$. Therefore $\sigma(I \vee J) \subseteq \sigma(I) \vee \sigma(J)$. Then σ is a homomorphism from the lattice of ideals of L into the lattice of ideals of $M_{\circ\circ}(L)$. □

Theorem 3.4. *The map $\overleftarrow{\sigma}\sigma(I) : I(L) \to I(L)$ is a closure operator, that is*

(1) $I \subseteq \overleftarrow{\sigma}\sigma(I)$,

(2) $I \subseteq H$ implies $\overleftarrow{\sigma}\sigma(I) \subseteq \overleftarrow{\sigma}\sigma(H)$,

(3) $\overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(I)\} = \overleftarrow{\sigma}\sigma(I)$, for any $I, H \in I(L)$

Proof. (1). Let $x \in I$, then $(x]_{\circ\circ} \in \sigma(I)$. Since $\sigma(I)$ is an ideal of $M_{\circ\circ}(L)$, then $x \in \overleftarrow{\sigma}\sigma(I)$. Therefore $I \subseteq \overleftarrow{\sigma}\sigma(I)$.

(2). Suppose $I \subseteq H$. Let $x \in \overleftarrow{\sigma}\sigma(I)$. Hence $(x]_{\circ\circ} \in \sigma(I)$. We have $(x]_{\circ\circ} = (y]_{\circ\circ}$ for some $y \in I \subseteq H$. Then $(x]_{\circ\circ} = (y]_{\circ\circ} \in \sigma(H)$. Since $\sigma(H)$ is an ideal of $M_{\circ\circ}(L)$, then $x \in \overleftarrow{\sigma}\sigma(H)$. Therefore $\overleftarrow{\sigma}\sigma(I) \subseteq \overleftarrow{\sigma}\sigma(H)$.

(3). We have $\overleftarrow{\sigma}\sigma(I) \subseteq \overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(I)\}$ as $\overleftarrow{\sigma}\sigma(I)\}$ is an ideal of $I(L)$. Conversely, let $x \in \overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(I)\}$. Then $(x]_{\circ\circ} \in \sigma\{\overleftarrow{\sigma}\sigma(I)\}$. Hence $(x]_{\circ\circ} = (y]_{\circ\circ}$ for some $y \in \overleftarrow{\sigma}\sigma(I)$. Thus $(x]_{\circ\circ} = (y]_{\circ\circ} \in \sigma(I)$. So $x \in \overleftarrow{\sigma}\sigma(I)$. □

Corollary 3.5. *For any two ideals I,H of an MS-algebra L, we have the following:*

$$\overleftarrow{\sigma}\sigma(I\cap H)=\overleftarrow{\sigma}\sigma(I)\cap\overleftarrow{\sigma}\sigma(H)$$

Proof. Clearly $\overleftarrow{\sigma}\sigma(I\cap H)\subseteq\overleftarrow{\sigma}\sigma(I)\cap\overleftarrow{\sigma}\sigma(H)$. Conversely, let $x\in\overleftarrow{\sigma}\sigma(I)\cap\overleftarrow{\sigma}\sigma(H)$. Then $(x]_{\circ\circ}\in\sigma(I)\cap\sigma(H)=\sigma(I\cap H)$ as σ is a homomorphism. Then we have $x\in\overleftarrow{\sigma}\sigma(I\cap H)$. Therefore $\overleftarrow{\sigma}\sigma(I)\cap\overleftarrow{\sigma}\sigma(H)\subseteq\overleftarrow{\sigma}\sigma(I\cap H)$. □

Definition 3.6. An ideal I of an MS-algebra L is called a closure ideal if $\overleftarrow{\sigma}\sigma(I)=I$.

Theorem 3.7. *Let I be an ideal of an MS-algebra L. Then the following conditions are equivalent:*

(1) I is a closure ideal,

(2) For all $x,y\in L,(x]_{\circ\circ}=(y]_{\circ\circ}$ and $x\in I$ imply $y\in I$,

(3) $I=I_{\circ\circ}$,

(4) $I=\bigcup_{i\in I}(i]_{\circ\circ}$,

(5) $x\in I$ implies $(x]_{\circ\circ}\subseteq I$.

Proof. (1)$\Rightarrow$(2) : Assume that I is a closure ideal. Let $x,y\in L$ be such that $(x]_{\circ\circ}=(y]_{\circ\circ}$. Suppose that $x\in I=\overleftarrow{\sigma}\sigma(I)$. Then $(x]_{\circ\circ}=(y]_{\circ\circ}\in\sigma(I)$. Hence $y\in\overleftarrow{\sigma}\sigma(I)=I$.

(2) $\Rightarrow$(3) : By Lemma 3.2(1), $I\subseteq I_{\circ\circ}$. Let $x\in I_{\circ\circ}$, $x\leq i^{\circ\circ}$, for some $i\in I$. Then $(x]_{\circ\circ}\subseteq(i^{\circ\circ}]_{\circ\circ}=(i]_{\circ\circ},i\in I$ imply $i^{\circ\circ}\in I$ by condition (2). Hence $x\in I$. Therefore $I_{\circ\circ}\subseteq I$ and $I_{\circ\circ}=I$.

(3)$\Rightarrow$(4) : By Theorem 3.9, it is clear.

(4)$\Rightarrow$(5) : Assume the condition (4). Let $x\in I$, then $x\in(x]\subseteq(x]_{\circ\circ}$. Hence by condition (4) we get $x\in\bigcup_{i\in I}(i]_{\circ\circ}=I$.

(5) $\Rightarrow$ (1) : Assume the condition (5). Clearly, $I\subseteq\overleftarrow{\sigma}\sigma(I)$. Conversely, let $x\in\overleftarrow{\sigma}\sigma(I)$. Then $(x]_{\circ\circ}\in\sigma(I)$. Hence $(x]_{\circ\circ}=(y]_{\circ\circ}$ for some $y\in I$. Since $y\in I$, by condition (5), it yields $x\in(x]_{\circ\circ}=(y]_{\circ\circ}\subseteq I$. □

Lemma 3.8. *The following conditions are hold in any MS-algebra L*

(1) $(a]$ is a closure ideal if and only if $a\in L^{\circ\circ}$,

(2) For any ideal I of L, $\overleftarrow{\sigma}\sigma(I)=I_{\circ\circ}$,

(3) For any ideal I of L, $I_{\circ\circ}$ is a closure ideal,

(4) The map $\overleftarrow{\sigma}\sigma(I) : I(L) \to I(L)$ is a (0,1)-homomorphism.

Proof. (1). For all $a \in L^{\circ\circ}$, we have $(a]_{\circ\circ} = (a]$ by Lemma 3.7(6). Then

$$\begin{aligned}
\sigma((a]_{\circ\circ}) &= \sigma((a]) = \{(x]_{\circ\circ} : x \in (a]\} \\
&= \{(x]_{\circ\circ} : x \leq a\} \\
&= \{(x]_{\circ\circ} : (x]_{\circ\circ} \subseteq (a]_{\circ\circ}\} \\
&= ((a]_{\circ\circ}], \\
\overleftarrow{\sigma}\sigma((a]_{\circ\circ}) &= \overleftarrow{\sigma}\{((a]_{\circ\circ}]\} \\
&= \{x \in L : (x]_{\circ\circ} \in ((a]_{\circ\circ}]\} \\
&= \{x \in L : (x^{\circ\circ}]_{\circ\circ} = (x]_{\circ\circ} \subseteq (a]\} \\
&= \{x \in L : x^{\circ\circ} \in (a]\} \\
&= \{x \in L : x \leq x^{\circ\circ} \leq a\} \\
&= (a] = (a]_{\circ\circ}.
\end{aligned}$$

Then any principal ideal of an MS-algebra L generated by a closed element is a closure ideal. Conversely, let $I = (a]$ be a closure ideal of L. By Lemma 3.7(2), we have $(a]_{\circ\circ} = (a^{\circ\circ}]_{\circ\circ}$. Then $a \in (a]$ implies $a^{\circ\circ} \in (a]$ (by Theorem 4.7(2)). So, $a^{\circ\circ} \leq a$. But $a \leq a^{\circ\circ}$. Therefore $a = a^{\circ\circ}$ and a is a closed element of L.

(2). For any ideal I of L, $I_{\circ\circ} = \bigcup_{i\in I}(i]_{\circ\circ}$ (See Theorem 3.9). Then we prove that $\overleftarrow{\sigma}\sigma(I) = \bigcup_{i\in I}(i]_{\circ\circ}$. Let $x \in \bigcup_{i\in I}(i]_{\circ\circ}$. Then we get $x \in (i]_{\circ\circ}$ for some $i \in I$. Hence $(x]_{\circ\circ} \subseteq (i]_{\circ\circ}$. Now, $i \in I$ implies $(i]_{\circ\circ} \in \sigma(I)$. So, $(x]_{\circ\circ} \in \sigma(I)$ as $\sigma(I)$ is an ideal of $M_{\circ\circ}(L)$. Then $x \in \overleftarrow{\sigma}\sigma(I)$. Therefore $\bigcup_{i\in I}(i]_{\circ\circ} \subseteq \overleftarrow{\sigma}\sigma(I)$. Conversely, let $x \in \overleftarrow{\sigma}\sigma(I)$. Then we get $(x]_{\circ\circ} \in \sigma(I)$. Hence $(x]_{\circ\circ} = (i]_{\circ\circ}$ for some $i \in I$. Thus $x \in (x]_{\circ\circ} = (i]_{\circ\circ} \subseteq \bigcup_{i\in I}(i]_{\circ\circ}$, which yields that $\overleftarrow{\sigma}\sigma(I) \subseteq \bigcup_{i\in I}(i]_{\circ\circ}$. Hence $\overleftarrow{\sigma}\sigma(I) = \bigcup_{i\in I}(i]_{\circ\circ}$. Then $\overleftarrow{\sigma}\sigma(I) = I_{\circ\circ}$.

(3). From (2), we have $\overleftarrow{\sigma}\sigma(I_{\circ\circ}) = \{I_{\circ\circ}\}_{\circ\circ}$. By Lemma 3.2(3), $\{I_{\circ\circ}\}_{\circ\circ} = I_{\circ\circ}$. Then $\overleftarrow{\sigma}\sigma(I_{\circ\circ}) = I_{\circ\circ}$. Therefore $I_{\circ\circ}$ is a closure ideal of L.

(4). Firstly, we observe

$$\overleftarrow{\sigma}\sigma((0]) = \overleftarrow{\sigma}\{(0]\} = \{0\} \text{ and } \overleftarrow{\sigma}\sigma(L) = \overleftarrow{\sigma}\{M_{\circ\circ}(L)\} = L$$

Let $I, J \in I(L)$. Then by (2) and Lemma 3.3(2),(3) we get

$$\begin{aligned}
\overleftarrow{\sigma}\sigma(I \cap J) &= (I \cap J)_{\circ\circ} \\
&= I_{\circ\circ} \cap J_{\circ\circ} \\
&= \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(J), \\
\overleftarrow{\sigma}\sigma(I \vee J) &= (I \cap J)_{\circ\circ} \\
&= I_{\circ\circ} \vee J_{\circ\circ} \\
&= \overleftarrow{\sigma}\sigma(I) \vee \overleftarrow{\sigma}\sigma(J).
\end{aligned}$$

Therefore $\overleftarrow{\sigma}\sigma$ is a (0,1)-homomorphism. □

Now, for any MS-algebra L, let $I_c(L)$ denote to the set of all closure ideals of L. We will prove that the set $I_c(L)$ forms a bounded distributive lattice.

Theorem 3.9. *Let L be an MS-algebra, the set $I_c(L)$ forms a bounded distributive lattice.*

Proof. We have to show that the set $I_c(L) = \{I \in I(L) : \overleftarrow{\sigma}\sigma(I) = I\}$ is a sublattice of $I(L)$. Clearly, $\{0\}$ and L are closure ideals. Let $I, J \in I_c(L)$. Then by Corollary 4.5, we get

$$\overleftarrow{\sigma}\sigma(I \cap J) = \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(I) = I \cap J$$

Then $I \cap J \in I_c(L)$. Now Lemma 4.7(3) gives $I = I_{\circ\circ}$ and $J = J_{\circ\circ}$ and Lemma 3.3(3) gives $(I \vee J)_{\circ\circ} = I_{\circ\circ} \vee J_{\circ\circ}$. Then we have

$$\overleftarrow{\sigma}\sigma(I \vee J) = (I \vee J)_{\circ\circ} = I_{\circ\circ} \vee J_{\circ\circ} = I \vee J$$

Then $I \vee J$ is a closure ideal of L. Then $I_c(L)$ is a bounded sublattice of $I(L)$. Consequently, $(I_c(L), \vee, \cap, \{0\}, L)$ is a bounded distributive lattice. □

Theorem 3.10. *Let L be an MS-algebra. Then there is an isomorphism of the lattice of closure ideals of L onto the ideal lattice of $M_{\circ\circ}(L)$. Under this isomorphism the prime closure ideals corresponds to prime ideals of $M_{\circ\circ}(L)$.*

Proof. Define the mapping $g : I_c(L) \to I(M_{\circ\circ}(L))$ by $g(I) = \sigma(I)$. Obviously, $g(\{0\}) = \{0\}$ and $g(L) = M_{\circ\circ}(L)$. Now for any two closure ideals I and J of L. Since σ is a homomorphism, we get

$$\begin{aligned}
g(I \cap J) &= \sigma(I \cap J) = \sigma(I) \cap \sigma(J) = g(I) \cap g(J), \\
g(I \vee J) &= \sigma(I \vee J) = \sigma(I) \vee \sigma(J) = g(I) \vee g(J).
\end{aligned}$$

Then g is a (0,1)-lattice homomorphism. Let $\sigma(I)=\sigma(J)$, then $I=\overleftarrow{\sigma}\sigma(I)=\overleftarrow{\sigma}\sigma(J)=J$. Hence g is an injective map. Now we prove that g is a surjective map. Let $\tilde{I}\in I(M_{\circ\circ}(L))$. Then $\overleftarrow{\sigma}(\tilde{I})$ is an ideal of L (by Lemma 4.3(2)) and $\sigma\overleftarrow{\sigma}(\tilde{I})=\tilde{I}$ (by Lemma 4.3(4)). We observe that $\overleftarrow{\sigma}(\tilde{I})\in CI(L)$ because of $\overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}(\tilde{I})\}=\overleftarrow{\sigma}(\tilde{I})$. Then we have $g(\overleftarrow{\sigma}(\tilde{I}))=\sigma\overleftarrow{\sigma}(\tilde{I})=\tilde{I}$. Therefore $\tilde{I}$ is a surjective map. Therefore g is an isomorphism. We have obtained a one-to-one correspondence between prime closure ideals of L and the prime ideals of $M_{\circ\circ}(L)$. Let I be a prime closure ideal of L. Suppose $(x\wedge y]_{\circ\circ}=(x]_{\circ\circ}\wedge(y]_{\circ\circ}\in\sigma(I)$. Then $x\wedge y\in\overleftarrow{\sigma}\sigma(I)=I$ (because I is a closure ideal). Since I is prime ideal, we get $x\in I$ or $y\in I$, which implies $(x]_{\circ\circ}\in\sigma(I)$ or $(y]_{\circ\circ}\in\sigma(I)$. Therefore $g(I)=\sigma(I)$ is a prime ideal of $M_{\circ\circ}(L)$. Conversely, suppose that $\tilde{I}$ is a prime ideal in $M_{\circ\circ}(L)$. Since g is a surjective map, there exists a closure ideal I in L such that $\tilde{I}=g(I)=\sigma(I)$. Let $x,y\in L$ such that $x\wedge y\in I$. Then $(x]_{\circ\circ}\cap(y]_{\circ\circ}=(x\wedge y]_{\circ\circ}\in\sigma(I)$. Hence $(x]_{\circ\circ}\in\sigma(I)$ or $(y]_{\circ\circ}\in\sigma(I)$. Since I is a closure ideal, we get $x\in\overleftarrow{\sigma}\sigma(I)=I$ or $y\in\overleftarrow{\sigma}\sigma(I)=I$. Therefore I is a prime ideal of L. □

4 Closure ideals and homomorphisms

In this section, some properties of the homomorphic images and the inverse images of closure ideals are studied. By a homomorphism on an MS-algebra L, we mean a lattice homomorphism h which preserves $^\circ$, that is, $(h(x))^\circ=h(x^\circ)$ for all $x\in L$.

Lemma 4.1. *Let L and M be two MS-algebras and $h:L\to M$ a homomorphism. Then we have the following:*

(1) for any non-empty subset A of L, $h(A_{\circ\circ})\subseteq\{h(A)\}_{\circ\circ}$

(2) for any non-empty subset B of M, $\{h^{-1}(B)\}_{\circ\circ}\subseteq h^{-1}(B_{\circ\circ})$.

Proof. (1). Let $x\in h(A_{\circ\circ})$. Then there exists $b\in A_{\circ\circ}$ such that $x=h(b)$ and $b\leq a^{\circ\circ}$ for some $a\in A$. Then we get

$$x=h(b)\leq h(a^{\circ\circ})=(h(a))^{\circ\circ}\in\{h(A)\}_{\circ\circ}$$

So $x\in\{h(A)\}_{\circ\circ}$ and $h(A_{\circ\circ})\subseteq\{h(A)\}_{\circ\circ}$.
(2). Let $x\in\{h^{-1}(B)\}_{\circ\circ}$. Then $x\leq a^{\circ\circ}$ for some $a\in h^{-1}(B)$. Then $h(a)\in B$ implies $h(x)\leq h(a^{\circ\circ})=(h(a))^{\circ\circ}\in B_{\circ\circ}$. Then $x\in h^{-1}(B_{\circ\circ})$. □

In the following example we show that $h(A_{\circ\circ}) \subseteq \{h(A)\}_{\circ\circ}$ and $h^{-1}(B_{\circ\circ}) \subseteq \{h^{-1}(B)\}_{\circ\circ}$ are not true in general.

Example 4.2. Let L be the five element chain $0 < a < b < d < 1$ and $a^{\circ} = b^{\circ} = b, d^{\circ} = 0$. Clearly L is an MS-algebra. Define $h : L \to L$ by $h(0) = 0, h(a) = h(b) = b, h(d) = d$ and $h(1) = 1$. Then clearly h is a homomorphism on L. Take $A = B = \{0, a\}$. Then clearly $A_{\circ\circ} = \{0, a, b\}$ and $h(A) = \{0, b\}$. Hence $h(A_{\circ\circ}) = \{0, b\}$ and $\{h(A)\}_{\circ\circ} = \{0, a, b\}$. Thus $\{h(A)\}_{\circ\circ}\} \nsubseteq h(A_{\circ\circ})$. Also, we have $h^{-1}(B) = \{0\}$ and $B_{\circ\circ} = \{0, a, b\}$. Then $\{h^{-1}(B)\}_{\circ\circ} = \{0\}$ and $h^{-1}(B_{\circ\circ}) = \{0, a, b\}$. Thus $h^{-1}(B_{\circ\circ}) \nsubseteq \{h^{-1}(B)\}_{\circ\circ}$.

We now define the concept of dominator ideal preserving homomorphism.

Definition 4.3. Let $h : L \to M$ be a homomorphism of an MS-algebra L into an MS-algebra M. Then h is called dominator ideal preserving if $h(I_{\circ\circ}) = \{h(I)\}_{\circ\circ}$, for any ideal I of L.

Theorem 4.4. *Let $h : L \to M$ be a homomorphism of an MS-algebra L onto an MS-algebra M. Then h is dominator ideal preserving.*

Proof. Let I be an ideal of L. Since h is an onto homomorphism, then $h(I)$ and $h(I_{\circ\circ})$ are ideals of M. Thus by Lemma 3.1 we have $h(I_{\circ\circ}) \subseteq \{h(I)\}_{\circ\circ}$. Conversely, let $y \in \{h(I)\}_{\circ\circ}$. Then we have

$$\begin{aligned} y \leq x^{\circ\circ}, \text{ for some } x \in h(I) \quad &\Rightarrow \quad y \leq x^{\circ\circ} = (h(a))^{\circ\circ}, a \in I \\ &\Rightarrow \quad y \leq h(a) \leq (h(a))^{\circ\circ} = h(a^{\circ\circ}) \in h(I_{\circ\circ}) \\ &\Rightarrow \quad y \in h(I_{\circ\circ}). \end{aligned}$$

Then $\{h(I)\}_{\circ\circ} \subseteq h(I_{\circ\circ})$. Therefore h is dominator ideal preserving. □

Now we study some properties of closure ideals of MS-algebras with respect to homomorphisms.

Theorem 4.5. *Let $h : L \to M$ be a homomorphism of an MS-algebra L onto an MS-algebra M. Then we have*

(1) for any $a \in L$, $h((a]_{\circ\circ}) = (h(a)]_{\circ\circ}$,

(2) for any closure ideal I of L, $h(I)$ is a closure ideal of M,

(3) for any closure ideal I of L, $h(I) = \bigcup_{i \in I} (h(i)]_{\circ\circ}$.

Proof. (1). For all $a \in L$, we get

$$\begin{aligned} h((a]_{\circ\circ}) &= h\{x \in L : x \leq a^{\circ\circ}\} \\ &= \{h(x) \in M : h(x) \leq (h(a))^{\circ\circ}\} \\ &= ((h(a))^{\circ\circ}]_{\circ\circ} = (h(a)]_{\circ\circ}. \end{aligned}$$

(2). For every ideal I of L, it is known that $h(I)$ is an ideal of M. By Theorem 3.7(1), we have $h(I) \subseteq \overleftarrow{\sigma}\sigma(h(I)$. So we have to prove only that $\overleftarrow{\sigma}\sigma(h(I)) \subseteq h(I)$

$$\begin{aligned} y \in \overleftarrow{\sigma}\sigma(h(I)) &\Rightarrow (y]_{\circ\circ} \in \sigma(h(I)) \\ &\Rightarrow (y]_{\circ\circ} = (z]_{\circ\circ}, \text{ for some } z \in h(I) \\ &\Rightarrow y \leq z^{\circ\circ} \in h(I_{\circ\circ}) = h(I) \\ &\Rightarrow y \in h(I). \end{aligned}$$

Then $h(I)$ is a closure ideal of M.

(3). For any closure ideal I of L, $I = I_{\circ\circ} = \bigcup_{i \in I}(i]_{\circ\circ}$. Now, for any $i \in I$, $(i]_{\circ\circ} \subseteq I$. Then $(h(i)]_{\circ\circ} \subseteq h(I)$. So $\bigcup_{i \in I}(h(i)]_{\circ\circ} \subseteq h(I)$. Conversely, let $y \in h(I)$. Then there exists $x \in I$ such that $y = h(i) \in (h(i)]_{\circ\circ} \subseteq \bigcup_{i \in I}(i]_{\circ\circ}$. □

Theorem 4.6. *Let L and M be two MS-algebras and $h : L \to M$ a homomorphism. Then we have the following :*

(1) for any closure ideal H of M, $h^{-1}(H)$ is a closure ideal of L,

(2) $\ker h$ is a closure ideal of L,

(3) for a closure ideal B of M, $h^{-1}(B_{\circ\circ}) = \{h^{-1}(B)\}_{\circ\circ}$.

Proof. (1). Since $h^{-1}(H)$ is an ideal of L, then by Theorem 3.7(1), we have $h^{-1}(H) \subseteq \overleftarrow{\sigma}\sigma(h^{-1}(H))$. Conversely,

$$\begin{aligned} x \in \overleftarrow{\sigma}\sigma(h^{-1}(H)) &\Rightarrow (x]_{\circ\circ} \in \sigma(h^{-1}(H)) \\ &\Rightarrow (y]_{\circ\circ} = (x]_{\circ\circ}, \text{ for some } y \in h^{-1}(H) \\ &\Rightarrow (h(y)]_{\circ\circ} = (h(x)]_{\circ\circ}), h(y) \in H \\ &\Rightarrow h(x) \in H \text{ by Lemma 4.2(2)} \\ &\Rightarrow x \in h^{-1}(H). \end{aligned}$$

Therefore $h^{-1}(H)$ is a closure ideal of L.

(2). Since $\ker h = \{x \in L : h(x) = 0\}$ is an ideal of L, then $\ker h \subseteq \overleftarrow{\sigma}\sigma(\ker h)$. For the converse, we get

$$\begin{aligned} x \in \overleftarrow{\sigma}\sigma(\ker h) &\Rightarrow (x]_{\circ\circ} \in \sigma(\ker h) \\ &\Rightarrow (x]_{\circ\circ} = (y]_{\circ\circ}, \text{ for some } y \in \ker h \\ &\Rightarrow x \leq y^{\circ\circ} \in \ker h \text{ as } h(y^{\circ\circ}) = 0 \\ &\Rightarrow x \in \ker h. \end{aligned}$$

Then $\overleftarrow{\sigma}\sigma(\ker h) \subseteq \ker h$. So $\ker h$ is a closure ideal of L.
(3). Since B is a closure ideal of M, then $B = B_{\circ\circ}$ and $h^{-1}(B)$ is a closure ideal of L. Then we get $h^{-1}(B_{\circ\circ}) = h^{-1}(B) = \{h^{-1}(B)\}_{\circ\circ}$. Therefore h^{-1} is dominator ideal preserving. □

Theorem 4.7. *Let $h : L \to L_1$ be an onto homomorphism between MS-algebras L and L_1. Then we have*

(1) $M_{\circ\circ}(L)$ is homomorphic of $M_{\circ\circ}(L_1)$,

(2) $I_c(L)$ is homomorphic of $I_c(L_1)$.

Proof. (1). Define $g : M_{\circ\circ}(L) \to M_{\circ\circ}(L_1)$ by $g(a]_{\circ\circ} = h(a]_{\circ\circ}$. For every

$(a]_{\circ\circ}, (b]_{\circ\circ} \in M_{\circ\circ}(L)$ we get,

$$\begin{aligned}
g((a]_{\circ\circ} \cap (b]_{\circ\circ}) &= g(a \wedge b]_{\circ\circ} \\
&= h(a \wedge b]_{\circ\circ} \\
&= (h(a \wedge b)]_{\circ\circ} \\
&= (h(a) \wedge h(b)]_{\circ\circ} \\
&= (h(a)]_{\circ\circ} \cap (h(b)]_{\circ\circ} \\
&= h(a]_{\circ\circ} \cap h(b]_{\circ\circ} \\
&= g(a]_{\circ\circ} \cap g(b]_{\circ\circ}, \\
g((a]_{\circ\circ} \vee (b]_{\circ\circ}) &= h(a \vee b]_{\circ\circ} = (h(a \vee b)]_{\circ\circ} \\
&= (h(a) \vee h(b)]_{\circ\circ} \\
&= (h(a)]_{\circ\circ} \vee (h(b)]_{\circ\circ} \\
&= h(a]_{\circ\circ} \vee h(b]_{\circ\circ} \\
&= g(a]_{\circ\circ} \vee g(b]_{\circ\circ}, \\
g(\overline{(a]}_{\circ\circ}) &= h(\overline{(a]}_{\circ\circ}) \\
&= h(a^{\circ}]_{\circ\circ} = ((h(a))^{\circ}]_{\circ\circ} \\
&= \overline{(h(a]_{\circ\circ})} = \overline{(g(a]_{\circ\circ})}.
\end{aligned}$$

Clearly $g(0_L]_{\circ\circ} = (0_{L_1}]_{\circ\circ}$ and $g(1_L]_{\circ\circ} = (1_{L_1}]_{\circ\circ}$, where $0_L, 0_{L_1}$ are the smallest elements of L and L_1 respectively and $1_L, 1_{L_1}$ are the greatest elements of L and L_1 respectively. Therefore g is a de Morgan homomorphism.

(2). Define the map $\pi : I_c(L) \to I_c(L_1)$ by $\pi(I) = h(I)$. It is clear that $\pi\{0_1\} = \{0_{L_1}\}$ and $\pi(L) = L_1$. Also, we get

$$\begin{aligned}
\pi(I \vee J) &= h(I \vee J) \\
&= h(I) \vee h(J) \\
&= \pi(I) \vee \pi(J), \\
\pi(I \cap J) &= h(I \cap J) \\
&= h(I) \cap h(J) \\
&= \pi(I) \cap \pi(J).
\end{aligned}$$

Therefore π is a (0,1)-lattice homomorphism. □

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