Chamchuri Journal of Mathematics
VOLUME 6(2014), 89–96
http://www.math.sc.chula.ac.th/cjm

On Banach Algebras Induced by a Certain Product

Ali Reza Khoddami

Received 17 July 2014
Revised 12 November 2014
Accepted 19 November 2014

Abstract: We obtain characterization of bounded approximate identities for Banach algebras induced by Lau product of Banach algebras defined by a Banach algebra homomorphism. Also we characterize the unitization and minimal idempotents of these algebras. Finally we study the ideal structure of these algebras. Also we extend a result of Sangani Monfared.

Keywords: bounded approximate identity, $T-$Lau product, minimal idempotent, unitization

2010 Mathematics Subject Classification: 46H05, 46J05

1 introduction

Let A and B be Banach algebras and let A be commutative. Suppose that $T: B \to A$ is an algebra homomorphism with $\|T\| \leq 1$. Then the direct product $A \times B$ equipped with the ℓ^1-norm and the algebra multiplication

$$(a,b) \cdot (c,d) = (ac + T(d)a + T(b)c, bd), \quad (a,c \in A,\ b,d \in B),$$

is an associative Banach algebra which is called the $T-$Lau product of A and B and will be denoted by $A \times_T B$. Some properties of this algebra such as, Arens regularity, amenability, weak amenability, character inner amenability are investigated in [1].

$n-$ weak amenability and character amenability of $A\times_T B$ are investigated in [3, 4]. For a Banach algebra B with $\theta \in \triangle(B) =$(the spectrum of B) and for an arbitrary unital Banach algebra A define $T: B \to A$ as $T(x)=\theta(x)e$ $(x \in B)$. Then the above product coincides with the product investigated by Sangani Monfared [5]. It is called the $\theta-$Lau product. Also the $\theta-$Lau product of A and B is denoted by $A\times_\theta B$. Character inner amenability of $A\times_\theta B$ is investigated in [2].
For an arbitrary Banach algebra A, a left (right) bounded approximate identity is a bounded net $(a_\alpha)_\alpha$ in A such that $\|a_\alpha a - a\| \to 0$ $(\|aa_\alpha - a\| \to 0)$ $(a \in A)$. A bounded approximate identity is a bounded net that is left and right approximate identity.
Recall that a non-zero element $\eta \in B$ is called a minimal idempotent if $\eta^2=\eta$ and $\eta B \eta = \mathbb{C}\eta$.

2 bounded approximate identities

In this section we characterize bounded approximate identities of $A\times_T B$. Also we obtain the unitization of $A\times_T B$.

Theorem 2.1. *Let A be a commutative Banach algebra, let B be a Banach algebra, and let $T: B\to A$ be an algebra homomorphism with $\|T\| \le 1$. Then*

i. $A\times_T B$ is commutative if and only if B is commutative.

ii. $(\tilde{a},\tilde{b})$ is an identity for $A\times_T B$ if and only if $\tilde{b}$ is an identity for B and $\tilde{a}+T(\tilde{b})$ is an identity for A.

iii. $((a_\alpha,b_\alpha))_\alpha$ is a bounded left (right, or two-sided) approximate identity for $A\times_T B$ if and only if $(a_\alpha+T(b_\alpha))_\alpha$ is a bounded approximate identity for A and $(b_\alpha)_\alpha$ is a bounded left (right, or two-sided) approximate identity for B. A similar statement is true for unbounded approximate identities.

Proof. $i)$ is clear. For $ii)$ let $(\tilde{a},\tilde{b})$ be an identity for $A\times_T B$. So for each $(a,b)\in A\times_T B$;

$$(a,b)(\tilde{a},\tilde{b}) = (\tilde{a},\tilde{b})(a,b) = (a,b).$$

It follows that

$$b\tilde{b} = \tilde{b}b = b; \tag{1}$$

$$a(\tilde{a}+T(\tilde{b})) + T(b)\tilde{a} = a. \tag{2}$$

Upon substituting $b=0$ in (2), we obtain $a(\tilde{a}+T(\tilde{b}))=a$. This implies that $\tilde{a}+T(\tilde{b})$ is an identity for A.

Conversely, let $\tilde{b}$ be an identity for B and let $\tilde{a}+T(\tilde{b})$ be an identity for A. Since $b\tilde{b}=b$ so $T(b)T(\tilde{b})=T(b)$. It follows that $T(b)\tilde{a}=T(b)(\tilde{a}+T(\tilde{b}))-T(b)T(\tilde{b})=T(b)-T(b)=0,\ (b\in B)$. Hence

$$(a,b)(\tilde{a},\tilde{b})=(a(\tilde{a}+T(\tilde{b}))+T(b)\tilde{a},b\tilde{b})=(a,b).$$

Similarly $(\tilde{a},\tilde{b})(a,b)=(a,b)$. So $(\tilde{a},\tilde{b})$ is an identity for $A\times_T B$.

iii) We only prove the left version. Let $((a_\alpha,b_\alpha))_\alpha$ be a bounded left approximate identity for $A\times_T B$. So for each $(a,b)\in A\times_T B$,

$$\|(a_\alpha,b_\alpha)(a,b)-(a,b)\|=\|((a_\alpha+T(b_\alpha))a+T(b)a_\alpha-a,b_\alpha b-b)\|\longrightarrow 0.$$

It follows that

$$\|(a_\alpha+T(b_\alpha))a+T(b)a_\alpha-a\|\longrightarrow 0; \tag{3}$$

$$\|b_\alpha b-b\|\longrightarrow 0. \tag{4}$$

(4) implies that $(b_\alpha)_\alpha$ is a bounded left approximate identity for B. Also upon substituting $b=0$ in (3) we conclude that $(a_\alpha+T(b_\alpha))_\alpha$ is a bounded approximate identity for A.

Conversely, let $(b_\alpha)_\alpha$ be a bounded left approximate identity for B and let $(a_\alpha+T(b_\alpha))_\alpha$ be a bounded approximate identities for A. So for each $b\in B$, $T(b_\alpha)T(b)\longrightarrow T(b)$ and also $(a_\alpha+T(b_\alpha))T(b)\longrightarrow T(b)$. It follows that

$$T(b)a_\alpha=(a_\alpha+T(b_\alpha))T(b)-T(b_\alpha)T(b)\longrightarrow 0. \tag{5}$$

For each $(a,b)\in A\times_T B$,

$$\begin{aligned}\|(a_\alpha,b_\alpha)(a,b)-(a,b)\| &= \|((a_\alpha+T(b_\alpha))a+T(b)a_\alpha-a,b_\alpha b-b)\|\\ &= \|(a_\alpha+T(b_\alpha))a+T(b)a_\alpha-a\|+\|b_\alpha b-b\|\\ &\le \|(a_\alpha+T(b_\alpha))a-a\|+\|T(b)a_\alpha\|+\|b_\alpha b-b\|\\ &\longrightarrow 0.\end{aligned}$$

Hence $(a_\alpha,b_\alpha)_\alpha$ is a left bounded approximate identity. □

Proposition 2.2. *Let A be a commutative Banach algebra, let B be a Banach algebra, and let $T:B\to A$ be an algebra homomorphism with $\|T\|\le 1$. Then*

$(A \times_T B)^\sharp \cong A \times_{\tilde{T}} B^\sharp$ *(isometrically isomorphism), where* $\tilde{T} : B^\sharp \to A^\sharp$ *is the algebra homomorphism defined by* $\tilde{T}((b,\lambda)) = (T(b),\lambda),\ (b \in B, \lambda \in \mathbb{C})$.

Proof. Define $\varphi : (A \times_T B)^\sharp \to A \times_{\tilde{T}} B^\sharp$ by $\varphi(((a,b),\lambda)) = (a,(b,\lambda))$. Clearly φ is a bijective bounded linear map. we shall show that φ is an algebra isomorphism. $\varphi(((a,b),\lambda)((c,d),\mu))$

$$\begin{aligned}
&= \varphi(((ac + T(b)c + T(d)a, bd) + (\lambda c, \lambda d) + (\mu a, \mu b), \lambda\mu)) \\
&= \varphi(((ac + T(b)c + T(d)a + \lambda c + \mu a, bd + \lambda d + \mu b), \lambda\mu)) \\
&= (ac + T(b)c + T(d)a + \lambda c + \mu a, (bd + \mu b + \lambda d, \lambda\mu)) \\
&= (a,(b,\lambda))(c,(d,\mu)) = \varphi(((a,b),\lambda))\varphi(((c,d),\mu)).
\end{aligned}$$

Also

$$\begin{aligned}
\|\varphi(((a,b),\lambda))\| &= \|(a,(b,\lambda))\| \\
&= \|a\| + \|(b,\lambda)\| = \|a\| + \|b\| + |\lambda| = \|(a,b)\| + |\lambda| \\
&= \|((a,b),\lambda)\|.
\end{aligned}$$

□

3 minimal idempotents

In this section we characterize minimal idempotents of $A \times_T B$.

Theorem 3.1. *An element* $(\tilde{a},\tilde{b})$ *is a minimal idempotent of* $A \times_T B$ *if and only if one of the following statements are hold:*

i. $\tilde{b} = 0$ *and* $\tilde{a}$ *is a minimal idempotent of* A.

ii. $\tilde{a}^2 = -\tilde{a}$, $\tilde{b}$ *is a minimal idempotent of* B, *and* $(\tilde{a} + T(\tilde{b}))a = 0,\quad (a \in A)$.

Proof. Let $(\tilde{a},\tilde{b})$ be a minimal idempotent. So the condition of $(\tilde{a},\tilde{b})^2 = (\tilde{a},\tilde{b})$ implies that

$$\tilde{b}^2 = \tilde{b}, \tag{6}$$

$$\tilde{a}^2 + 2T(\tilde{b})\tilde{a} = \tilde{a}. \tag{7}$$

For each $(a,b) \in A \times_T B$ there exists $\mu_{a,b} \in \mathbb{C}$ such that

$$(\tilde{a},\tilde{b})(a,b)(\tilde{a},\tilde{b}) = \mu_{a,b}(\tilde{a},\tilde{b}),$$

equivalently, the fact that A is commutative and also the equality $\tilde{b}^2 = \tilde{b}$ imply that

$$\tilde{a}^2 a + 2T(\tilde{b})a\tilde{a} + 2T(\tilde{b})T(b)\tilde{a} + T(b)\tilde{a}^2 + T(\tilde{b})a = \mu_{a,b}\tilde{a}, \tag{8}$$

$$\tilde{b}b\tilde{b} = \mu_{a,b}\tilde{b}. \tag{9}$$

The comparison of (9) and (6) shows that either $\tilde{b} = 0$, or $\tilde{b}$ is a minimal idempotent of B with $\mu_{a,b} = \mu_{0,b}$ for all $a \in A$.
The assumption $\tilde{b} = 0$ accompanied with (7) imply $\tilde{a}^2 = \tilde{a}$ and (8) implies that $\tilde{a}^2 a + T(b)\tilde{a}^2 = \mu_{a,b}\tilde{a}$, $(a, b) \in A \times_T B$.
Upon substituting $b = 0$, we conclude that $\tilde{a}a\tilde{a} = \tilde{a}^2 a = \mu_{a,0}\tilde{a}$, $(a \in A)$. It follows that $\tilde{a}$ is a minimal idempotent that provides statement $i)$.
If we assume $\tilde{b} \neq 0$ is a minimal idempotent, and $\mu_{a,b} = \mu_{0,b}$ for all $a \in A$, as $\mu_{a,0} = 0$ for all $a \in A$, upon substituting $b = 0$ in (8) we conclude that $\tilde{a}^2 a + 2T(\tilde{b})a\tilde{a} + T(\tilde{b})a = 0$. It follows that

$$0 = (\tilde{a}^2 + 2T(\tilde{b})\tilde{a})a + T(\tilde{b})a = \tilde{a}a + T(\tilde{b})a = (\tilde{a} + T(\tilde{b}))a, \quad (a \in A).$$

So $T(\tilde{b})\tilde{a} = -\tilde{a}^2$. Applying (7) we obtain that $\tilde{a}^2 = -\tilde{a}$, providing $ii)$. Conversely, Let $\tilde{a}$ be a minimal idempotent of A. So $\tilde{a}^2 = \tilde{a}$ and $\tilde{a}a\tilde{a} = \mu_a\tilde{a}$ for all $a \in A$. We shall show that $(\tilde{a}, 0)$ is a minimal idempotent of $A \times_T B$. Clearly $(\tilde{a}, 0)^2 = (\tilde{a}, 0)$. For each $(a, b) \in A \times_T B$,

$$\begin{aligned}(\tilde{a}, 0)(a, b)(\tilde{a}, 0) &= (\tilde{a}a + T(b)\tilde{a}, 0)(\tilde{a}, 0)\\ &= (\tilde{a}a\tilde{a} + T(b)\tilde{a}^2, 0) = (\tilde{a}a\tilde{a}, 0) + (\tilde{a}T(b)\tilde{a}, 0)\\ &= (\mu_a\tilde{a}, 0) + (\mu_{T(b)}\tilde{a}, 0) = (\mu_a + \mu_{T(b)})(\tilde{a}, 0).\end{aligned}$$

It follows that $(\tilde{a}, 0)$ is a minimal idempotent of $A \times_T B$.
Similarly let $\tilde{a} \in A$, $\tilde{b} \in B$ and let the conditions of $ii)$ hold. we shall show that $(\tilde{a}, \tilde{b})$ is a minimal idempotent of $A \times_T B$. As $\tilde{b}$ is a minimal idempotent so $\tilde{b}^2 = \tilde{b}$ and for each $b \in B$ there exists $\mu_b \in \mathbb{C}$ such that $\tilde{b}b\tilde{b} = \mu_b\tilde{b}$.
Since by hypothesis for each $a \in A$, $(\tilde{a} + T(\tilde{b}))a = 0$, we have $T(\tilde{b})\tilde{a} = -\tilde{a}^2 = \tilde{a}$. So

$$(\tilde{a}, \tilde{b})^2 = (\tilde{a}^2 + 2T(\tilde{b})\tilde{a}, \tilde{b}^2) = (\tilde{a}^2 - 2\tilde{a}^2, \tilde{b}^2) = (\tilde{a}, \tilde{b}).$$

Since $(\tilde{a}+T(\tilde{b}))T(b)\tilde{a}=0$, it follows that

$$\begin{aligned}-T(b)\tilde{a} &= -T(\tilde{b})T(b)\tilde{a}\\ &= -T(\tilde{b}b\tilde{b})\tilde{a}\\ &= -\mu_b T(\tilde{b})\tilde{a} = -\mu_b(-\tilde{a}^2)\\ &= -\mu_b\tilde{a}.\end{aligned}$$

So $T(b)\tilde{a}=\mu_b\tilde{a}$. For each $(a,b)\in A\times_T B$,

$$\begin{aligned}(\tilde{a},\tilde{b})(a,b)(\tilde{a},\tilde{b}) &= (\tilde{a}a+T(\tilde{b})a+T(b)\tilde{a},\tilde{b}b)(\tilde{a},\tilde{b})\\ &= ((\tilde{a}+T(\tilde{b}))a+T(b)\tilde{a},\tilde{b}b)(\tilde{a},\tilde{b})\\ &= (T(b)\tilde{a},\tilde{b}b)(\tilde{a},\tilde{b}) = (T(b)\tilde{a}^2+2T(b)T(\tilde{b})\tilde{a},\tilde{b}b\tilde{b})\\ &= (-T(b)\tilde{a}+2T(b)(-\tilde{a}^2),\mu_b\tilde{b}) = (-T(b)\tilde{a}+2T(b)\tilde{a},\mu_b\tilde{b})\\ &= (T(b)\tilde{a},\mu_b\tilde{b}) = (\mu_b\tilde{a},\mu_b\tilde{b})\\ &= \mu_b(\tilde{a},\tilde{b}).\end{aligned}$$

This shows that $(\tilde{a},\tilde{b})$ is a minimal idempotent. □

4 ideal structures of $A\times_T B$

In this section we characterize the ideal structures of $A\times_T B$.

Theorem 4.1. *Let A be a commutative Banach algebra, let B be a Banach algebra, and let $T:B\longrightarrow A$ be an algebra homomorphism with $\|T\|\leq 1$. Suppose that I is an ideal of A, J is a left (right or two-sided) ideal of B, and $M=I\times J$. Then*
M is a left (right or two-sided) ideal of $A\times_T B$ if and only if $T(J)A\subseteq I$.

Proof. We only prove the left-version. Let I be an ideal of A and let J be a left ideal of B, then M is a left ideal of $A\times_T B$ if and only if $(a,b)(x,y)\in M$ for all $(a,b)\in A\times_T B$, $(x,y)\in M$. It follows that $(ax+T(b)x+T(y)a,by)\in M$, which is equivalent to $T(y)a\in I, y\in J, a\in A$. Equivalently, $T(J)A\subseteq I$. □

As an immediate consequence of Theorem 4.1 we present a result of Sangani-Monfared [5, Proposition 2.6].

Corollary 4.2. *Let A be a unital Banach algebra and let B be a Banach algebra such that $\theta \in \triangle(B)$. Suppose that I is a left (right or two-sided) ideal of A, J is a left (right or two-sided) ideal of B, and $M = I \times J$. Then M is a left (right or two-sided) ideal of $A \times_\theta B$ if and only if $J \subseteq ker\theta$ or $I = A$.*

For the converse of Theorem 4.1 suppose that M is a left (right or two-sided) ideal of $A \times_T B$ and let,

i. $I = \{a \in A : (a,b) \in M$ for some $b \in B\}$;

ii. $J = \{b \in B : (a,b) \in M$ for some $a \in A\}$.

In general neither I is a left (right or two-sided) ideal, nor the equality $M = I \times J$ holds [5, Examples 2.8]. In this situation we can present the following result.

Proposition 4.3. *Let $M, I,$ and J be as above. Then*

i. J is a left (right or two-sided) ideal of B;

ii. If $T(J)A \subseteq I$ then I is an ideal of A.

Proof. We only prove part ii) for the left version. Let M be a left ideal of $A \times_T B$ and let $\tilde{a} \in I$. There exists $\tilde{b} \in B$ such that $(\tilde{a}, \tilde{b}) \in M$. It follows that $\tilde{b} \in J$ and for each $a \in A$, $(a,0)(\tilde{a}, \tilde{b}) \in M$. This implies that $a\tilde{a} + T(\tilde{b})a \in I$. As $T(\tilde{b})a \in I$ it follows that $a\tilde{a} \in I$. Since A is commutative so I is an ideal of A. □

Acknowledgements: The author would like to thank the referee for careful reading of the paper and giving some useful suggestions.

References

[1] S.J. Bhatt and P.A. Dabhi, Arens regularity and amenability of Lau product of Banach algebras defined by a Banach algebra morphism, *Bull. Aust. Math. Soc.*, (2012), 1–12.

[2] H.R. Ebrahimi Vishki and A.R. Khoddami, Character inner amenability of certain Banach algebras, *Colloq. Math.*, **122** (2011), 225–232.

[3] A.R. Khoddami, $n-$weak amenability of $T-$Lau product of Banach algebras, *Chamchuri J. Math.*, **5** (2013), 57–65.

[4] A.R. Khoddami, On character amenability of $T-$Lau product of Banach algebras, to appear.

[5] M. Sangani-Monfared, On certain products of Banach algebras with applications to harmonic analysis, *Studia Math.*, **178** (2007), 277–294.

Ali Reza Khoddami
Department of Pure Mathematics
University of Shahrood
P.O. BOX 3619995161-316, Shahrood, Iran.
Email: `khoddami.alireza@shahroodut.ac.ir`