

Generalized composition operators of higher order on weighted Hardy Spaces

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Abstract: In the present paper, the notion of generalized composition operators is discussed along with the description of its properties.

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1 Introduction and Preliminaries

The theory of composition operators establishes a contact with ergodic theory, entropy theory and classical mechanics. Also it touches differentiable dynamics and the theory of distributions. Unilateral shifts, bilateral shifts and translations on the space L^2 are well known examples of composition operators. The study of composition operators was started by E. Nordgren [11] on spaces of analytic functions in 1968. Schwartz [12] wrote his Ph.D. dissertation in 1969 on composition operators on H^p -spaces. After that, the study of composition operators is considered by several mathematicians in several directions. Composition operators are discussed over Hardy spaces by many mathematician, we refer [3] and [4]. For the

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study of these operators over Bergman spaces and Dirichlet spaces, we refer [9-10] and the references therein. For the study of composition operators on various function spaces, we suggest the references [1], [2], [6], [13] and the references therein. Recently the appearance of composition operators is seen in the study of Toeplitz and Hankel operators in [6] and [8]. It is shown in [6] that composition operators provide a bridge between the classes of λ -Hankel operators and (λ, μ) -Hankel operators.

A new dimension is added to the theory of composition operators after the appearance of weighted Hardy spaces. Weighted Hardy spaces have the capacity to cover the classical Hardy, Bergman and Dirichlet spaces for the specific choices of the weights (see [16]). Zorboska [17] in the year 1989 initiated the study of composition operators on the weighted Hardy spaces. In the present paper, the study of composition operators is made in the connection of the notion of derivatives in the weighted Hardy spaces.

Let $\langle \beta_n \rangle_{n \geq 0}$ be a sequence of positive real numbers with $\beta_0 = 1$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the formal power series (whether or not the series converges for any values of z). For $1 \leq p < \infty$, define $\|f\|_{\beta}$ as $\|f\|_{\beta}^p = \sum_{n=0}^{\infty} |a_n|^p \beta_n^p$. The collection of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_{\beta}^p = \sum_{n=-\infty}^{\infty} |a_n|^p \beta_n^p < \infty$ is denoted by $H^p(\beta)$.

Then $H^p(\beta)$ is a Banach space under the norm $\|f\|_{\beta}^p = \sum_{n=0}^{\infty} |f_n|^p \beta_n^p < \infty$. For $p = 2$, the space $H^2(\beta)$ is a Hilbert space with norm induced by the inner product defined as $\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n \beta_n^2$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are elements of $H^2(\beta)$. The set $\{e_n\}_{n \geq 0}$, where $e_n(z) = \frac{z^n}{\beta_n}$, forms an orthonormal basis for the space $H^2(\beta)$. Symbol $H^{\infty}(\beta)$ denotes the set of formal power series ϕ such that $\phi(H^2(\beta)) \subseteq H^2(\beta)$. Weighted shift operators and multiplication operators are studied in detail in the work of Shields [16]. We refer to [16] as well as the references therein, for the details and applications of these spaces.

The study of composition operators from Lebesgue spaces has been lifted to various other function spaces like, Lorentz space, Orlicz space and Lorentz-Karamata spaces in the recent years (see [1-5] and [7]). In the pursuance, in the present paper, the notion of composition operators is extended to k^{th} -order generalized composition operators in the weighted Hardy spaces $H^2(\beta)$ and some properties of these operators are described. By operator we mean a bounded linear mapping on $H^2(\beta)$ and we reserve the symbol $B(H^2(\beta))$ to denote the collection of all bounded operators on $H^2(\beta)$.

2 Generalized Composition Operator

In this section, we introduce the notion of generalized composition operators. To begin with, we introduce the notion of derivatives in weighted Hardy spaces, which coincide with classical derivatives in the Hardy spaces.

Definition 2.1. For a natural number m and an element f of $H^2(\beta)$ with expression $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we define $f^{(m)}$ as

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} a_n z^{n-m},$$

and call it as the m^{th} -derivative of f .

The product of two functions $f, g \in H^2(\beta)$, denoted by $f \cdot g$, is given by $(f \cdot g)(z) = \sum_{k=0}^{\infty} c_k z^k$, where $c_k = \sum_{t=0}^k a_t b_{k-t}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. For a positive integer m , by f^m we simply mean $f \cdot f \cdots f$ (m times). If no confusion arises, we simply write fg for $f \cdot g$. For $m = 1$, f^m is simply written as f . For $\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\beta)$ and $g \in H^2(\beta)$, the composition of ϕ and g is denoted by $\phi \circ g$ and given by $(\phi \circ g)(z) = \phi(g(z)) = \sum_{n=0}^{\infty} a_n g^n(z)$.

Definition 2.2. Let $k \geq 1$ be a fixed natural number. If ϕ is a formal power series such that the mapping $f \mapsto f^{(k)} \circ \phi$ for each $f \in H^2(\beta)$ is continuous on $H^2(\beta)$, where $f^{(k)}$ denotes the k^{th} -order derivative of f , then this mapping is denoted by $C_{\phi, k}$ and is called a k^{th} -order generalized composition operators on $H^2(\beta)$ induced by ϕ .

If $k = 1$, then $C_{\phi, k}$ is simply called a generalized composition operator, which are discussed in [15].

The following examples verify the existence of generalized composition operators on $H^2(\beta)$.

Example 2.3. Consider $H^2(\beta)$, where $\langle \beta_n \rangle$ is an increasing sequence. Let $\phi(z) = \frac{z}{2}$ and $k \geq 1$ be a fixed integer. Then we find that for each $f \in H^2(\beta)$ with $f = \sum_{n=0}^{\infty} a_n z^n$,

$$\begin{aligned} \|f^{(k)} \circ \phi\|_{\beta}^2 &= \left\| \sum_{n=k}^{\infty} \alpha_n a_n \phi^{n-k} \right\|_{\beta}^2 = \left\| \sum_{n=0}^{\infty} \frac{\alpha_{n+k}}{2^n} a_{n+k} z^n \right\|_{\beta}^2 \\ &= \sum_{n=0}^{\infty} A_n^2 |a_{n+k}|^2 \beta_n^2 \end{aligned}$$

where for each $n \geq 0$, $A_n = \frac{\alpha_{n+k}}{2^n}$. As A_n tends to zero as n approaches to ∞ , we can find a natural number m and a positive real number p such that $A_n^2 \leq p^2$ for $n \leq (m - 1)$ and $A_n^2 < 1$ for $n \geq m$. This along with the fact that $\langle \beta_n \rangle$ is an increasing sequence provides that $\sum_{n=0}^{\infty} A_n^2 |a_{n+k}|^2 \beta_n^2 \leq (p^2 + 1) \|f\|_{\beta}^2$ and hence we have $\|C_{\phi,k} f\|_{\beta} \leq (p + 1) \|f\|_{\beta}$ for each $f \in H^2(\beta)$. Thus ϕ induces a generalized composition operator $C_{\phi,k}$ of k^{th} -order on $H^2(\beta)$.

Example 2.4. Let the sequence $\langle \beta_n \rangle$ be defined as $\beta_0 = 1$ and $\beta_{n+1} = \frac{n+1}{2} \beta_n$ for $n \geq 0$. It is a decreasing sequence and a simple computation shows that $\phi(z) = \frac{z}{2}$ induces the generalized composition operator of order one.

Example 2.5. Let $k \geq 2$ be a fixed natural number. Let $\langle \beta_n \rangle$ be a sequence with $\beta_0 = 1$, $\beta_{n+k} = \frac{(n+1)(n+2)\dots(n+k)}{2^n} \beta_n$ for all $n \geq 0$ and β'_i 's, $1 \leq i \leq k - 1$ are such that $\langle \beta_n \rangle$ is neither an increasing sequence nor a decreasing sequence. For $\phi(z) = \frac{z}{2}$, it can be shown by calculation as in the case of Example 2.3 that $C_{\phi,k}$ is a generalized composition operator of k^{th} - order.

Example 2.6. Let $k \geq 1$ be a fixed natural number. Let $\langle \beta_n \rangle$ be a sequence with $\beta_0 = 1$, and $k\beta_{n+k} \leq \beta_n \alpha_{n+k} \leq 1$ for $n \geq 0$ and β'_i 's, $1 \leq i \leq k - 1$ are any positive real numbers. Consider $\phi(z) = az$, where $a \in \mathbb{C}$. Then for ϕ to induce the operator $C_{\phi,k}$ on $H^2(\beta)$, there exists a positive integer $M > 0$ such that $\|e_{n+k}^{(k)} \circ \phi\|_{\beta} \leq M$ which implies $\|\frac{\alpha_{n+k}}{\beta_{n+k}} \beta_n a^n e_n\|_{\beta} \leq M$ for each $n \geq 0$, which gives $|a| \leq 1$. Further if $|a| \leq 1$ then $\|C_{\phi,k} f\|_{\beta}^2 = \|\sum_{n=k}^{\infty} a_n \alpha_n a^{n-k} z^{n-k}\|_{\beta}^2 = \sum_{n=0}^{\infty} |a_{n+k}|^2 |\alpha_{n+k}|^2 |a|^{2n} \beta_n^2 \leq \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 = \|f\|_{\beta}^2$. Hence we conclude that $\phi(z) = az$ induces $C_{\phi,k}$ on $H^2(\beta)$ if and only if $|a| \leq 1$.

An important aspect in the study of composition operator is to discuss its continuity, which is described on L^p spaces in [14] by Takagi and Yokouchi. The same result is extended to Orlicz spaces by Cui, Hudzik and Kumar in [5]. In next result, we discuss the continuity of generalized composition operator of k^{th} -order on weighted Hardy spaces and obtain the following.

Theorem 2.7. *Suppose a formal power series is such that $\{\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$. Then the mapping $C_{\phi,k} : (f \rightarrow f^{(k)} \circ \phi)$ on $H^2(\beta)$ is bounded if and only if there exists a real number $M > 0$ such that $\|\phi^n\|_{\beta} \leq M \frac{\beta_{n+k}}{\alpha_{n+k}}$ for each $n \geq 0$.*

Proof. Let $C_{\phi,k}$ be a bounded operator and suppose that $\|C_{\phi,k} f\|_{\beta} \leq M \|f\|_{\beta}$ for all $f \in H^2(\beta)$, for some $M > 0$. In particular, $\|C_{\phi,k} e_n\|_{\beta} \leq M$ for each $n \geq 0$.

This provides that for each $n \geq k$,

$$\frac{\alpha_n}{\beta_n} \|\phi^{n-k}\|_\beta = \|C_{\phi,k}(\frac{z^n}{\beta_n})\|_\beta = \|C_{\phi,k}e_n\|_\beta \leq M.$$

As a consequence, $\|\phi^n\|_\beta \leq M \frac{\beta_{n+k}}{\alpha_{n+k}}$ for all $n \geq 0$.

Conversely, if there exists a real number $M > 0$ satisfying $\|\phi^n\|_\beta \leq M \frac{\beta_{n+k}}{\alpha_{n+k}}$ for all $n \geq 0$ then for each $f = \sum_{n=0}^\infty f_n z^n \in H^2(\beta)$,

$$\begin{aligned} \|C_{\phi,k}f\|_\beta^2 &= \langle \sum_{n=k}^\infty f_n \alpha_n \phi^{n-k}, \sum_{p=k}^\infty f_p \alpha_p \phi^{p-k} \rangle \\ &= \sum_{n=k}^\infty |f_n|^2 \alpha_n^2 \|\phi^{n-k}\|_\beta^2 \leq M^2 \|f\|_\beta^2. \end{aligned}$$

This completes the proof. □

It can be verified that the product of two generalized composition operators need not be a generalized composition operator. For, we take $\phi(z) = z^k$ for integer $k \geq 1$ and show that for no $\psi \in H^2(\beta)$ and $p \geq 1$, $C_{\phi,k}^2 = C_{\psi,p}$. Let $\phi(z) = z^k$. Then $C_{\phi,k}^2(e_{k+1}) = \frac{(k+1)!k!}{\beta_{k+1}}$ and $C_{\phi,k}^2(e_k) = 0$. But

$$C_{\psi,p}(e_{k+1}) = \begin{cases} 0 & \text{if } p > k + 1 \\ (p + 1)! & \text{if } p = k + 1 \end{cases}$$

and

$$C_{\psi,p}(e_k) = \begin{cases} \frac{k!}{\beta_k} & \text{if } p = k \\ \frac{p!}{\beta_p} & \text{if } p < k. \end{cases}$$

These computations ensure that $C_{\phi,k}^2 \neq C_{\psi,p}$ for any p .

We can also verify that the adjoint of a generalized composition operator is not a generalized composition operator from the observation that $\langle C_{\phi,k}^*e_k, e_0 \rangle = 0$ for $\phi \in H^2(\beta)$ and $\langle C_{\psi,k}e_k, e_0 \rangle = k!$ for $\psi \in H^2(\beta)$.

Motivated by the paper of Zorboska [17], we have derived the condition for the compactness of the k^{th} – order generalized composition operator on weighted Hardy spaces.

Theorem 2.8. *Let $\phi = \sum_{n=0}^\infty \phi_n z^n$ is such that $\{\phi^n : n \geq 0\}$ is an orthogonal family in $H^2(\beta)$ and induces the k^{th} – order generalized composition operator $C_{\phi,k} \in B(H^2(\beta))$. Then $C_{\phi,k}$ is a compact operator if and only if $\frac{\alpha_n}{\beta_n} \|\phi^{n-k}\|_\beta \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let $C_{\phi,k}$ be a compact operator on $H^2(\beta)$. As $e_n \rightarrow 0$ weakly, we have $C_{\phi,k}e_n \rightarrow 0$ as $n \rightarrow \infty$. This implies that $\frac{\alpha_n}{\beta_n} \|(\beta_{n-k}e_{n-k}) \circ \phi\|_{\beta} \rightarrow 0$ as $n \rightarrow \infty$.

Conversely, assume that ϕ satisfies $\frac{\alpha_n}{\beta_n} \|\phi^{n-k}\|_{\beta} \rightarrow 0$ as $n \rightarrow \infty$. Then for $\epsilon > 0$, there exists a positive integer $n_0 (\geq k)$ such that $\frac{\alpha_n}{\beta_n} \|\phi^{n-k}\|_{\beta} \leq \epsilon$ for all $n \geq n_0$. Define a sequence $\langle A_m \rangle$ of compact operators on $H^2(\beta)$ as $A_m f = \sum_{n=0}^m \beta_n f_n C_{\phi,k} e_n$ for $f = \sum_{n=0}^{\infty} \beta_n f_n e_n \in H^2(\beta)$. Then

$$\begin{aligned} \|C_{\phi,k}f - A_m f\|_{\beta}^2 &= \left\| \sum_{n=m+1}^{\infty} f_n \beta_n C_{\phi,k} e_n \right\|_{\beta}^2 \\ &= \left\| \sum_{n=m+1}^{\infty} \alpha_n f_n \phi^{n-k} \right\|_{\beta}^2 \\ &= \sum_{n=m+1}^{\infty} \alpha_n^2 |f_n|^2 \|\phi^{n-k}\|_{\beta}^2 \end{aligned}$$

Now for $m \geq n_0$, $\|C_{\phi,k}f - A_m f\|_{\beta}^2 \leq \sum_{n=m+1}^{\infty} \epsilon^2 \beta_n^2 |f_n|^2$. Therefore, $\|A_m - C_{\phi,k}\|_{\beta} \leq \epsilon$ for all $m \geq n_0$. Hence $C_{\phi,k}$ is compact. \square

Definition 2.9. Let $\langle \beta_n \rangle$ be a sequence of positive real numbers with $\beta_0 = 1$. If the mapping $f \rightarrow D_{a,k}f$ be continuous on $H^2(\beta)$, where

$$D_{a,k}(f) = \sum_{n=0}^{\infty} \frac{f_n}{\gamma_n} z^{n+k},$$

$\gamma_n = (n+1)(n+2)\dots(n+k)$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n$ then $D_{a,k}$ is called an anti-differential operator of order k on $H^2(\beta)$.

Existence of anti-differential operators can be seen in abundance.

Example 2.10. Let $\beta = \langle \beta_n \rangle$ be a decreasing sequence of positive real numbers with $\beta_0 = 1$. Then for $f(z) = \sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$, $\|D_{a,k}(f)\|_{\beta}^2 = \left\| \sum_{n=0}^{\infty} \frac{f_n}{\gamma_n} z^{n+k} \right\|_{\beta}^2 = \sum_{n=0}^{\infty} \frac{|f_n|^2}{\gamma_n^2} \beta_{n+k}^2 \leq \|f\|_{\beta}^2$. Hence $D_{a,k} \in B(H^2(\beta))$.

Example 2.11. Consider the sequence $\langle \beta_n \rangle$, where $\beta_0 = 1$ and $\beta_n = \frac{1}{2^{i+m}}$ for $n = mk + i$. A simple computation verifies that $D_{a,k} \in B(H^2(\beta))$.

Theorem 2.12. Let $\langle \beta_n \rangle$ be a sequence such that $D_{a,k} \in B(H^2(\beta))$. Then $D_{a,k}$ is compact if and only if $\frac{\beta_{n+k}}{\beta_n \gamma_n} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. For $D_{a,k}$ being compact, $\|D_{a,k}e_n\|_{\beta} \rightarrow 0$ as $n \rightarrow \infty$, which gives $\left\| \frac{z^{n+k}}{\beta_n \gamma_n} \right\|_{\beta} \rightarrow 0$ as $n \rightarrow \infty$. Hence $\frac{\beta_{n+k}}{\beta_n \gamma_n} \rightarrow 0$ as $n \rightarrow \infty$.

For the converse, assume that the sequence $\langle \beta_n \rangle$ satisfies $\frac{\beta_{n+k}}{\beta_n \gamma_n} \rightarrow 0$ as $n \rightarrow \infty$. Now define a sequence of operators $\langle A_m \rangle$ as

$$A_m f = \sum_{n=0}^m f_n D_{a,k} z^n$$

for $f \in H^2(\beta)$. Now from the given condition for $\epsilon > 0$, we find a positive integer n_0 satisfying $\frac{\beta_{n+k}}{\beta_n \gamma_n} \leq \epsilon$ for all $n \geq n_0$. Now for $m \geq n_0$, it can be shown that $\|A_m f - D_{a,k} f\|_\beta^2 < \epsilon^2 \|f\|_\beta^2$. As a consequence $\|A_m - D_{a,k}\|_\beta < \epsilon$ for all $m \geq n_0$. Hence $D_{a,k}$ is a compact operator. \square

Remark 2.13. Suppose the sequence $\beta = \langle \beta_n \rangle$ is such that $\phi(z) = z$ induces $C_{\phi,k}$ on $H^2(\beta)$. A simple computation provides that $C_{\phi,k} D_{a,k} f = f$ for each $f \in H^2(\beta)$.

If the sequence $\beta = \langle \beta_n \rangle$ is such that $\phi(z) = z$ induces $C_{\phi,k}$ on $H^2(\beta)$ then both $C_{\phi,k}$ and $D_{a,k}$ are non compact operators on $H^2(\beta)$.

Let $\langle \beta_n \rangle$ be a decreasing sequence of positive real numbers with $\beta_0 = 1$. Then from Theorem 2.8, it can be verified that $D_{a,k}$ is a compact operator and hence $\phi(z) = z$ can not induce generalized composition operator on $H^2(\beta)$ of k^{th} -order.

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