Chamchuri Journal of Mathematics
VOLUME 10(2018), 1–13
http://www.math.sc.chula.ac.th/cjm

Necessary and Sufficient Conditions for Existence of an Equilibrium and a Periodic of Prime Period 2 Solution of a Certain Rational Difference Equation

Ratinan Boonklurb, Julaluk Neammai, Vasana Sukkrasanti, and Theeruth Tantasuparuk*

Received 20 April 2018
Revised 28 May 2018
Accepted 14 June 2018

Abstract: This article studies the necessary and sufficient conditions for the existence of positive equilibrium solutions and positive periodic solutions of prime period 2 of the following rational difference equation.

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + \cdots + B_k x_{n-k}}, \text{ for } n \in \{0, 1, 2, ...\}$$

where the parameters $\alpha > 0$ and β, A, B_0, B_1, ..., B_k and the initial conditions x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_{-1}, x_0 are nonnegative real numbers such that the denominator is always positive.

Keywords: Equilibrium, Periodic, Prime Period, Rational difference equation

2000 Mathematics Subject Classification: 39A23

1 Introduction

Most of the dynamical phenomena in the real world involve discrete independence. Thus, difference equations are appropriate to model such dynamical systems.

* *Corresponding author*

Definition 1.1. [2, 5] A *difference equation of order* $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}), n \in \{0, 1, 2, ...\}, \tag{1.1}$$

where f is a function which maps I^{k+1} into I and I is an interval of real numbers.

Definition 1.2. [2, 6] A *solution* of (1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies (1.1) for all $n \geq 0$. Moreover, if we give a set of initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, then there exists a unique solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1).

Definition 1.3. [5, 6] An *equilibrium point* of (1.1) is a point $\bar{x} \in I$ that satisfies the condition $\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, ..., \bar{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \bar{x}$ for all $n \geq -k$ is a solution of (1.1), or equivalently $\bar{x} \in I$ is a fixed point of f.

However, fully nonlinear difference equations are very difficult to study. Many researchers turn their attention to the rational difference equations which are the difference equation that the function in Definition 1.1 is in terms of fraction. For example, in 2001, Kulenović et al. [8] investigated the global asymptotic stability of the positive equilibrium of the equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{A + x_{n-1}}, n \in \{0, 1, 2, ...\},$$

where the parameters α, β and A and the initial conditions x_{-1} and x_0 are nonnegative real numbers.

Some rational difference equations with appropriate initial conditions can have solutions that possess a special behavior. They repeat after several steps.

Definition 1.4. [6] A solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) is called *periodic with period p* (or a period-p solution) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \text{ for all } n \geq -k. \tag{1.2}$$

A solution is called *periodic with prime period p* if p is the smallest positive integer for which (1.2) holds. In this case, a p-tuple

$$(x_{n+1}, x_{n+2}, x_{n+3}, ..., x_{n+p})$$

of any p consecutive values of the solution is called a p-*cycle* of (1.1).

In 2001, DeVault et al. [4] studied the rational difference equation of order $(k+1)$

$$x_{n+1} = \frac{p + x_{n-k}}{qx_n + x_{n-k}}, n \in \{0, 1, 2, ...\},$$

with positive parameters and positive initial conditions and obtained the existence of period-two solutions. In the same year, Kulenović and Ladas [7] studied the second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, n \in \{0, 1, 2, ...\},$$

with nonnegative parameters and nonnegative initial conditions. In 2005, Camouzis and Ladas [1] presented several results, open problem and conjectures on period-three solutions of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, n \in \{0, 1, 2, ...\},$$

with nonnegative parameters and nonnegative initial conditions.

Recently, Chaihao [3] investigated some behaviors of nonnegative solutions and periodic solutions of the rational difference equation of the form

$$x_{n+1} = \frac{x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + \cdots + B_k x_{n-k}}, n \in \{0, 1, 2, ...\}, \tag{1.3}$$

where the parameters A, B_0, B_1, ..., B_k and the initial conditions x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_{-1}, x_0 are nonnegative real numbers such that the denominator in (1.3) is always positive. She obtained necessary and sufficient conditions for the existence of nonnegative prime period 2 and 3 solutions.

Our article considers nonnegative equilibrium solutions and periodic solutions of prime period 2 of a more general rational difference equation than that of Chaihao [3] which is

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + \cdots + B_k x_{n-k}}, n \in \{0, 1, 2, ...\} \tag{1.4}$$

where the parameters $\alpha > 0$ and β, A, B_0, B_1, ..., B_k and the initial conditions x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_{-1}, x_0 are nonnegative real numbers such that the denominator in (1.4) is always positive.

In Section 2, the necessary and sufficient conditions on the existence of the positive equilibrium solution are presented. In Section 3, the necessary and sufficient conditions on the existence of the positive prime period 2 solution is given. Finally, some discussion on future research and conclusion of our project are provided in Section 4.

2 Equilibrium Solution

In this section, we establish the existence of positive equilibrium solution of (1.4). Some numerical examples are also provided. First, let

$$b = B_0 + B_1 + B_2 + \cdots + B_k \geq 0.$$

Theorem 2.1. *(1.4) has a positive equilibrium solution if and only if* $b > 0$ *or* $(b = 0$ *and* $0 \leq \beta < A)$.

Proof. Assume that there is a positive equilibrium solution, namely, $..., \phi, \phi, \phi, ...$ of (1.4). Since from the definition, $b \geq 0$, we only need to check the condition when $b = 0$. Then, for $n = 0$, it follows from (1.4) that

$$\phi = \frac{\alpha + \beta\phi}{A + B_0\phi + B_1\phi + B_2\phi + B_3\phi + \cdots + B_k\phi} = \frac{\alpha + \beta\phi}{A + b\phi}.$$

That is, $\phi((A-\beta) + b\phi) = \alpha$. Since $\phi > 0$ and $\alpha > 0$, when $b = 0$ will give that $0 \leq \beta < A$.

Conversely, assume that $b > 0$ or ($b = 0$ and $0 \leq \beta < A$).

Case 1 $b > 0$. Then, choose

$$\phi = \frac{-(A-\beta) + \sqrt{(A-\beta)^2 + 4b\alpha}}{2b}.$$

Since $\sqrt{(A-\beta)^2 + 4b\alpha} > |A - \beta| \geq A - \beta$ and $b > 0$, we have $\phi > 0$. Now, by direct calculation, we can show that

$$b\phi^2 + (A-\beta)\phi - \alpha = 0,$$

which imples that

$$\phi = \frac{\alpha + \beta\phi}{A + b\phi} = \frac{\alpha + \beta\phi}{A + B_0\phi + B_1\phi + B_2\phi + B_3\phi + \cdots + B_k\phi}.$$

Thus, ϕ is the positive equilibrium solution of (1.4).

Case 2 $b = 0$ and $0 \leq \beta < A$. Then, choose

$$\phi = \frac{\alpha}{A - \beta}.$$

Since $\alpha > 0$ and $A - \beta > 0$, we have $\phi > 0$. By direct calculation, we have $(A-\beta)\phi - \alpha = 0$ which implies

$$\phi = \frac{\alpha + \beta\phi}{A + 0(\phi)} = \frac{\alpha + \beta\phi}{A + (0)\phi + (0)\phi + (0)\phi + (0)\phi + \cdots + (0)\phi}.$$

Thus, ϕ is the positive equilibrium solution of (1.4).

□

Example 2.1 Consider the difference equation of order 5

$$x_{n+1} = \frac{3 + 2x_{n-4}}{4 + x_{n-1} + x_{n-2} + x_{n-3} + 5x_{n-4}}, n \in \{0, 1, 2, ...\} \tag{2.1}$$

with the initial conditions $x_{-4} = x_{-3} = x_{-2} = x_{-1} = x_0 = \frac{1}{2}$. In this example, $k = 4$, $\alpha = 3$, $\beta = 2$, $A = 4$ and $b = 0 + 1 + 1 + 1 + 5 = 8$. This falls into case 1 of Theorem 2.1. Then, (2.1) has an equilibrium solution of the form $..., \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...$ as in Figure 1.

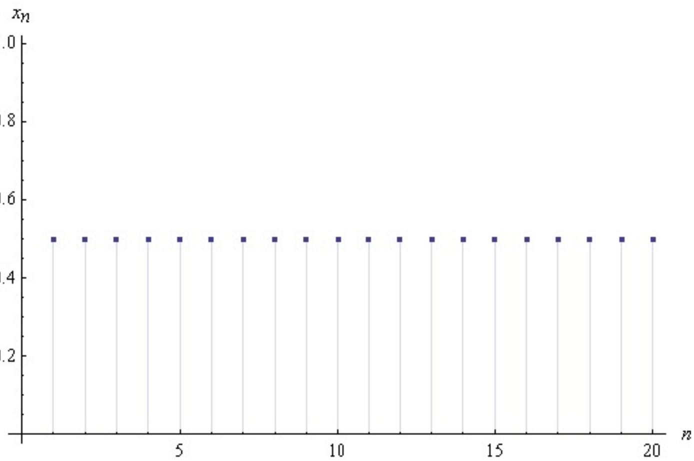

Figure 1: Equilibrium solution of Example 2.1

Example 2.2 Consider the difference equation of order 5

$$x_{n+1} = \frac{3 + 2x_{n-4}}{4}, n \in \{0, 1, 2, ...\} \tag{2.2}$$

with the initial conditions $x_{-4} = x_{-3} = x_{-2} = x_{-1} = x_0 = \frac{3}{2}$. In this example, $k = 4$, $\alpha = 3$, $\beta = 2$ and $A = 4$ and $b = 0$. This falls into case 1 of Theorem 2.1. Then, (2.1) has an equilibrium solution of the form $..., \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, ...$ as in Figure 2.

Example 2.3 Consider the difference equation of order 5

$$x_{n+1} = \frac{3 + 2x_{n-4}}{2 + x_{n-1} + x_{n-2} + x_{n-3} + 24x_{n-4}}, n \in \{0, 1, 2, ...\} \tag{2.3}$$

with the initial conditions $x_{-4} = x_{-3} = x_{-2} = x_{-1} = x_0 = \frac{1}{3}$. In this example, $k = 4$, $\alpha = 3$, $\beta = 2 = A$ and $b = 0 + 1 + 1 + 1 + 24 = 27$. This falls into case 2 of Theorem 2.1. Then, (2.1) has an equilibrium solution of the form $..., \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, ...$ as in Figure 3.

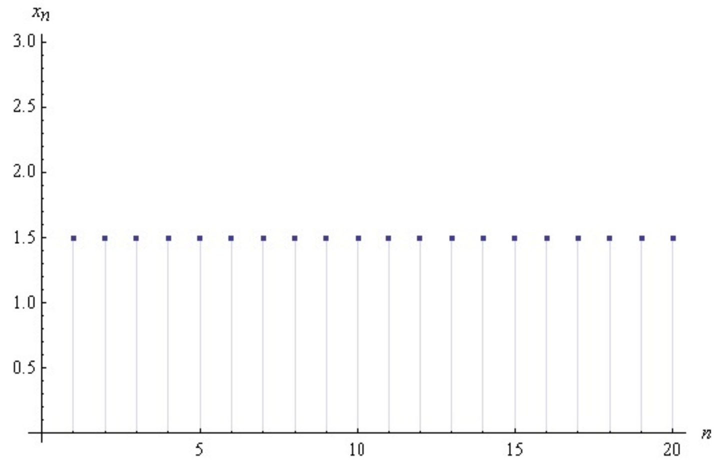

Figure 2: Equilibrium solution of Example 2.2

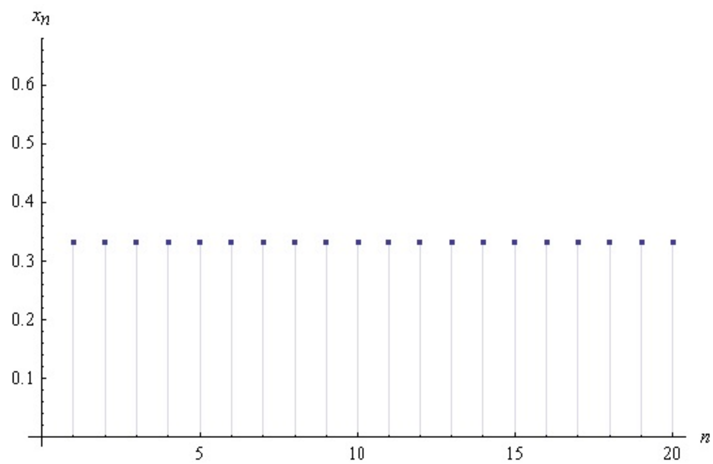

Figure 3: Equilibrium solution of Example 2.3

3 Period-Two Solution

In this section, we establish the existence of positive periodic solution with prime period two of (1.4). Some numerical examples are also provided. First, let

$$b_0 = B_0 + B_2 + B_4 + ... \text{ and } b_1 = B_1 + B_3 + B_5 + ...$$

Theorem 3.1. *Let k be an even integer. Assume that $\beta > 0$. Then, (1.4) has no positive solution of prime period two.*

Proof. Let k be an even integer. Assume that there is a positive solution of prime period two, namely, $..., \phi, \psi, \phi, \psi, ...$ of (1.4). Since k is even, then $x_n = x_{n-k}$. For $n = 0$, it follows from (1.4) that

$$\begin{aligned}\phi &= \frac{\alpha + \beta\psi}{A + B_0\psi + B_1\phi + B_2\psi + B_3\phi + \cdots + B_k\psi} \\ &= \frac{\alpha + \beta\psi}{A + (B_0 + B_2 + \cdots + B_k)\psi + (B_1 + B_3 + \cdots + B_{k-1})\phi} \\ &= \frac{\alpha + \beta\psi}{A + b_0\psi + b_1\phi}.\end{aligned}$$

On the other hand, for $n = 1$, it follows from (1.4) that

$$\begin{aligned}\psi &= \frac{\alpha + \beta\phi}{A + B_0\phi + B_1\psi + B_2\phi + B_3\psi + \cdots + B_k\phi} \\ &= \frac{\alpha + \beta\phi}{A + (B_0 + B_2 + \cdots + B_k)\phi + (B_1 + B_3 + \cdots + B_{k-1})\psi} \\ &= \frac{\alpha + \beta\phi}{A + b_0\phi + b_1\psi}.\end{aligned}$$

That is,

$$A\phi + b_0\phi\psi + b_1\phi^2 = \alpha + \beta\psi \text{ and } A\psi + b_0\phi\psi + b_1\psi^2 = \alpha + \beta\phi.$$

Thus,

$$\begin{aligned}A(\phi - \psi) + b_1(\phi^2 - \psi^2) &= (A\phi + b_0\phi\psi + b_1\phi^2) - (A\psi + b_0\phi\psi + b_1\psi^2) \\ &= -\beta(\phi - \psi).\end{aligned}$$

Since $\phi \neq \psi$, it follows that $A + b_1(\phi + \psi) = -\beta$. Since β, A, b_1, ϕ and ψ are all nonnegative, this is a contradiction. □

Theorem 3.2. *Let k be an odd integer. Assume that $b_1 > 0$. Then, (1.4) has a positive prime period-two solution if and only if $0 \leq A < \beta$, $b_0 > b_1$ and $\left(\frac{\beta - A}{b_1}\right)^2 > \frac{4\alpha}{b_0 - b_1}$.*

Proof. Let k be an odd integer and $b_1 > 0$.

First, suppose that there exists a positive prime period-two solution, namely, $..., \phi, \psi, \phi, \psi, ...$ of (1.4). Since k is odd, for $n = 0$, it follows from (1.4) that

$$\begin{aligned}\phi &= \frac{\alpha + \beta\phi}{A + B_0\psi + B_1\phi + B_2\psi + B_3\phi + \cdots + B_k\phi} \\ &= \frac{\alpha + \beta\phi}{A + (B_0 + B_2 + \cdots + B_{k-1})\psi + (B_1 + B_3 + \cdots + B_k)\phi} \\ &= \frac{\alpha + \beta\phi}{A + b_0\psi + b_1\phi}.\end{aligned}$$

On the other hand, for $n = 1$, it follows from (1.4) that

$$\begin{aligned}\psi &= \frac{\alpha + \beta\psi}{A + B_0\phi + B_1\psi + B_2\phi + B_3\psi + \cdots + B_k\psi} \\ &= \frac{\alpha + \beta\psi}{A + (B_0 + B_2 + \cdots + B_{k-1})\phi + (B_1 + B_3 + \cdots + B_k)\psi} \\ &= \frac{\alpha + \beta\psi}{A + b_0\phi + b_1\psi}.\end{aligned}$$

That is,

$$A\phi + b_0\phi\psi + b_1\phi^2 = \alpha + \beta\phi \text{ and } A\psi + b_0\phi\psi + b_1\psi^2 = \alpha + \beta\psi.$$

Thus,

$$\begin{aligned}A(\phi - \psi) + b_1(\phi^2 - \psi^2) &= (A\phi + b_0\phi\psi + b_1\phi^2) - (A\psi + b_0\phi\psi + b_1\psi^2) \\ &= \beta(\phi - \psi).\end{aligned}$$

Since $\phi \neq \psi$, it follows that

$$\phi + \psi = \frac{\beta - A}{b_1}. \tag{3.1}$$

Since ϕ, ψ and b_1 are positive, it implies that $0 \leq A < \beta$. On the other hand,

$$\begin{aligned}A(\phi + \psi) + 2b_0\phi\psi + b_1(\phi^2 + \psi^2) &= (A\phi + b_0\phi\psi + b_1\phi^2) + (A\psi + b_0\phi\psi + b_1\psi^2) \\ &= 2\alpha + \beta(\phi + \psi).\end{aligned}$$

The relation $\phi^2 + \psi^2 = (\phi + \psi)^2 - 2\phi\psi$ implies

$$A(\phi + \psi) + 2b_0\phi\psi + b_1(\phi + \psi)^2 - 2b_1\phi\psi = 2\alpha + \beta(\phi + \psi).$$

That is,

$$\phi\psi = \frac{\alpha}{(b_0 - b_1)} + \frac{(\beta - A)(\phi + \psi)}{2(b_0 - b_1)} - \frac{b_1(\phi + \psi)^2}{2(b_0 - b_1)}.$$

Since $\phi+\psi=\frac{\beta-A}{b_1}$, $\frac{(\beta-A)(\phi+\psi)}{2(b_0-b_1)}=\frac{b_1(\phi+\psi)^2}{2(b_0-b_1)}$. Thus,

$$\phi\psi=\frac{\alpha}{b_0-b_1}. \tag{3.2}$$

Since $\phi,\ \psi,\ \alpha>0$, this is enough to say that $b_0-b_1>0$. Now, from (3.1) and (3.2), we have $\left(\frac{\beta-A}{b_1}\right)^2=\phi^2+\psi^2+\dfrac{2\alpha}{b_0-b_1}$. since $(\phi-\psi)^2>0$, (3.2) implies

$$\begin{aligned}\left(\frac{\beta-A}{b_1}\right)^2 &> 2\phi\psi+\frac{2\alpha}{b_0-b_1}\\ &=\frac{4\alpha}{b_0-b_1}.\end{aligned}$$

Conversely, let us suppose that $0\le A<\beta$, $b_0>b_1$ and $\left(\frac{\beta-A}{b_1}\right)^2>\frac{4\alpha}{b_0-b_1}$. Choose

$$\phi=\frac{\beta-A}{2b_1}+\frac{1}{2}\sqrt{\left(\frac{\beta-A}{b_1}\right)^2-\frac{4\alpha}{b_0-b_1}} \tag{3.3}$$

and

$$\psi=\frac{\beta-A}{2b_1}-\frac{1}{2}\sqrt{\left(\frac{\beta-A}{b_1}\right)^2-\frac{4\alpha}{b_0-b_1}}. \tag{3.4}$$

We will show that $\{...,\phi,\psi,\phi,\psi,...\}$ is a positive prime period-two solution of (1.4). Notice that $\phi+\psi=\frac{\beta-A}{b_1}$ or $A+b_1(\phi+\psi)=\beta$. That is,

$$\begin{aligned}A\phi-A\psi+b_1\phi^2-b_1\psi^2+b_0\phi\psi-b_0\phi\psi &= A(\phi-\psi)+b_1(\phi+\psi)(\phi-\psi)\\ &=\beta(\phi-\psi)\\ &=\beta\phi-\beta\psi+\alpha-\alpha.\end{aligned}$$

Thus,

$$(A\phi+b_0\phi\psi+b_1\phi^2)-(A\psi+b_0\phi\psi+b_1\psi^2)=(\alpha+\beta\phi)-(\alpha+\beta\psi). \tag{3.5}$$

Multiplying (3.3) and (3.4), we obtain

$$\phi\psi=\frac{\alpha}{b_0-b_1}=\frac{\alpha}{b_0-b_1}+\frac{(\beta-A)^2}{2b_1(b_0-b_1)}-\frac{(\beta-A)^2}{2b_1(b_0-b_1)}.$$

Since $\frac{\beta-A}{b_1}=\phi+\psi$, we have $\frac{(\beta-A)^2}{2b_1(b_0-b_1)}=\frac{(\beta-A)(\phi+\psi)}{2(b_0-b_1)}$ and since $\left(\frac{\beta-A}{b_1}\right)^2=(\phi+\psi)^2$, we have $\frac{(\beta-A)^2}{2b_1(b_0-b_1)}=\frac{b_1(\phi+\psi)^2}{2(b_0-b_1)}$. That is,

$$\phi\psi=\frac{2\alpha+(\beta-A)(\phi+\psi)-b_1(\phi+\psi)^2}{2(b_0-b_1)}.$$

By direct calculation, we have

$$(A\phi + b_0\phi\psi + b_1\phi^2) + (A\psi + b_0\phi\psi + b_1\psi^2) = (\alpha + \beta\phi) + (\alpha + \beta\psi). \tag{3.6}$$

Therefore, (3.5) and (3.6) implies $2(A\phi + b_0\phi\psi + b_1\phi^2) = 2(\alpha + \beta\phi)$ and $-2(A\psi + b_0\phi\psi + b_1\psi^2) = -2(\alpha + \beta\psi)$. That is,

$$\phi = \frac{\alpha + \beta\phi}{A + b_0\psi + b_1\phi} \text{ and } \psi = \frac{\alpha + \beta\psi}{A + b_0\phi + b_1\psi}.$$

Hence, $\{..., \phi, \psi, \phi, \psi, ...\}$ is a positive prime period-two solution of (1.4). $\square$

Example 3.1 Consider the difference equation of order 4

$$x_{n+1} = \frac{\frac{5}{4} + 2x_{n-3}}{1 + 93x_n + 3x_{n-1}}, n \in \{0, 1, 2, ...\} \tag{3.7}$$

with the initial conditions $x_{-3} = x_{-1} = \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{18}}$ and $x_{-2} = x_0 = \frac{1}{6} - \frac{1}{2}\sqrt{\frac{1}{18}}$. In this example, $k = 3$, $\alpha = \frac{5}{4}$, $\beta = 2$, $A = 1$, $b_0 = 93 + 0 = 93$ and $b_1 = 3 + 0 = 3$. That means the conditions of Theorem 3.2 are satisfied. Then, (3.7) has a prime period-2 solution of the form $..., \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{18}}, \frac{1}{6} - \frac{1}{2}\sqrt{\frac{1}{18}}, \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{18}}, \frac{1}{6} - \frac{1}{2}\sqrt{\frac{1}{18}}, \frac{1}{6} + \frac{1}{2}\sqrt{\frac{1}{18}}, \frac{1}{6} - \frac{1}{2}\sqrt{\frac{1}{18}}, ...$ as in Figure 4.

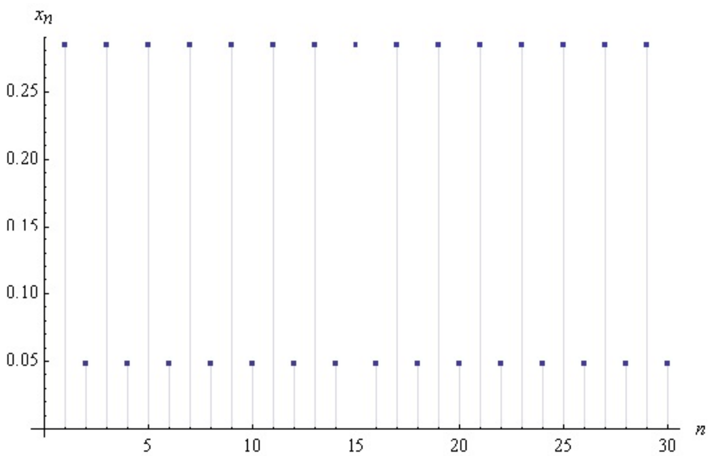

Figure 4: prime period-2 solution of Example 3.1

4 Conclusion and Disucssion

This article considers positive equilibrium solution and positive periodic solution of prime period 2 of a rational difference equation (1.4) which is

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + \cdots + B_k x_{n-k}}, n \in \{0, 1, 2, ...\}$$

where the parameters α, β, A, B_0, B_1, ..., B_k and the initial conditions x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_{-1}, x_0 are nonnegative real numbers such that the denominator (1.4) is always positive. We obtain necessary and sufficient conditions concerning the parameters of (1.4) for the existence of an equilibrium solution and a prime period-2 solution as shown in the following table.

Conditions	Solution
$b > 0$	$..., \phi, \phi, \phi, ...,$ where $\phi = \frac{-(A-\beta)+\sqrt{(A-\beta)^2+4b\alpha}}{2b}$.
$0 \le \beta < A$ and $b = 0$	$..., \phi, \phi, \phi, ...,$ where $\phi = \frac{\alpha}{A-\beta}$.
k is odd, $0 \le A < \beta, b_0 > b_1$ and $\left(\frac{\beta-A}{b_1}\right)^2 > \frac{4\alpha}{b_0-b_1}$	$..., \phi, \psi, \phi, \psi, \phi, \psi, ...,$ where $\phi = \frac{\beta-A}{2b_1} + \frac{1}{2}\sqrt{\left(\frac{\beta-A}{b_1}\right)^2 - \frac{4\alpha}{b_0-b_1}}$ and $\psi = \frac{\beta-A}{2b_1} - \frac{1}{2}\sqrt{\left(\frac{\beta-A}{b_1}\right)^2 - \frac{4\alpha}{b_0-b_1}}$.

Actually, for $3|(k+1)$, we have necessary and sufficient conditions for the existence of a periodic solution of prime period-3 of (1.4) as shown in the following table.

Conditions	Solution
($\alpha = 0$ and $0 < A < \beta$) or ($\alpha = 0, A = 0, b_0 > 0$ and $b_1 > 0$).	$..., 0, 0, \gamma, 0, 0, \gamma, ...,$ where $\gamma = \frac{\beta-A}{b_2}$.
($\alpha = 0$, $0 < A < \beta$, $b_0 > b_2$ and $b_1 > b_2$) or $\alpha = 0$, $0 < A < \beta$ and $0 \le b_0, b_1 < b_2$.	$..., 0, \psi, \gamma, 0, \psi, \gamma, ...,$ where $\psi = \frac{(\beta-A)(b_2-b_1)}{b_2^2-b_0b_1}$ and $\gamma = \frac{(\beta-A)(b_2-b_0)}{b_2^2-b_0b_1}$.

However, the results coincide with those obtained by Chaihao [3]. Thus, as for the future research, one may try to find necessary and sufficient conditions for the existence of prime period-3 solution of (1.4) of the form $..., \phi, \psi, \gamma, \phi, \psi, \gamma, \phi, \psi, \gamma, ...$ or the existence of prime period-p solution of (1.4) for a positive integer $p \ge 4$. Also, one can consider the local or global stability of the equilibrium or the prime period-p solution of (1.4).

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Ratinan Boonklurb
Department of Mathematics and Computer Science, Faculty of Science,
Chulalongkorn University, Thailand
Email: ratinan.b@chula.ac.th

Julaluk Neammai
Department of Mathematics and Statistics, Faculty of Science,
Prince of Songkla University, Thailand
Email: nuchgifted@gmail.com

Vasana Sukkrasanti
Department of Mathematics and Computer Science, Faculty of Science,

Chulalongkorn University, Thailand
Email: vasana.c@chula.ac.th

Theeruth Tantasuparuk
Department of Mathematics and Computer Science, Faculty of Science,
Chulalongkorn University, Thailand
Email: Theeruth_bill@windowslive.com