Chamchuri Journal of Mathematics
VOLUME 8(2016), 1–17
http://www.math.sc.chula.ac.th/cjm

Some improvements on weak convergence theorems of Chuang and Takahashi in Hilbert spaces

Chanitnan Jaipranop* and Satit Saejung†

Received 9 May 2016
Revised 26 December 2016
Accepted 26 December 2016

Abstract: Chuang and Takahashi [3] recently proved three weak convergence theorems for a family of firmly nonexpansive mappings with generalized parameters. We discuss these three results for a family of k-demicontractive mappings where $k \leq 1$. Obviously, the class of k-demicontractive mappings contains all firmly nonexpansive mappings. The situation $k = 1$ is extensively studied by means of the Ishikawa iteration and the extragradient method of Korpelevič. Some numerical results for $k = 1$ are presented and further discussed.

Keywords: fixed point, k-demicontractive mapping, Mann iteration, Ishikawa iteration, extragradient method

2000 Mathematics Subject Classification: 47H09, 47H10

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty subset of $\mathcal{H}$. An element $x \in C$ is called a *fixed point* of a mapping $T : C \to \mathcal{H}$ if $x = Tx$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

* *The author is supported by the Human Resource Development in Science Project (Science Achievement Scholarship of Thailand, SAST)*
† *Corresponding author*

Our work is inspired by the recent work of Chuang and Takahashi [3]. They proved three weak convergence theorems for a family of firmly nonexpansive mappings. Their results are interesting because their iterations are established with generalized parameters. In the previous work of Mann [8], the parameter is taken in $[0,1]$ while Chuang and Takahashi's work allows the wider interval of parameters in $[0,2]$. We continue the study of these works and extend the class of firmly nonexpansive mappings to that of k-demicontractive mappings where $k \leq 1$. Note that every firmly nonexpansive mapping with a fixed point is just (-1)-demicontractive. Hence our work includes theorems of Chuang and Takahashi as a special case. We also discuss the 1-demicontractive case. This class is very interesting and beyond the scope of the work of Chuang and Takahashi. We use two techniques in the work of Kraikaew and Saejung [7] in this situation. Some numerical results are also presented and discussed.

2 Preliminaries

Throughout this paper, we use $\to$ and $\rightharpoonup$ for the strong and weak convergences, respectively. We write $x_n \equiv x$ for the statement $x_n = x$ for all $n \geq 1$.

Definition 2.1. [4] Let C be a nonempty subset of $\mathcal{H}$ and k be a real number. We say that a mapping $T : C \to \mathcal{H}$ is k*-pseudocontractive* if $\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2$ for all $x, y \in C$. If T is 1-pseudocontractive, then it is simply called *pseudocontractive.* If T is k-pseudocontractive where $k < 1$, then it is usually called *strictly pseudocontractive.* If T is 0-pseudocontractive, then it is called *nonexpansive.* If T is (-1)-pseudocontractive, then it is called *firmly nonexpansive.*

Definition 2.2. [4] Let C be a nonempty subset of $\mathcal{H}$ and k be a real number. We say that a mapping $T : C \to \mathcal{H}$ is k*-demicontractive* if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - p\|^2 \leq \|x-p\|^2 + k\|x - Tx\|^2$ for all $p \in \mathrm{Fix}(T), x \in C$. If T is 0-demicontractive, then it is called *quasi-nonexpansive.* If T is (-1)-demicontractive, then it is called *quasi-firmly nonexpansive.*

Remark 2.3.

1. Every k-pseudocontractive mapping with a fixed point is k-demicontractive.
2. Let C be a nonempty, closed and convex subset of $\mathcal{H}$. If $T : C \to \mathcal{H}$ is quasi-nonexpansive, then $\mathrm{Fix}(T)$ is closed and convex.

Lemma 2.4. *Let C be a nonempty subset of $\mathcal{H}$. Let $T : C \to \mathcal{H}$ be a k-demicontractive mapping. Let $S := (1-\alpha)I + \alpha T$ where α is a nonnegative real number and I is an identity mapping. Then for all $x \in C$ and $p \in \mathrm{Fix}(T)$,*

$$\|Sx - p\|^2 \leq \|x-p\|^2 - \alpha(1-k-\alpha)\|x - Tx\|^2.$$

In addition, if $\alpha \in \,]0, 1-k[$, then $\mathrm{Fix}(S) = \mathrm{Fix}(T)$ and S is quasi-nonexpansive.

Proof. Let $x \in C$ and $p \in \mathrm{Fix}(T)$. We have

$$\begin{aligned} \|Sx-p\|^2 &= \|(1-\alpha)(x-p) + \alpha(Tx-p)\|^2 \\ &= (1-\alpha)\|x-p\|^2 + \alpha\|Tx-p\|^2 - \alpha(1-\alpha)\|x-Tx\|^2 \\ &\leq \|x-p\|^2 - \alpha(1-k-\alpha)\|x-Tx\|^2. \end{aligned}$$

If $\alpha \in \,]0, 1-k[$, then $\mathrm{Fix}(T) = \mathrm{Fix}(S)$ and S is quasi-nonexpansive. □

The following conditions are studied in [3].

Definition 2.5. Let C be a nonempty subset of $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ be a sequence of mappings and $\mathcal{T}$ be a family of mappings from C into $\mathcal{H}$. Suppose that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. We say that

1. $\{T_n\}_{n=1}^{\infty}$ satisfies the *resolvent property* if there exists a nonexpansive mapping $T : C \to \mathcal{H}$ and $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ and there exist $n_0, k \geq 1$ such that $\|x - Tx\| \leq k\|x - T_n x\|$ for all $x \in C$ and for all $n \geq n_0$. In this situation, we also say that $\{T_n\}_{n=1}^{\infty}$ satisfies the *resolvent property with a nonexpansive mapping T*.

2. $\{T_n\}_{n=1}^{\infty}$ satisfies the *AKTT-condition* if the following two conditions are satisfied:

   (a) $\sum_{n=1}^{\infty} \sup_{x\in B} \|T_{n+1}x - T_n x\| < \infty$ for each nonempty and bounded subset B of C. (In particular, the sequence $\{T_n x\}_{n=1}^{\infty}$ is Cauchy for all $x \in C$.)

   (b) The mapping $T : C \to \mathcal{H}$ given by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in C$ satisfies the property $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

   In this situation, we also say that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the *AKTT-condition.*

3. $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ satisfies the *NST-condition* if

(a) $\mathrm{Fix}(\mathcal{T}) := \bigcap_{T\in\mathcal{T}} \mathrm{Fix}(T) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

(b) For each bounded sequence $\{z_n\}_{n=1}^{\infty} \subset C$, $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ implies that $\lim_{n\to\infty} \|z_n - T z_n\| = 0$ for all $T \in \mathcal{T}$.

Remark 2.6. 1. If $\{T_n\}_{n=1}^{\infty}$ satisfies the resolvent property with a nonexpansive mapping T, then $(\{T_n\}_{n=1}^{\infty}, \{T\})$ satisfies the NST-condition.

2. If $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition , then $(\{T_n\}_{n=1}^{\infty}, \{T\})$ satisfies the NST-condition.

Proof. Let $\{z_n\}_{n=1}^{\infty}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. Since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition, $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in \{z_n\}\} = 0$. In particular, $\lim_{n\to\infty} \|Tz_n - T_n z_n\| = 0$. This implies that

$$\limsup_{n\to\infty} \|z_n - Tz_n\| \leq \lim_{n\to\infty} \|z_n - T_n z_n\| + \lim_{n\to\infty} \|T_n z_n - T z_n\| = 0.$$

Hence $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. □

Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Then for each $x \in \mathcal{H}$, there is a unique element $\widehat{x} \in C$ such that

$$\|x - \widehat{x}\| = \min_{y\in C} \|x - y\|.$$

Set $P_C x = \widehat{x}$. The mapping P_C is called the *metric projection* from $\mathcal{H}$ onto C.

Lemma 2.7. *[10] Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Then, for all $x \in \mathcal{H}$ and $y \in C$, $y = P_C x$ if and only if $\langle y - x, z - y\rangle \geq 0$ for all $z \in C$.*

The following is the most general result amongst the three weak convergence theorems of Chuang and Takahashi [3].

Theorem 2.8. *[3] Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to C\}_{n=1}^{\infty}$ be a sequence of firmly nonexpansive mappings. Let $\mathcal{T}$ be a family of nonexpansive mappings of C into itself, which satisfies NST-condition. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 2[$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

We recall the following facts which are of interest and play an important role in this paper.

Lemma 2.9 (Opial's property)**.** *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}$ *such that* $x_n \rightharpoonup x \in \mathcal{H}$. *Then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in \mathcal{H}$ *with* $y \neq x$.

Definition 2.10. Let F be a nonempty subset of $\mathcal{H}$. A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is *Fejér monotone with respect to* F if $\|x_{n+1} - p\| \leq \|x_n - p\|$ for all $n \geq 1$ and $p \in F$.

Lemma 2.11. *[11] Let* F *be a nonempty, closed and convex subset of* $\mathcal{H}$ *and* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}$. *If* $\{x_n\}_{n=1}^{\infty}$ *is Fejér monotone with respect to* F, *then* $\{P_F x_n\}_{n=1}^{\infty}$ *is convergent.*

Lemma 2.12. *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, *and* $\{c_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers such that* $a_{n+1} \leq a_n - c_n b_n$ *for all* $n \geq 1$ *and* $\liminf_{n\to\infty} c_n > 0$. *Then* $\lim_{n\to\infty} a_n$ *exists and* $\sum_{n=1}^{\infty} b_n < \infty$. *In particular,* $\lim_{n\to\infty} b_n = 0$.

Proof. The proof of this lemma is rather simple but it is given here for the sake of completeness. Note that $a_{n+1} \leq a_n$ for all $n \geq 1$. Thus $\lim_{n\to\infty} a_n$ exists. Moreover, $c_n b_n \leq a_n - a_{n+1}$. This yields $\sum_{n=1}^{k} c_n b_n \leq a_1 - a_{k+1} \leq a_1$. So $\sum_{n=1}^{\infty} c_n b_n \leq a_1 < \infty$. Since $\liminf_{n\to\infty} c_n > 0$, there are an integer $n_0 \geq 1$ and a positive real number b such that $b \leq c_n$ for all $n \geq n_0$. Thus $b\sum_{n=n_0}^{\infty} b_n \leq \sum_{n=n_0}^{\infty} c_n b_n < \infty$. Then $\sum_{n=1}^{\infty} b_n < \infty$ and hence $\lim_{n\to\infty} b_n = 0$. □

3 Results

Definition 3.1. Let C be a nonempty subset of $\mathcal{H}$. A mapping $T : C \to \mathcal{H}$ satisfies the *demiclosedness property* if $x = Tx$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$.

We say that a family $\mathcal{T}$ mappings from C into $\mathcal{H}$ satisfies the *demiclosedness property* if T satisfies the demiclosedness property for all $T \in \mathcal{T}$.

Lemma 3.2. *Let* C *be a nonempty, closed and convex subset of* $\mathcal{H}$. *Let* $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ *be a sequence of mappings. Let* $\mathcal{T}$ *be a family of mappings of* C *into* $\mathcal{H}$ *satisfying the demiclosedness property. Assume that* $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ *satisfies the*

NST-condition. Let $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* C. *If* $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ *and* $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$, *then* $x_n \rightharpoonup \overline{x}$ *for some* $\overline{x} \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Proof. First, we show that all weak cluster points of $\{x_n\}_{n=1}^{\infty}$ belong to the set $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. To see this, let $\{x_{n_k}\}_{k=1}^{\infty}$ be a weakly convergent subsequence of $\{x_n\}_{n=1}^{\infty}$. (Such a subsequence exists because $\{x_n\}_{n=1}^{\infty}$ is bounded.) We assume that $x_{n_k} \rightharpoonup u$ for some $u \in C$. Let $T \in \mathcal{T}$. Since $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ satisfies the NST-condition, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and hence $\lim_{k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0$. Since $\mathcal{T}$ satisfies the demiclosedness property, $u \in \mathrm{Fix}(T)$. This implies that $u \in \mathrm{Fix}(\mathcal{T}) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Finally, we show that the whole sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to some element in the set $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. To see this, it suffices to prove that the set of all weak cluster points of $\{x_n\}_{n=1}^{\infty}$ is a singleton. Suppose that $\{x_{m_j}\}_{j=1}^{\infty}$ and $\{x_{p_k}\}_{k=1}^{\infty}$ are two subsequences of $\{x_n\}_{n=1}^{\infty}$ which converge weakly to u and v, respectively. From the first part of the proof, we obtain that $u, v \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. In particular, both limits $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. Suppose that $u \neq v$. By Opial's property, we obtain the following contradiction:

$$\begin{aligned}\liminf_{j\to\infty}\|x_{m_j} - u\| &< \lim_{j\to\infty}\|x_{m_j} - v\| \\ &= \lim_{k\to\infty}\|x_{p_k} - v\| \\ &< \lim_{k\to\infty}\|x_{p_k} - u\| \\ &= \liminf_{j\to\infty}\|x_{m_j} - u\|.\end{aligned}$$

So $u = v$. Hence $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$, as desired. □

3.1 k-demicontractive mappings where $k < 1$

Theorem 3.3. *Let* C *be a nonempty, closed and convex subset of* $\mathcal{H}$. *Let* $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ *be a sequence of* k_n*-demicontractive mappings where* $k_n < 1$ *for all* $n \geq 1$. *Let* $\mathcal{T}$ *be a family of mappings of* C *into* $\mathcal{H}$ *satisfying the demiclosedness property. Assume that* $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ *satisfies NST-condition. Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $]0, 1 - k_n[$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* C *defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_nT_nx_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n((1-k_n)-\alpha_n) > 0$, *then* $x_n \rightharpoonup \overline{x}$ *for some* $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. *Moreover,* $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Let $S_n := (1-\alpha_n)I + \alpha_n T_n$ for all $n \geq 1$. By Lemma 2.4, we get the following statements:

$$\|x_{n+1}-p\|^2 \leq \|S_n x_n - p\|^2 \leq \|x_n - p\|^2 - \alpha_n((1-k_n)-\alpha_n)\|x_n - T_n x_n\|^2,$$

$\mathrm{Fix}(S_n) = \mathrm{Fix}(T_n)$, and S_n is quasi-nonexpansive for all $n \geq 1$. By Lemma 2.12 with $a_n \equiv \|x_n - p\|^2$, $b_n \equiv \|x_n - T_n x_n\|^2$ and $c_n \equiv \alpha_n((1-k_n)-\alpha_n)$, we get that $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. By Lemma 3.2, we have $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Note that $\{x_n\}_{n=1}^{\infty}$ is Fejér monotone with respect to $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Since $\mathrm{Fix}(T_n)$ is closed and convex for all $n \geq 1$, it follows that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is closed and convex. By Lemma 2.11, $\{P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n\}_{n=1}^{\infty}$ converges to a point q in $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. It follows from Lemma 2.7 that

$$\left\langle x_n - P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n, P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n - \overline{x} \right\rangle \geq 0.$$

Since $x_n \rightharpoonup \overline{x}$ and $P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n \to q$, we have

$$\langle x_n - P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n, P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n - \overline{x} \rangle \to \langle \overline{x} - q, q - \overline{x} \rangle = -\|\overline{x} - q\|^2 \geq 0.$$

This implies that $\overline{x} = q$. Hence $\lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n = \overline{x}$. □

Set $k_n \equiv -1$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let* C *be a nonempty, closed and convex subset of* $\mathcal{H}$. *Let* $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ *be a sequence of quasi-firmly nonexpansive mappings. Let* $\mathcal{T}$ *be a family of mappings of* C *into* $\mathcal{H}$ *satisfying the demiclosedness property. Assume that* $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ *satisfies NST-condition. Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $]0,2[$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* C *defined by*

$$\begin{cases} x_1 \in C \ \textit{arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, *then* $x_n \rightharpoonup \overline{x}$ *for some* $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ *and* $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

Remark 3.5. Our Corollary 3.4 improves Theorem 3.3 of [3] in the following ways.

(a) Since every firmly nonexpansive mapping with a fixed point is quasi-firmly nonexpansive, Corollary 3.4 deals with a wider class of mappings.

(b) The family $\mathcal{T}$ in our Corollary 3.4 is more general than the family $\mathcal{T}$ of nonexpansive mappings in Theorem 3.3 of [3]. In fact, it is known that every nonexpansive mapping satisfies the demiclosedness property.

(c) All mappings in Theorem 3.3 of [3] are self-mappings while in our Corollary 3.4 they are nonself.

(d) We obtain a further information about the weak limit $\overline{x}$ of the sequence $\{x_n\}_{n=1}^{\infty}$. In fact, we can conclude that $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)} x_n$.

From Remark 2.6 and Theorem 3.3, we obtain the following two corollaries which improve Theorems 3.1 and 3.2 of [3], respectively.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ be a sequence of k_n-demicontractive mappings where $k_n < 1$ for all $n \geq 1$. Assume that $\{T_n\}_{n=1}^{\infty}$ satisfies the resolvent property with a nonexpansive mapping T. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 1-k_n[$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n((1-k_n)-\alpha_n) > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)} x_n$.

Corollary 3.7. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to \mathcal{H}\}_{n=1}^{\infty}$ be a sequence of k_n-demicontractive mappings where $k_n < 1$ for all $n \geq 1$. Assume that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition and T satisfies the demiclosedness property. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 1-k_n[$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} := P_C\left((1-\alpha_n)x_n + \alpha_n T_n x_n\right) \quad \forall n \geq 1. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n((1-k_n)-\alpha_n) > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)} x_n$.

3.2 1-demicontractive mappings

Definition 3.8. Let C be a nonempty subset of $\mathcal{H}$ and let L be a positive real number. A mapping $T : C \to \mathcal{H}$ is *L-Lipschitzian* if $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$.

It is known that the sequence $\{x_n\}_{n=1}^{\infty}$ defined in Theorem 3.3 fails to converge even $T_n \equiv T$ where T is 1-demicontractive and L-Lipschitzian (see [2]). We modify the iteration in Theorem 3.3 to obtain two weak convergence theorems, that is, Theorems 3.10 and 3.12. We now restrict ourselves from the nonself mappings to the self ones. The first result is based on the Ishikawa iteration [5]. The following lemma is modified from [7].

Lemma 3.9. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $T : C \to C$ be an L-Lipschitzian and 1-demicontractive mapping. Let $\alpha, \beta \in [0, 1]$. Define the mappings S and U by $S := (1 - \alpha)I + \alpha T$ and $U := (1 - \beta)I + \beta TS$. Then for all $x \in C$ and $p \in \mathrm{Fix}(T)$,*

$$\|Ux - p\|^2 \leq \|x - p\|^2 + \alpha\beta(L^2\alpha^2 + 2\alpha - 1)\|x - Tx\|^2 + \beta(\beta - \alpha)\|x - TSx\|^2.$$

In addition, if $0 < \beta \leq \alpha < \frac{1}{\sqrt{L^2+1}+1}$, then $\mathrm{Fix}(U) = \mathrm{Fix}(T)$ and U is quasi-nonexpansive.

Proof. Let $x \in C$ and $p \in \mathrm{Fix}(T)$. Then

$$\begin{aligned}\|Ux - p\|^2 &= \|(1 - \beta)(x - p) + \beta(TSx - p)\|^2 \\ &= (1 - \beta)\|x - p\|^2 + \beta\|TSx - p\|^2 - (1 - \beta)\beta\|x - TSx\|^2.\end{aligned}$$

Since T is 1-demicontractive, $\|TSx - p\|^2 \leq \|Sx - p\|^2 + \|Sx - TSx\|^2$. Note that

$$\begin{aligned}\|Sx - p\|^2 &= \|(1 - \alpha)(x - p) + \alpha(Tx - p)\|^2 \\ &= (1 - \alpha)\|x - p\|^2 + \alpha\|Tx - p\|^2 - (1 - \alpha)\alpha\|x - Tx\|^2 \\ &\leq (1 - \alpha)\|x - p\|^2 + \alpha\|x - p\|^2 + \alpha\|x - Tx\|^2 - (1 - \alpha)\alpha\|x - Tx\|^2 \\ &= \|x - p\|^2 + \alpha^2\|x - Tx\|^2;\end{aligned}$$

and

$$\begin{aligned}\|Sx - TSx\|^2 &= \|(1-\alpha)(x - TSx) + \alpha(Tx - TSx)\|^2 \\ &= (1-\alpha)\|x - TSx\|^2 + \alpha\|Tx - TSx\|^2 - (1-\alpha)\alpha\|x - Tx\|^2 \\ &\le (1-\alpha)\|x - TSx\|^2 + \alpha L^2\|x - Sx\|^2 - (1-\alpha)\alpha\|x - Tx\|^2 \\ &= (1-\alpha)\|x - TSx\|^2 + \alpha^3 L^2\|x - Tx\|^2 - (1-\alpha)\alpha\|x - Tx\|^2 \\ &= (1-\alpha)\|x - TSx\|^2 + \alpha(L^2\alpha^2 + \alpha - 1)\|x - Tx\|^2.\end{aligned}$$

So $\|TSx - p\|^2 \le \|x - p\|^2 + \alpha(L^2\alpha^2 + 2\alpha - 1)\|x - Tx\|^2 + (1-\alpha)\|x - TSx\|^2$. We get that

$$\|Ux - p\|^2 \le \|x - p\|^2 + \alpha\beta(L^2\alpha^2 + 2\alpha - 1)\|x - Tx\|^2 + \beta(\beta - \alpha)\|x - TSx\|^2.$$

If $0 < \beta \le \alpha < \frac{1}{\sqrt{L^2+1}+1}$, then $L^2\alpha^2 + 2\alpha - 1 < 0$. This implies that $\mathrm{Fix}(T) = \mathrm{Fix}(U)$ and U is quasi-nonexpansive. □

Theorem 3.10. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to C\}_{n=1}^{\infty}$ be a sequence of L-Lipschitzian and 1-demicontractive mappings. Let $\mathcal{T}$ be a family of mappings of C into itself satisfying the demiclosedness property. Assume that $(\{T_n\}_{n=1}^{\infty}, \mathcal{T})$ satisfies NST-condition. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $]0, 1/(\sqrt{L^2+1}+1)[$ and $\{\beta_n\}_{n=1}^{\infty}$ be a sequence in $]0, \alpha_n]$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ y_n := (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ x_{n+1} := (1-\beta_n)x_n + \beta_n T_n y_n \quad \forall n \ge 1. \end{cases}$$

If $\liminf_{n\to\infty} \beta_n(1 - 2\alpha_n - L^2\alpha_n^2) > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)} x_n$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Let $S_n := (1-\alpha_n)I + \alpha_n T_n$ and $U_n := (1-\beta_n)I + \beta_n T_n S_n$ for all $n \ge 1$. Note that $y_n = S_n x_n$ and $x_{n+1} = U_n x_n$ for all $n \ge 1$. By Lemma 3.9, $\|U_n x_n - p\|^2 \le \|x_n - p\|^2 + \beta_n^2(L^2\alpha_n^2 + 2\alpha_n - 1)\|x_n - T_n x_n\|^2$ and U_n is quasi-nonexpansive and $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$. Thus

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + \beta_n^2(L^2\alpha_n^2 + 2\alpha_n - 1)\|x_n - T_n x_n\|^2.$$

Note that $L^2\alpha_n^2 + 2\alpha_n - 1 < 0$ for all $\alpha_n \in]0, 1/(\sqrt{L^2+1}+1)[$. By Lemma 2.12, we get that $\lim_{n\to\infty}\|x_n - p\|$ exists and $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. By Lemma 3.2, we have $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Since $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$ and U_n is quasi-nonexpansive, $\mathrm{Fix}(T_n)$ is closed and convex for all $n \geq 1$. So $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is closed and convex. Note that $\{x_n\}_{n=1}^{\infty}$ is Fejér monotone with respect to $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. The rest of the proof is essentially the same as that of Theorem 3.3, so it is omitted. □

Next, we use the extragradient technique of Korpelevič [6] for this situation. We observe the following inequality which plays an important role in the next theorem. Note that this result is more general than the one in [7].

Lemma 3.11. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $T : C \to C$ be an L-Lipschitzian and 1-demicontractive mapping. Let $\alpha \in [0,1]$. Define the mappings S and U by $S := (1-\alpha)I + \alpha T$ and $U := P_C(I - \alpha S + \alpha TS)$. Then for all $x \in C$ and $p \in \mathrm{Fix}(T)$,*

$$\|Ux - p\|^2 \leq \|x - p\|^2 - (1 - \alpha^2(1+L)^2)\alpha^2\|x - Tx\|^2.$$

In addition, if $\alpha \in \left]0, \frac{1}{1+L}\right[$, then $\mathrm{Fix}(U) = \mathrm{Fix}(T)$ and U is quasi-nonexpansive.

Proof. Let $x \in C$ and $p \in \mathrm{Fix}(T)$.

$$\begin{aligned} \|Ux - p\|^2 &\leq \|x - \alpha Sx + \alpha TSx - p\|^2 - \|x - \alpha Sx + \alpha TSx - Ux\|^2 \\ &= \|x - p - \alpha(Sx - TSx)\|^2 - \|x - Ux - \alpha(Sx - TSx)\|^2 \\ &= \|x - p\|^2 - \|x - Ux\|^2 + 2\alpha\langle p - Ux, Sx - TSx\rangle. \end{aligned}$$

Since T is 1-demicontactive, $\langle p - Sx, Sx - TSx\rangle \leq 0$. So

$$\begin{aligned} \langle p - Ux, Sx - TSx\rangle &= \langle Sx - Ux, Sx - TSx\rangle + \langle p - Sx, Sx - TSx\rangle \\ &\leq \langle Sx - Ux, Sx - TSx\rangle. \end{aligned}$$

Note that $\|x - Ux\|^2 = \|x - Sx\|^2 + 2\langle x - Sx, Sx - Ux\rangle + \|Sx - Ux\|^2$. Then

$$\begin{aligned} \|Ux - p\|^2 \leq\ & \|x - p\|^2 - \|x - Sx\|^2 - 2\langle x - Sx, Sx - Ux\rangle - \|Sx - Ux\|^2 \\ & + 2\alpha\langle Sx - Ux, Sx - TSx\rangle \\ =\ & \|x - p\|^2 - \|x - Sx\|^2 - \|Sx - Ux\|^2 \\ & + 2\langle Sx - Ux, \alpha(Sx - TSx) - (x - Sx)\rangle. \end{aligned}$$

We consider

$$\begin{aligned}
&2\langle Sx - Ux, \alpha(Sx - TSx) - (x - Sx)\rangle \\
&= 2\alpha\langle Sx - Ux, Sx - TSx - (x - Tx)\rangle \\
&\leq 2\alpha\|Sx - Ux\|\|Sx - TSx - (x - Tx)\| \\
&\leq 2\alpha\|Sx - Ux\|(\|x - Sx\| + \|Tx - TSx\|) \\
&\leq 2\alpha(1 + L)\|Sx - Ux\|\|x - Sx\| \\
&\leq \|Sx - Ux\|^2 + \alpha^2(1 + L)^2\|x - Sx\|^2.
\end{aligned}$$

We have that

$$\begin{aligned}
\|Ux - p\|^2 &\leq \|x - p\|^2 - (1 - \alpha^2(1 + L)^2)\|x - Sx\|^2 \\
&= \|x - p\|^2 - (1 - \alpha^2(1 + L)^2)\alpha^2\|x - Tx\|^2. \qquad (1)
\end{aligned}$$

If $\alpha \in]0, 1/(1 + L)[$, then $1 - \alpha^2(1 + L)^2 \geq 0$. From (1) , we get that $\mathrm{Fix}(T) = \mathrm{Fix}(U)$ and U is quasi-nonexpansive. □

Theorem 3.12. *Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\{T_n : C \to C\}_{n=1}^\infty$ be a sequence of L-Lipschitzian and 1-demicontractive mappings. Let $\mathcal{T}$ be a family of mappings of C into itself satisfying the demiclosedness property. Assume that $(\{T_n\}_{n=1}^\infty, \mathcal{T})$ satisfies NST-condition. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $]0, 1/(1 + L)[$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in C defined by*

$$\begin{cases}
x_1 \in C \text{ arbitrarily chosen,} \\
y_n := (1 - \alpha_n)x_n + \alpha_n T_n x_n, \\
x_{n+1} := P_C(x_n - \alpha_n y_n + \alpha_n T_n y_n) \quad \forall n \geq 1.
\end{cases}$$

If $\liminf_{n\to\infty}(1 - \alpha_n(1 + L))\alpha_n > 0$, then $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^\infty \mathrm{Fix}(T_n)$ and $\overline{x} = \lim_{n\to\infty} P_{\bigcap_{n=1}^\infty \mathrm{Fix}(T_n)} x_n$.

Proof. Let $p \in \bigcap_{n=1}^\infty \mathrm{Fix}(T_n)$. Let $S_n := (1 - \alpha_n)I + \alpha_n T_n$ and $U_n := P_C(I - \alpha_n)S_n + \alpha_n T_n S_n$ for all $n \geq 1$. Note that $y_n = S_n x_n$ and $x_{n+1} = U_n x_n$ for all $n \geq 1$. By Lemma 3.11, we get that $\|U_n x_n - p\|^2 \leq \|x_n - p\|^2 - (1 - \alpha_n^2(1 + L)^2)\alpha_n^2\|x_n - T_n x_n\|^2$ and U_n is quasi-nonexpansive and $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$. Thus

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1 - \alpha_n^2(1 + L)^2)\alpha_n^2\|x_n - T_n x_n\|^2.$$

By Lemma 2.12, we get that $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. By Lemma 3.2, we have $x_n \rightharpoonup \overline{x}$ for some $\overline{x} \in \bigcap_{n=1}^\infty \mathrm{Fix}(T_n)$.

Since $\mathrm{Fix}(U_n) = \mathrm{Fix}(T_n)$ and U_n is quasi-nonexpansive, $\mathrm{Fix}(T_n)$ is closed and convex for all $n \geq 1$. So $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is closed and convex. Note that $\{x_n\}_{n=1}^{\infty}$ is Fejér monotone with respect to $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. The rest of the proof is essentially the same as that of Theorem 3.3, so it is omitted. □

4 Numerical Results

Finally, we show some numerical results for Theorems 3.10 and 3.12. The following example is taken from [2]. Let $\mathcal{H}$ be the two-dimensional Euclidean space $\mathbb{R}^2$. If $x = (a, b) \in \mathcal{H}$, define $x^{\perp} \in \mathcal{H}$ to be $(b, -a)$. Let $K := K_1 \cup K_2$ where

$$K_1 := \{x \in \mathcal{H} : \|x\| \leq 1/2\} \text{ and } K_2 := \{x \in \mathcal{H} : 1/2 \leq \|x\| \leq 1\}.$$

Define $T : K \to K$ by

$$Tx = \begin{cases} x + x^{\perp} & \text{if } x \in K_1, \\ \frac{x}{\|x\|} - x + x^{\perp} & \text{if } x \in K_2. \end{cases}$$

Then K is a closed and convex subset of $\mathcal{H}$. Moreover, T is 5-Lipschitzian and 1-demicontractive mapping with $\mathrm{Fix}(T) = \{(0,0)\}$. For computational purposes, it is of interest to know

(a) how the convergence behaviour of $\{x_n\}_{n=1}^{\infty}$ depends on the choice of $\{\alpha_n\}_{n=1}^{\infty}$ in Theorems 3.10 and 3.12;

(b) which of the iterations in Theorems 3.10 and 3.12 is more efficient.

To illustrate (a), we discuss Theorem 3.10 with $x_1 = (1, 0)$ and $\alpha_n = \beta_n \equiv \alpha$. To guarantee the convergence of $\{x_n\}_{n=1}^{\infty}$, we are allowed to choose $\alpha \in \left]0, 1/(\sqrt{26}+1)\right[$. Figures 1 and 2 show that the larger choice α, the closer the term x_n is to the fixed point $(0, 0)$. For Theorem 3.12, we set $x_1 = (1, 0)$ and $\alpha_n \equiv \alpha \in]0, 1/6[$.

To illustrate (b), let $x_1 = x'_1 = (0.1, 0)$ and let $\{x_n\}_{n=2}^{\infty}$ and $\{x'_n\}_{n=2}^{\infty}$ be defined by the iterations in Theorem 3.10 with $\alpha_n = \beta_n \equiv \alpha$ and Theorem 3.12 with $\alpha_n \equiv \alpha$, respectively. Note that $\left]0, 1/(\sqrt{26}+1)\right[\subset]0, 1/6[$. Figure 3 shows that in this situation the iteration in Theorem 3.12 is more efficient than the one in Theorem 3.10.

Table 1: The value of $\|x_n-(0,0)\|$ where x_n is defined by the iteration in Theorem 3.10

n	$\alpha_n=\beta_n\equiv\alpha$				
	0.004	0.007	0.080	0.160	0.163
1	1	1	1	1	1
2	9.96e−1	9.93e−1	9.23e−1	8.49e−1	8.46e−1
50	8.38e−1	7.51e−1	4.67e−1	3.61e−1	3.57e−1
100	7.26e−1	6.24e−1	4.09e−1	2.41e−1	2.36e−1
500	5.09e−1	5.00e−1	1.42e−1	9.37e−3	8.46e−3

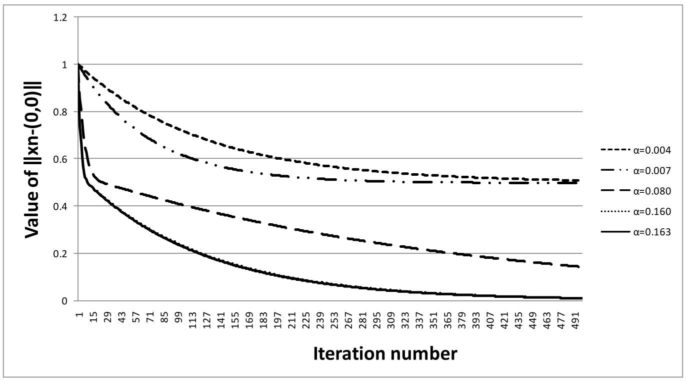

Figure 1: The behaviour of $\|x_n-(0,0)\|$ in Theorem 3.10 and the choice of $\{\alpha_n\}_{n=1}^{\infty}$

Table 2: The value of $\|x_n-(0,0)\|$ where x_n is defined by the iteration in Theorem 3.12

n	$\alpha_n\equiv\alpha$				
	0.004	0.010	0.080	0.160	0.166
1	1	1	1	1	1
2	9.96e−1	9.90e−1	9.29e−1	8.70e−1	8.66e−1
50	8.38e−1	6.88e−1	4.62e−1	3.04e−1	2.91e−1
100	7.27e−1	5.69e−1	3.94e−1	1.62e−1	1.47e−1
500	5.09e−1	5.00e−1	1.10e−1	1.03e−3	6.45e−4

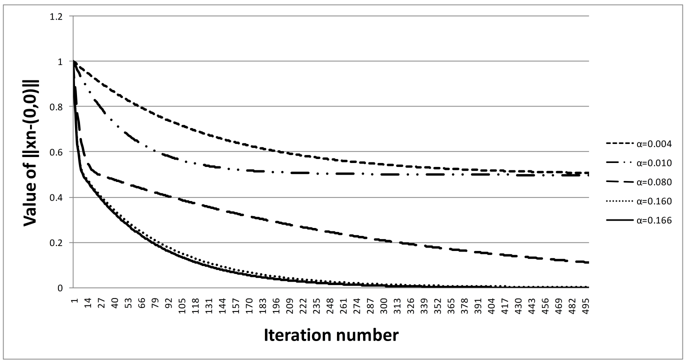

Figure 2: The behaviour of $\|x_n - (0,0)\|$ in Theorem 3.12 and the choice of $\{\alpha_n\}_{n=1}^{\infty}$

Table 3: The values of $\|x_n - (0,0)\|$ and $\|x'_n - (0,0)\|$

n	$\alpha_n \equiv \beta_n \equiv \alpha$ (Theorem 3.10)		$\alpha_n \equiv \alpha$ (Theorem 3.12)	
	0.160	0.163	0.160	0.166
1	1e−1	1e−1	1e−1	1e−1
2	9.92e−2	9.92e−2	9.87e−2	9.87e−2
50	6.72e−2	6.65e−2	5.39e−2	5.14e−2
100	4.48e−2	4.39e−2	2.86e−2	2.61e−2
500	1.74e−3	1.58e−3	1.83e−4	1.14e−4

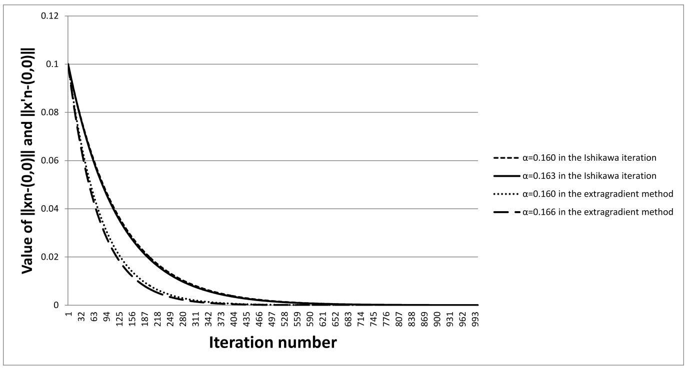

Figure 3: Comparative values of $\|x_n - (0,0)\|$ and $\|x'_n - (0,0)\|$

Acknowledgements The first author is supported by the Human Resource Development in Science Project (Science Achievement Scholarship of Thailand, SAST). The second author is supported by the Thailand Research Fund and Khon Kaen University under grant RSA5980006.

References

[1] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, Approximation of Common Fixed Points of a Countable Family of Nonexpansive Mappings in a Banach Space, *Nonlinear Anal.*, **67**(2007), 2350–2360.

[2] C. Chidume and S. Mutangadura, An Example on the Mann Iteration Method for Lipschitz Pseudocontractions, *Proc. Amer. Math. Soc.*, **129**(2001), 2359–2363.

[3] C.S. Chuang and W. Takahashi, Weak Convergence Theorems for Families of Nonlinear Mappings with Generalized Parameters, *Numer. Funct. Anal. Optim.*, **36**(2015), 41–54.

[4] T.L. Hicks and J.D. Kubicek, On the Mann Iteration Process in a Hilbert Space, *J. Math. Anal. Appl.*, **59**(1977), 498–504.

[5] S. Ishikawa, Fixed Points by a New Iteration Method, *Proc. Amer. Math. Soc.*, **44**(1974), 147–150.

[6] G.M. Korpelevič, An Extragradient Method for Finding Saddle Points and for Other Problems, (Russian) *Èkonom. i Mat. Metody.*, **12**(1976), 747–756.

[7] R. Kraikaew and S. Saejung, On a Hybrid Extragradient-Viscosity Method for Monotone Operators and Fixed Point Problems, *Numer. Funct. Anal. Optim.*, **35**(2014), 32–49.

[8] W.R. Mann, Mean Value Methods in Iteration, *Proc. Amer. Math. Soc.*, **4**(1953), 506–510.

[9] K. Nakajo, K. Shimoji, and W. Takahashi, Strong Convergence to Common Fixed Points of Families of Nonexpansive Mappings in Banach Spaces, *J. Nonlinear Convex Anal.*, **8**(2007), 11–34.

[10] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, 2009.

[11] W. Takahashi and M. Toyoda, Weak Convergence Theorems for Nonexpansive Mappings and Monotone Mappings, *J. Optim. Theory Appl.*, **118**(2003), 417–428.

Chanitnan Jaipranop
Department of Mathematics,
Faculty of Science, Khon Kaen University,
Khon Kaen, 40002, Thailand
Email: `chanitnanjaipranop@kkumail.com`

Satit Saejung
Department of Mathematics,
Faculty of Science, Khon Kaen University,
Khon Kaen, 40002, Thailand
Email: `saejung@kku.ac.th`