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$\mu-$filters of Almost Distributive Lattices

Noorbhasha Rafi* and Ravi Kumar Bandaru

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Abstract: The concept of $\mu-$filters is introduced in an Almost Distributive Lattice(ADL) and studied their properties in terms of dual annihilator filters of an ADL. Observed that the set of all dual annihilator filters of an ADL forms a complete Boolean algebra. Derived equivalent conditions for every filter of an ADL becomes a dual annihilator filter by assuming the property that every proper filter is non co-dense. Also, observed that μ is homomorphism of $F(L)$ in to $I(\mathfrak{A}^+(L))$. Characterized $\mu-$filter in element wise and verified that every minimal prime filter of an ADL is a $\mu-$filter. Finally, we proved that the intersection of all prime $\mu-$filters is the set of all maximal elements of an ADL.

Keywords: Almost Distributive Lattice(ADL), $\mu-$filter, Co-dense, minimal prime filter, dual annihilator filter, Boolean algebra

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1 Introduction

U. M. Swamy and G. C. Rao [5] have introduced the notion of an Almost Distributive Lattice (ADL). An ADL $(L, \vee, \wedge, 0)$ satisfies all the axioms of distributive lattice, except possibly the commutativity of the operations $\wedge$ and $\vee$. It is known that, in any ADL the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$. In [4], M.S. Rao and Badawy, Abd El-Mohsen introduced the notion of $\mu-$filters and then characterized with the help of

* *Corresponding author*

co-annihilator filters. It is proved that the class of all $\mu-$filters of a lattice forms a complete distributive lattice. An isomorphism is obtained between the lattice of $\mu-$filters of a distributive lattice and the lattice of all co-annihilator filters. In this paper,we extended this concept of $\mu-$filters to an Almost Distributive Lattice(ADL), analogously and studied their properties in terms of dual annihilator filters of an ADL. Derived equivalent conditions for every filter of an ADL becomes a dual annihilator filter by assuming the property that every proper filter is non co-dense. Characterized $\mu-$filter in element wise and verified that every minimal prime filter of an ADL is a $\mu-$filter.

2 Preliminaries

First, we recall certain definitions and properties of ADLs that are required in the paper. We begin with ADL definition as follows.

Definition 2.1. [5] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x \vee 0 = x$, for all $x, y, z \in L$.

Example 2.2. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.3. *[5] If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1). $a \vee b = a \Leftrightarrow a \wedge b = b$
(2). $a \vee b = b \Leftrightarrow a \wedge b = a$
(3). $\wedge$ *is associative in* L
(4). $a \wedge b \wedge c = b \wedge a \wedge c$
(5). $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(7). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(8). $a \wedge (a \vee b) = a, \quad (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(9). $a \leq a \vee b$ *and* $a \wedge b \leq b$
(10). $a \wedge a = a$ *and* $a \vee a = a$
(11). $0 \vee a = a$ *and* $a \wedge 0 = 0$
(12). *If* $a \leq c, \quad b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
(13). $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

Theorem 2.4. *[5] Let* $(L, \vee, \wedge, 0)$ *be an ADL with 0. Then the following are equivalent:*

1). $(L, \vee, \wedge, 0)$ *is a distributive lattice*
2). $a \vee b = b \vee a$*, for all* $a, b \in L$
3). $a \wedge b = b \wedge a$*, for all* $a, b \in L$
4). $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*, for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.5. *[5] Let* L *be an ADL and* $m \in L$*. Then the following are equivalent:*

1). m *is maximal with respect to* $\leq$
2). $m \vee a = m$*, for all* $a \in L$
3). $m \wedge a = a$*, for all* $a \in L$
4). $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [1, 2], a non-empty subset I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. A proper filter G of L is called a prime filter of L if, for any $x, y \in L$, $x \vee y \in G \Rightarrow x \in G$ or $y \in G$. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L \text{ and } n \in N\}$ is the smallest filter containing S. If $S = \{s\}$, we write $[s)$ instead of $[S)$. The set $F(L)$ of all filters of L forms a bounded distributive lattice, where $F \cap G$ is the infimum and $F \vee G = \{a \wedge b \mid a \in F, \ b \in G\}$ is the supremum in $F(L)$.

Theorem 2.6. *[5] For any x, y in L the following are equivalent:*

1). $(x] \subseteq (y]$
2). $y \wedge x = x$
3). $y \vee x = y$
4). $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

Theorem 2.7 ([3])**.** *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

3 $\mu-$filters of ADLs

In [4], the concept of $\mu-$filters is introduced and studied their important properties. In this paper, extended the notion of $\mu-$filters to an ADL, analogously.

Observed that the set of all dual annihilator filters of an ADL forms a complete Boolean algebra. Equivalent conditions are established for every filter of an ADL becomes a dual annihilator filter by assuming the property that every proper filter is non co-dense. Characterized $\mu-$filter in element wise and verified that every minimal prime filter of an ADL is a $\mu-$filter. Finally, we proved that the intersection of all prime $\mu-$filters is the set of all maximal elements of an ADL. Though many results look similar, the proofs are not similar because we do not have the properties like commutativity of $\vee$ commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ in an ADL.
We begin with the following definition.

Definition 3.1. For any subset S of an ADL L with maximal elements, define $S^+ = \{x \in L \mid s \vee x$ is a maximal element, for all $s \in S\}$.
Here we say that S^+ is a dual annihilator of S. For $S = \{x\}$, then we denote simply $(x)^+$ for $(\{x\})^+$. It is clear that $L^+ = \mathcal{M}_{max.elt}$, where $\mathcal{M}_{max.elt}$ is the set of all maximal elements of an ADL L, for any maximal element m of an ADL L, we have $m^+ = L$ and it is easy to verify that S^+ is a filter of an ADL L.

The following result is a direct consequence of definition.

Lemma 3.2. *Let L be an ADL with maximal elements and S, T be any non-empty subsets of L. Then the following conditions hold:*

(1). $[S) \cap S^+ = \mathcal{M}_{max.elt}$

(2). $S \subseteq S^{++}$

(3). If $S \subseteq T$, *then* $T^+ \subseteq S^+$

(4). $S^{+++} = S^+$.

Lemma 3.3. *Let L be an ADL with maximal elements. For any two filters F and G of L, we have the following conditions:*

(1). $(F \vee G)^+ = F^+ \cap G^+$

(2). $F^+ = L$ *iff* $F = \mathcal{M}_{max.elt}$

(3). $F \cap G = \mathcal{M}_{max.elt}$ *iff* $F \subseteq G^+$.

Proof. 1. Clearly, $(F \vee G)^+ \subseteq F^+ \cap G^+$. Let $x \in F^+ \cap G^+$. Then $x \vee y$ is maximal, for all $y \in F$ and $x \vee z$ is maximal, for all $z \in G$. Let t be any element of L.

Now $(x \vee (y \wedge z)) \wedge t = t$, since $x \vee y$ and $x \vee z$ are maximal elements of L. Hence $x \vee (y \wedge z)$ is a maximal element of L, for all $y \wedge z \in F \vee G$. That implies $x \in (F \vee G)^+$. Therefore $F^+ \cap G^+ \subseteq (F \vee G)^+$. Thus $(F \vee G)^+ = F^+ \cap G^+$.
2. Assume that $F^+ = L$. Then $0 \in F^+$. That implies $0 \vee x$ is maximal, for all $x \in F$. Hence every element of F is a maximal element. Therefore $F \subseteq \mathcal{M}_{max.elt}$. Thus $F = \mathcal{M}_{max.elt}$. Conversely, assume that $F = \mathcal{M}_{max.elt}$. Let x be any element of L. Then $x \vee y$ is maximal, for all $y \in F$. That implies $x \in F^+$. Therefore $F^+ = L$.
3. Assume that $F \cap G = \mathcal{M}_{max.elt}$. Let $x \in F$. Then $x \vee y$ is maximal, for all $y \in G$. That implies $x \in G^+$. Therefore $F \subseteq G^+$. Conversely, assume that $F \subseteq G^+$. Let $x \in F \cap G$. Then $x \in F$ and $x \in G$. By our assumption, we have $x \in G^+$. It implies that $x \vee y$ is maximal, for all $y \in G$. Since $x \in G$, we have $x \vee x = x$ is maximal. Therefore $x \in \mathcal{M}_{max.elt}$ and hence $F \cap G \subseteq \mathcal{M}_{max.elt}$. Thus $F \cap G = \mathcal{M}_{max.elt}$. □

Corollary 3.4. *Let L be an ADL with maximal elements. For any $x, y \in L$, the following conditions hold:*

(1). If $x \leq y$ then $x^+ \subseteq y^+$

(2). $(x \wedge y)^+ = x^+ \cap y^+$

(3). $x^+ = L$ iff x is a maximal element.

we have the following definition.

Definition 3.5. Let L be an ADL with maximal elements. A filter F of L is said to be a direct factor of L, if there exists a proper filter G such that $F \cap G = \mathcal{M}_{max.elt}$ and $F \vee G = L$.

Theorem 3.6. *Let L be an ADL with maximal elements. For any $x \in L$, x^+ is a direct factor of L if and only if $x^+ \vee x^{++} = L$.*

Proof. Assume that x^+ is a direct factor of L. Then there exists a proper filter G of L such that $x^+ \cap G = \mathcal{M}_{max.elt}$ and $x^+ \vee G = L$. Since $x^+ \cap G = \mathcal{M}_{max.elt}$, we have $G \subseteq x^{++}$ and hence $x^+ \vee x^{++} = L$. Conversely, assume that $x^+ \vee x^{++} = L$. We have to show that $x^+ \cap x^{++} = \mathcal{M}_{max.elt}$. Let $t \in x^+ \cap x^{++}$. Then $t \in x^+$ and $t \in x^{++}$. By our assumption, we have t is a maximal element. Therefore $x^+ \cap x^{++} = \mathcal{M}_{max.elt}$ and hence x^+ is a direct factor of L. □

Definition 3.7. Let L be an ADL with maximal elements. A filter F of L is called a dual annihilator filter if $F = F^{++}$ or equivalently, $F = S^{+}$, for some non-empty subset S of L.

It is easy to verify that the set $\mathcal{DA}(L)$ of dual annihilator filters of an ADL L forms a complete Boolean algebra.

Definition 3.8. Let L be an ADL with maximal elements. A filter F of L is said to be co-dense if $F^{+} = \mathcal{M}_{max.elt}$.

We have the following result.

Theorem 3.9. *Let L be an ADL with maximal elements, in which every proper filter is non co-dense. Then the following conditions are equivalents:*

(1). Every filter is a dual annihilator filter

(2). Every prime filter is a dual annihilator filter

(3). Every prime filter is minimal

(4). Every prime filter is maximal.

Proof. $(1) \Rightarrow (2)$: Clear
$(2) \Rightarrow (3)$: Assume that every prime filter is a dual annihilator filter. Let P be a prime filter. Then by our assumption $P = P^{++}$. Suppose P is not minimal. Then there exists a prime filter Q such that $Q \subsetneq P$. Choose an element $x \in P$ such that $x \notin Q$. Since $x \in P^{++}$, we have that $x \vee a$ is maximal, for all $a \in P^{+}$. Since $a \in P^{+}$, we get that $a \vee b$ is maximal, for all $b \in P$. That implies $x \vee a$ is maximal and hence $x \vee a \in Q$. Since $x \notin Q$, we get that $a \in Q \subset P$. That implies $P^{+} \subseteq P$. So that $P^{+} = P^{+} \cap P = \mathcal{M}_{max.elt}$. That implies $P^{++} = L$ and hence $P = P^{++} = L$, which is a contradiction. Therefore P is minimal.
$(3) \Rightarrow (4)$: Clear
$(4) \Rightarrow (1)$: Assume that every prime filter is maximal. Let F be a filter of L. Clearly, $F \subseteq F^{++}$. Suppose that $F^{++} \nsubseteq F$. Choose an element $x \in F^{++}$ such that $x \notin F$. Since $x \notin F$, there exists a prime filter P such that $F \subseteq P$ and $x \notin P$. That implies $P^{+} \subseteq F^{+}$ and $x^{+} \subseteq P$. By our assumption, P is maximal. Since $x \notin P$, we get that $P \vee [x) = L$. That implies $(P \vee [x))^{+} = \mathcal{M}_{max.elt}$. That implies $P^{+} \cap [x)^{+} = \mathcal{M}_{max.elt}$. Since $x \in F^{++}$, we have $F^{+++} \subseteq x^{+}$. That implies $F^{+} \subseteq x^{+}$. That implies $P^{+} \cap F^{+} \subseteq P^{+} \cap [x)^{+} = \mathcal{M}_{max.elt}$. That implies that $P^{+} \cap F^{+} = \mathcal{M}_{max.elt}$ and hence $P^{+} = \mathcal{M}_{max.elt}$, since $P^{+} \subseteq F^{+}$. That

implies $P^{++} = L$, which is a contradiction. Hence $x \in F$. Therefore $F^{++} \subseteq F$. Thus $F^{++} = F$. □

Lemma 3.10. *Let L be an ADL with maximal element. For any two filters F, G of L, $(F \cap G)^{++} = F^{++} \cap G^{++}$.*

Proof. Clearly, $(F \cap G)^{++} \subseteq F^{++} \cap G^{++}$. Let $x \in F^{++} \cap G^{++}$. Then $x \in F^{++}$ and $x \in G^{++}$. Let $y \in (F \cap G)^{+}$. Then $y \vee z$ is maximal, for all $z \in F \cap G$. That implies $y \vee (f \vee g)$ is maximal, since $z = f \vee g$, for some $f \in F$ and $g \in G$. So that $y \vee f \in G$ and hence $y \vee f \vee x$ is maximal. Since $x \in G^{++}$, we have $x \vee y \in F^{+}$. That implies $x \vee y$ is maximal, since $x \in F^{++}$. Therefore $x \in (F \cap G)^{++}$ and hence $(F \cap G)^{++} = F^{++} \cap G^{++}$. □

Corollary 3.11. *For any $x, y \in L$, $(x \vee y)^{++} = x^{++} \cap y^{++}$.*

Proof. Clearly, $x^{++} \cap y^{++} \subseteq (x \vee y)^{++}$. Let $t \in (x \vee y)^{++}$. Then $t \vee s$ is maximal, for all $s \in (x \vee y)^{+}$. Let $a \in x^{+} \cap y^{+}$. Then $x \vee a$ is maximal and $a \vee y$ is maximal and hence $a \vee x \vee y$ is maximal. Therefore $a \in (x \vee y)^{+}$. So that $t \vee a$ is maximal, since $t \in (x \vee y)^{+}$. That implies $t \in x^{++} \cap y^{++}$. Thus $(x \vee y)^{++} = x^{++} \cap y^{++}$. □

The following two result are verified easily.

Theorem 3.12. *Let L be an ADL with maximal elements. If $\mathfrak{A}^{+}(L) = \{x^{++} \mid x \in L\}$ then $\mathfrak{A}^{+}(L)$ forms a distributive lattice with respect to the operation $\cap$ and $\sqcup$ defined by $a^{++} \cap b^{++} = (a \vee b)^{++}$ and $a^{++} \sqcup b^{++} = (a^{+} \cap b^{+})^{+} = (a \wedge b)^{++}$.*

Theorem 3.13. *Let L be an ADL with maximal elements. Then $f : L \longrightarrow \mathfrak{A}^{+}(L)$ by $f(x) = x^{++}$ is a dual homomorphism.*

Definition 3.14. Let L be an ADL with maximal elements.

(1). For any filter F of L, define $\mu(F) = \{x^{++} \mid x \in F\}$

(2). For any ideal I of $\mathfrak{A}^{+}(L)$, define $\overleftarrow{\mu}(I) = \{x \in L \mid x^{++} \in I\}$.

We prove the following result.

Lemma 3.15. *Let L be an ADL with maximal elements. Then the following conditions hold:*

(1). For any filter F of L, $\mu(F)$ is an ideal in $\mathfrak{A}^{+}(L)$

(2). For any ideal I of $\mathfrak{A}^{+}(L)$, $\overleftarrow{\mu}(I)$ is a filter of L

(3). μ and $\overleftarrow{\mu}$ are isotones.

Proof. 1. Clearly, $\mu(F)$ is a non-empty set. Let $x^{++}, y^{++} \in \mu(F)$. Then $x, y \in F$. Since F is a filter of L, we have $x \wedge y \in F$. That implies that $(x \wedge y)^{++} \in \mu(F)$. Since $x^{++} \sqcup y^{++} = (x \wedge y)^{++}$, we have $x^{++} \sqcup y^{++} \in \mu(F)$. Let $x^{++} \in \mu(F)$ and $r^{++} \in \mathfrak{A}^{+}(L)$. Then $x \in F$ and hence $x \vee r \in F$. That implies that $(x \vee r)^{++} \in \mu(F)$. Therefore $x^{++} \cap r^{++} \in \mu(F)$, since $(x \vee r)^{++} = x^{++} \cap y^{++}$. Thus $\mu(F)$ is an ideal of $\mathfrak{A}^{+}(L)$.
2. Let I be an ideal of $\mathfrak{A}^{+}(L)$. Clearly, $\overleftarrow{\mu}(I)$ is a non-empty set. Let $x, y \in \overleftarrow{\mu}(I)$. Then $x^{++}, y^{++} \in I$. That implies $x^{++} \sqcup y^{++} \in I$ and hence $(x \wedge y)^{++} = x^{++} \sqcup y^{++} \in I$. Therefore $x \wedge y \in \overleftarrow{\mu}(I)$. Let $x \in \overleftarrow{\mu}(I)$ and $r \in L$. Then $x^{++} \in I$. That implies $x^{++} \cap r^{++} \in I$ and hence $(x \vee r)^{++} \in I$. Therefore $r \vee x \in \overleftarrow{\mu}(I)$. Thus $\overleftarrow{\mu}(I)$ is a filter of L.
3. Let F and G be two filters of L such that $F \subseteq G$. We have to prove that $\mu(F) \subseteq \mu(G)$. Let $x^{++} \in \mu(F)$. Then $x \in F \subseteq G$. That implies $x^{++} \in \mu(G)$. Therefore $\mu(F) \subseteq \mu(G)$. Let I, J be any two ideals of $\mathfrak{A}^{+}(L)$ with $I \subseteq J$. We have to show that $\overleftarrow{\mu}(I) \subseteq \overleftarrow{\mu}(J)$. Let $x \in \overleftarrow{\mu}(I)$. Then $x^{++} \in I \subseteq J$ and hence $x \in \overleftarrow{\mu}(J)$. Therefore $\overleftarrow{\mu}(I) \subseteq \overleftarrow{\mu}(J)$. Thus μ and $\overleftarrow{\mu}$ are isotones. □

Note that it is easy to verify that the set $I(\mathfrak{A}^{+}(L))$ of all ideals of $\mathfrak{A}^{+}(L)$ forms a distributive lattice.

Theorem 3.16. *The mapping μ is a homomorphism of $F(L)$ in to $I(\mathfrak{A}^{+}(L))$.*

Proof. Let F, G be any two filters of L. We have to prove that $\mu(F \vee G) = \mu(F) \sqcup \mu(G)$ and $\mu(F \cap G) = \mu(F) \cap \mu(G)$. Clearly, $\mu(F) \sqcup \mu(G) \subseteq \mu(F \vee G)$. Let $x^{++} \in \mu(F \vee G)$. Then $x^{++} = y^{++}$, for some $y \in F \vee G$. Since $y \in F \vee G$, we have $y = f \wedge g$, for some $f \in F$ and $g \in G$. Now, $x^{++} = y^{++} = (f \vee g)^{++} = f^{++} \sqcup g^{++} \in \mu(F) \sqcup \mu(G)$. Therefore $\mu(F \vee G) = \mu(F) \sqcup \mu(G)$. Clearly, $\mu(F \cap G) \subseteq \mu(F) \cap \mu(G)$. Let $x \in \mu(F) \cap \mu(G)$. Then $x \in F$ and $x \in G$. That implies $x \in F \cap G$. That implies $x^{++} \in \mu(F \cap G)$ and hence $\mu(F \cap G) = \mu(F) \cap \mu(G)$. Thus μ is a homomorphism. □

Lemma 3.17. *For any filter F of an ADL L, the mapping $F \longrightarrow \overleftarrow{\mu}\mu(F)$ is a closure operator on $F(L)$, that is*

(1). $F \subseteq \overleftarrow{\mu}\mu(F)$

(2). $\overleftarrow{\mu}\mu(\overleftarrow{\mu}\mu(F)) = \overleftarrow{\mu}\mu(F)$

(3). If $F \subseteq G$ then $\overleftarrow{\mu}\mu(F) \subseteq \overleftarrow{\mu}\mu(G)$.

Proof. 1. Let $x \in F$. Then $x^{++} \in \mu(F)$. Since $\mu(F)$ is an ideal of $\mathfrak{A}^+(L)$, and $x^{++} \in \mu(F)$, we have $x \in \overleftarrow{\mu}\mu(F)$. Therefore $F \subseteq \overleftarrow{\mu}\mu(F)$.
2. Clearly, $\overleftarrow{\mu}\mu(F) \subseteq \overleftarrow{\mu}\mu(\overleftarrow{\mu}\mu(F))$. Let $x \in \overleftarrow{\mu}\mu(\overleftarrow{\mu}\mu(F))$. Then $x^{++} \in \mu(\overleftarrow{\mu}\mu(F))$. That implies $x^{++} = y^{++}$, for some $y \in \overleftarrow{\mu}\mu(F)$ and hence $x^{++} = y^{++} \in \mu(F)$, since $y \in \overleftarrow{\mu}\mu(F)$. That implies $x \in \overleftarrow{\mu}\mu(F)$. Therefore $\overleftarrow{\mu}\mu(\overleftarrow{\mu}\mu(F)) = \overleftarrow{\mu}\mu(F)$.
3. Suppose that $F \subseteq G$. Then $\mu(F) \subseteq \mu(G)$ and hence $\overleftarrow{\mu}\mu(F) \subseteq \overleftarrow{\mu}\mu(G)$. □

Now, we introduce the definition of $\mu-$filter to an ADL.

Definition 3.18. A filter F of an ADL L is said to be a $\mu-$filter, if $\overleftarrow{\mu}\mu(F) = F$.

It is observed easily that the class of all $\mu-$filters of an ADL L with maximal elements forms a complete distributive lattice which is isomorphic to the lattice of ideals, ordered by set-inclusion, of the lattice $\mathfrak{A}^+(L)$. The infimum of a set of $\mu-$filters $\{F_i\}_{i\in\Delta}$ of an ADL is $\bigcap\limits_{i\in\Delta} F_i$, their set-theoretic intersection. The supremum is $\overleftarrow{\mu}\mu(\bigvee\limits_{i\in\Delta}(F_i))$, where $\bigvee\limits_{i\in\Delta} F_i$ is their supremum in the lattice of filters of L.

Theorem 3.19. *Let F be a filter of an ADL L. Then the following conditions are equivalent:*

(1). F is a $\mu-$filter

(2). For any $x, y \in L$, $x^+ = y^+$ and $x \in F$, then $y \in F$

(3). $F = \bigcup\limits_{x\in F} x^{++}$

(4). if $x \in F$, then $x^{++} \subseteq F$.

Proof. $(1) \Rightarrow (2)$: Assume that F is a $\mu-$filter of L. Let $x, y \in L$ with $x^+ = y^+$ and $x \in F$. Then $x \in \overleftarrow{\mu}\mu(F)$ and hence $x^{++} \in \mu(F)$. That implies $y^{++} \in \mu(F)$. Therefore$y \in \overleftarrow{\mu}\mu(F)$. Thus $y \in F$.
$(2) \Rightarrow (3)$: Assume (2). Let $x \in F$. Then $[x) \subseteq F$ and hence $F = \bigcup\limits_{x\in F}[x) \subseteq \bigcup\limits_{x\in F} x^{++}$. Let $y \in \bigcup\limits_{x\in F} x^{++}$. Then $y \in x^{++}$, for some $x \in F$. That implies $y^{++} \subseteq x^{++}$ and hence $y^{++} = y^{++} \cap x^{++} = (y \vee x)^{++}$. Since $y \vee x \in F$, we have $y \in F$. Therefore $F = \bigcup\limits_{x\in F} x^{++}$.
$(3) \Rightarrow (4)$: Let $x \in F$ and $t \in x^{++}$. Then $t \in \bigcup\limits_{x\in F} x^{++} = F$. That implies $t \in F$.

Therefore $x^{++} \subseteq F$.
(4) $\Rightarrow$ (1) : Clearly, $F \subseteq \overleftarrow{\mu}\mu(F)$. Let $x \in \overleftarrow{\mu}\mu(F)$. Then $x^{++} \in \mu(F)$. That implies $x^{++} = y^{++}$, for some $y \in F$. By our assumption, $y^{++} \subseteq F$ and hence $x^{++} \subseteq F$. Therefore $x \in F$.

□

Corollary 3.20. *Let L be an ADL with maximal elements. Every minimal prime filter of L is a $\mu-$filter.*

Proof. Let P be a minimal prime filter of L. Let $x, y \in L$ with $x^+ = y^+$ and $x \in P$. Since P is a minimal prime filter, there exists an element $a \notin P$ such that $x \vee a$ is maximal. That implies $a \in x^+ = y^+$ and hence $a \vee y$ is maximal. Therefore $y \in P$. Thus P is a $\mu-$filter of L. □

Corollary 3.21. *Let L be an ADL with maximal elements. Every minimal prime $\mu-$filter of L is a minimal prime filter.*

Lemma 3.22. *Let L be an ADL and m be any maximal element of L. Then $\{m\}$ is a $\mu-$filter of L.*

Proof. Clearly, $\{m\} \subseteq \overleftarrow{\mu}\mu(\{m\})$. Let $x \in \overleftarrow{\mu}\mu(\{m\})$. Then $x^{++} \in \mu(\{m\})$. Since $x^{++} \in \mu(\{m\})$, we have $x^{++} = y^{++}$, for some $y \in \{m\}$. That implies $x^{++} = m^{++}$. Therefore $x^+ = m^+ = L$ and hence x is a maximal element. Thus $\{m\}$ is a $\mu-$filter of L. □

Lemma 3.23. *Let L be an ADL with maximal elements. For any F and G of an ADL L, we have $\overleftarrow{\mu}\mu(F \cap G) = \overleftarrow{\mu}\mu(F) \cap \overleftarrow{\mu}\mu(G)$.*

Proof. It is clear that $\overleftarrow{\mu}\mu(F \cap G) \subseteq \overleftarrow{\mu}\mu(F) \cap \overleftarrow{\mu}\mu(G)$. Let $x \in \overleftarrow{\mu}\mu(F) \cap \overleftarrow{\mu}\mu(G)$. Then $x \in \overleftarrow{\mu}\mu(F)$ and $x \in \overleftarrow{\mu}\mu(G)$. That implies $x^{++} \in \mu(F)$ and $x^{++} \in \mu(G)$. That implies $x^{++} = y^{++}$, for some $y \in F$ and $x^{++} = z^{++}$, for some $z \in G$. Therefore $x^{++} = y^{++} \cap z^{++} = (y \vee z)^{++}$. Since $y \vee z \in F \cap G$, we get that $(y \vee z)^{++} \in \mu(F \cap G)$. Implies that $x^{++} \in \mu(F \cap G)$ and hence $x \in \overleftarrow{\mu}\mu(F \cap G)$. Thus $\overleftarrow{\mu}\mu(F \cap G) = \overleftarrow{\mu}\mu(F) \cap \overleftarrow{\mu}\mu(G)$. □

Theorem 3.24. *Let F be a $\mu-$filter and I an ideal of an ADL L such that $F \cap I = \emptyset$. Then there exists a prime $\mu-$filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Consider $\mathfrak{F} = \{G \mid G \text{ is a } \mu - \text{filter}, F \subseteq G \text{ and } G \cap I = \emptyset\}$. Clearly, $f \in \mathfrak{F}$ and hence $\mathfrak{F}$ is a non-empty set. It is easy to verify that $\mathfrak{F}$ satisfies hypothesis of Zorn's lemma. Then $\mathfrak{F}$ has a minimal element say M. We prove that M is prime $\mu-$filter. Let $x, y \in L$ such that $x \vee y \in M$. Suppose that $x \notin M$ and $y \notin M$. Then $M \vee [x) \subseteq \overleftarrow{\mu}\mu(M \vee [x))$ and $M \vee [y) \subseteq \overleftarrow{\mu}\mu(y)$. Clearly, $\overleftarrow{\mu}\mu(M \vee [x))$ is a $\mu-$filter of L. Then $\overleftarrow{\mu}\mu(M \vee [x)) \cap I \neq \emptyset$. Similarly, we get that $\overleftarrow{\mu}\mu(M \vee [y)) \cap I \neq \emptyset$. Then then there exist elements $s, t \in L$ such that $s \in \overleftarrow{\mu}\mu(M \vee [x)) \cap I$ and $t \in \overleftarrow{\mu}\mu(M \vee [y)) \cap I$. That implies $s \vee t \in I$ and $s \vee t \in \overleftarrow{\mu}\mu(M \vee [x)) \cap \overleftarrow{\mu}\mu(M \vee [y)) = \overleftarrow{\mu}\mu((M \vee [y)) \cap (M \vee [y))) = \overleftarrow{\mu}\mu(M \vee [x \vee y))$. Therefore $s \vee t \in I$ and $s \vee t \in \overleftarrow{\mu}\mu(M \vee [x \vee y))$. Since $x \vee y \in M$, we get that $s \vee t \in \overleftarrow{\mu}\mu(M) = M$. That implies $s \vee t \in M \cap I$ and hence $I \cap M \neq \emptyset$, which is a contradiction. Therefore $x \in M$ or $y \in M$. Thus M is a prime $\mu-$filter of an ADL L. □

Corollary 3.25. *Let F be a $\mu-$filter of L and $x \notin F$. Then there exists a prime $\mu-$filter P of L such that $F \subseteq P$ and $x \notin P$.*

Corollary 3.26. *For any $\mu-$filter F of L, we have $F = \bigcap\{P \mid P$ is a prime $\mu-$filter of $L,\ F \subseteq P\}$.*

Corollary 3.27. *The intersection of all prime $\mu-$filters is equal to $\mathcal{M}_{max.elt}$.*

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Noorbhasha Rafi
Department of Mathematics,
Bapatla Engineering College
Bapatla, Andhra Pradesh, India-522 101
Email: rafimaths@gmail.com

Ravi Kumar Bandaru
Department of Engg. Mathematics,
GITAM University, Hyderabad Campus,
Telangana, India-502 329
Email: ravimaths83@gmail.com