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Argument Estimates Of Certain Meromorphiclly Mulivalent Functions Assoicated With The Multiplier Transformation

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Abstract: The object of this paper is to obtain some argument properties of meromorphically multivalent functions associated with the multiplier transformation. We also derive the integral preserving properties in a sector.

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1 Introduction

Let $\Sigma_{p,n}(n > -p)$ denote the class of all meromorphic functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U\backslash\{0\}$. For convenience, we write $\Sigma_{p,-p+1} = \Sigma_p$.

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as follows:

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$$f \prec g \text{ or } f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalent (cf., e.g., [5]; see also [10],[11, p. 4])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f_j(z) \in \Sigma_{p,n}$, given by

$$f_j(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,j} z^k \quad (j = 1, 2), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^n = (f_2 * f_1)(z). \tag{1.3}$$

Now, we define the operator $I_p^m(n, \lambda, \ell)$ $(\lambda \geq 0, \ell > 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ for a function $f(z) \in \Sigma_{p,n}$ given by (1.1) as follows:

$$I_p^m(n, \lambda, \ell) f(z) = z^{-p} + \sum_{k=n}^{\infty} \left[\frac{\lambda(k+p) + \ell}{\ell} \right]^m a_k z^k, \tag{1.4}$$

we can write (1.4) as follows:

$$I_p^m(n, \lambda, \ell) f(z) = (\Phi_{n,\lambda,\ell}^{p,m} * f)(z),$$

where

$$\Phi_{n,\lambda,\ell}^{p,m}(z) = z^{-p} + \sum_{k=n}^{\infty} \left[\frac{\lambda(k+p) + \ell}{\ell} \right]^m z^k. \tag{1.5}$$

It is easily verified from (1.4), that

$$\lambda z (I_p^m(n, \lambda, \ell) f(z))' = \ell I_p^{m+1}(n, \lambda, \ell) f(z) - (\lambda p + \ell) I_p^m(n, \lambda, \ell) f(z) \ (\lambda > 0). \tag{1.6}$$

The operator $I_p^m(n, \lambda, \ell)$ was introduced by El-Ashwah [9].
We note that:

$$I_p^0(n, \lambda, \ell) f(z) = f(z) \text{ and } I_p^1(n, 1, 1) f(z) = \frac{(z^{p+1} f(z))'}{z^p} = (p+1) f(z) + z f'(z).$$

Also by specilizing the parameters λ, ℓ, m and p, we obtain the following operators studied by various authors:

(i) $I_p^m(1,1)f(z) = D_p^m f(z)$ (see Aouf and Hossen [1]] and Srivastava and Patel [14]);

(ii) $I_1^m(1,\ell)f(z) = I(m,\ell)f(z)$ (see Cho et al. [6,7]);

(iii) $I_1^m(1,1) = I^m f(z)$ (see Uralegaddi and Somanatha [16]).

Making use of the principle of differential subordination as well as the linear operator $I_p^m(n,\lambda,\ell)$, we now introduce a subclass of the function class $\Sigma_{p,n}$ as follows:
Let $\Sigma_{p,n}^*[\lambda,\ell,m;A,B]$ be the class of functions $f(z) \in \Sigma_{p,n}$ defined by

$$\Sigma_{p,n}^*[\ell,\lambda,m;A,B] = \left\{ f \in \Sigma_{p,n} : -\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)f(z)} \prec p\frac{1+Az}{1+Bz}, \right.$$
$$\left. -1 \le B < A \le 1, \lambda > 0, \ell > 0, p \in N, n > -p, m \in \mathbb{N}_0, z \in U \right\}. \quad (1.7)$$

We note that:

(i) For $m = 0$, we note that $\Sigma_{p,n}^*[\ell,\lambda,0;1,-1] = \Sigma_{p,n}^*$, is the well-known class of meromorphically starlike functions;

(ii) For $m = 0$, $A = 1-\frac{2\alpha}{p}$, $0 \le \alpha < p$ and $B = -1$, we note that $\Sigma_{p,n}^*[\ell,\lambda,0;1-\frac{2\alpha}{p},-1] = \Sigma_{p,n}^*[\alpha]$, is the well-known class of meromorphically starlike functions of order α (see [2]).

From (1.7) and by using the result of Silverman and Silvia [15], we observe that a function $f(z)$ is in $\Sigma_{p,n}^*[\ell,\lambda,m;A,B]$ $(-1 < B < A \le 1,\ \lambda > 0,\ \ell > 0, p \in \mathbb{N}, m \in \mathbb{N})$ if and only if

$$\left| \frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)f(z)} + \frac{p(1-AB)}{1-B^2} \right| < \frac{p(A-B)}{1-B^2} \ (z \in U). \quad (1.8)$$

For a function $f(z) \in \Sigma_{p,n}$ and $\nu > 0$, the integral operator $F_{\nu,p}(f)(z)$: $\Sigma_{p,n} \to \Sigma_{p,n}$ is defined by

$$F_{\nu,p}(f)(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t)dt$$
$$= z^{-p} + \sum_{k=n}^{\infty} \left(\frac{\nu}{\nu+p+k} \right) a_k z^k * f(z) \ \ (\nu > 0; z \in U) \ . \quad (1.9)$$

It follows from (1.9) that

$$z\left(I_p^m(n,\lambda,\ell)\,F_{\nu,p}(f)(z)\right)^{'} = \nu I_p^m(n,\lambda,\ell)f(z) - (\nu+p)I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z). \quad (1.10)$$

The operator $F_{\nu,p}(f)(z)$ was investigated by many authors (see for example [1], [17] and [18]).

The object of the present paper is to give some argument properties of meromorphically functions belonging to $\Sigma_{p,n}$ and the integral preserving properties in connection with the operator $I_p^m(n,\lambda,\ell)$ defined by (1.4).

Many researches introduced and study the argument properties of meromorphically multivalent functions such as Aouf [3], cho et al. [6] and Qing Yang and Jin-Lin Liu [13].

2 Main Reuslts

Unless otherwise mentioned,we shall assume in the reminder of this paper that $\lambda \geq 0, \ell > 0, p \in \mathbb{N}$ *and* $m \in \mathbb{N}_0$. In order to show our main results, we need the following lemmas.

Lemma 2.1 ([8])**.** *Let* h *be convex univalent in* U *with* $h(0) = 1$ *and* $\operatorname{Re}\{\beta h(z) + \gamma\} > 0$ $(\beta, \gamma \in \mathbb{C})$. *If* q *is analytic in* U *with* $q(0) = 1$, *then*

$$q(z) + \frac{zq^{'}(z)}{\beta q(z) + \gamma} \prec h(z)$$

implies

$$q(z) \prec h(z).$$

Lemma 2.2 ([10])**.** *Let* h *be convex univalent in* U *and* $\lambda(z)$ *be analytic in* U $\operatorname{Re}\lambda(z) \geq 0$. *If* q *is analytic in* U *and* $q(0) = h(0)$, *then*

$$q(z) + \lambda(z)zq^{'}(z) \prec h(z)$$

implies

$$q(z) \prec h(z) \quad (z \in U)\ .$$

Lemma 2.3 ([12])**.** *Let* q *be analytic in* U *with* $q(0) = 1$ *and* $q(z) \neq 0$ *in* U. *Suppose that there exists a point* z_0 *in* U *such that*

$$|\arg q(z)| < \frac{\pi}{2}\alpha \quad for \quad |z| < |z_0| \quad (2.1)$$

and

$$|\arg q(z_0)| < \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\alpha, \tag{2.3}$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg q(z_0) = \frac{\pi}{2}\alpha, \tag{2.4}$$

$$k \geq \frac{-1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg q(z_0) = \frac{-\pi}{2}\alpha, \tag{2.5}$$

and

$$q(z_0)^{\frac{1}{\alpha}} = \pm i\alpha \qquad (\alpha > 0). \tag{2.6}$$

At first, with the help of Lemma 2.1, we obtain the following theorem.

Theorem 2.4. *Let h be convex univalent in U with $h(0) = 1$ and $\mathrm{Re}\{h\}$ be bounded in U. If $f(z) \in \Sigma_{p,n}$ satisfies the condition*

$$-\frac{z(I_p^{m+1}(n,\lambda,\ell)f(z))'}{pI_p^{m+1}(n,\lambda,\ell)f(z)} \prec h(z),$$

then

$$-\frac{z(I_p^{m}(n,\lambda,\ell)f(z))'}{pI_p^{m}(n,\lambda,\ell)f(z)} \prec h(z)$$

for $\max_{z\in U} \mathrm{Re}\, h(z) < \left(\frac{\frac{\ell}{\lambda}+p}{p}\right)$ *(provided $I_p^m(n,\lambda,\ell)f(z) \neq 0$ in U).*

Proof. Let

$$q(z) = -\frac{z(I_p^{m}(n,\lambda,\ell)f(z))'}{pI_p^{m}(n,\lambda,\ell)f(z)} \qquad (z \in U)\ .$$

By using (1.6), we have

$$q(z) - \left(\frac{\frac{\ell}{\lambda}+p}{p}\right) = -\frac{\frac{\ell}{\lambda}I_p^{m+1}(n,\lambda,\ell)f(z)}{pI_p^{m}(n,\lambda,\ell)f(z)}\ . \tag{2.7}$$

Differentiating (2.7) logarithmically with respect to z and multiplying by z, we get

$$\frac{zq'(z)}{-pq(z) + \left(\frac{\ell}{\lambda}+p\right)} + q(z) = -\frac{z(I_p^{m+1}(n,\lambda,\ell)f(z))'}{pI_p^{m+1}(n,\lambda,\ell)f(z)} \prec h(z) \quad (z \in U)\ .$$

From Lemma 2.1, it follows that $q(z) \prec h(z)$ for $\mathrm{Re}\left\{-h(z)+\left(\frac{\frac{\ell}{\lambda}+p}{p}\right)\right\} > 0$ $(z \in U)$, which means

$$-\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{pI_p^m(n,\lambda,\ell)f(z)} \prec h(z),$$

for $\max_{z\in U} \mathrm{Re}\, h(z) < \left(\frac{\frac{\ell}{\lambda}+p}{p}\right)$. □

Using Lemmas 2.1 and 2.2 and Theorem 2.4, we now derive the following Theorem.

Theorem 2.5. *Let $f(z) \in \Sigma_{p,n}$ and choose ℓ and λ such that $\frac{\ell}{\lambda} \geq \frac{p(A-B)}{1+B}$, where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If*

$$\left|\arg\left(-\frac{z(I_p^{m+1}(n,\lambda,\ell)f(z))'}{I_p^{m+1}(n,\lambda,\ell)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < p; 0 < \delta \leq 1)$$

for some $g \in \Sigma^*_{p,n}(\ell,\lambda,m+1;A,B)$, then

$$\left|\arg\left(-\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where $\alpha(0 < \alpha \leq 1)$ is the solution of

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha \sin\frac{\pi}{2}(1-t(A,B))}{\frac{\frac{\ell}{\lambda}(1-B)+p(A-B)}{(1-B)} + \alpha\cos\frac{\pi}{2}(1-t(A,B))}\right\}, \tag{2.8}$$

when

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left[\frac{p(A-B)}{\left(\frac{\ell}{\lambda}+p\right)(1-B^2) - p(1-AB)}\right]. \tag{2.9}$$

Proof. Let

$$q(z) = -\frac{1}{p-\gamma}\left(\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)g(z)} + \gamma\right) \qquad (z \in U).$$

By using the identity (1.6), we have

$$(p-\gamma)zq'(z)I_p^m(n,\lambda,\ell)g(z) + (p-\gamma)q(z)z(I_p^m(n,\lambda,\ell)g(z))' + \gamma z(I_p^m(n,\lambda,\ell)g(z))'$$

$$= \left(\frac{\ell}{\lambda} + p\right) z(I_p^m(n,\lambda,\ell)f(z))^{'} - \frac{\ell}{\lambda} z(I_p^{m+1}(n,\lambda,\ell)f(z))^{'} .$$

Dividing (2.10) by $I_p^m(n,\lambda,\ell)g(z)$ and simplifying, we obtain

$$q(z) + \frac{zq^{'}(z)}{-r(z) + \frac{\ell}{\lambda} + p} = -\frac{1}{p-\gamma}\left(\frac{z(I_p^{m+1}(n,\lambda,\ell)f(z))^{'}}{I_p^{m+1}(n,\lambda,\ell)g(z)} + \gamma\right) , \tag{2.11}$$

where

$$r(z) = -\frac{z(I_p^m(n,\lambda,\ell)g(z))^{'}}{I_p^m(n,\lambda,\ell)g(z)} .$$

Since $g(z) \in \Sigma_{p,n}^*(\ell,\lambda,m+1;A,B)$, from Theorem 1, we have

$$r(z) \prec p\frac{1+Az}{1+Bz} ,$$

using (1.8), we have

$$-r(z) + \left(\frac{\ell}{\lambda} + p\right) = \rho e^{i\frac{\pi}{2}\varphi},$$

where

$$\frac{\frac{\ell}{\lambda}(1+B) - p(A-B)}{1+B} < \rho < \frac{\frac{\ell}{\lambda}(1-B) + p(A-B)}{1-B},$$

$$-t(A,B) < \varphi < t(A,B),$$

where $t(A,B)$ is given by (2.9).

Let h be a function which maps U onto the angular domain $\left\{w : |\arg w| < \frac{\pi}{2}\delta\right\}$ with $h(0) = 1$. Applying Lemma 2.2 for this h with $\lambda(z) = \dfrac{1}{-r(z) + \frac{\ell}{\lambda} + p}$, we see that $\operatorname{Re} q(z) > 0$ in U and hence $q(z) \neq 0$ in U.

If there exists a point $z_0 \in U$ such that the conditions (2.1) and (2.2) are satisfied, then by Lemma 2.3, we have (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $q(z_0)^{\frac{1}{\alpha}} = ia\ (a > 0)$. Then we obtain

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(I_p^{m+1}(n,\lambda,\ell)f(z_0))'}{I_p^{m+1}(n,\lambda,\ell)g(z_0)}+\gamma\right)\right]\\
&=\arg\left[q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+\frac{\ell}{\lambda}+p}\right]\\
&=\frac{\pi}{2}\alpha+\arg\left(1+i\alpha k(\rho e^{i\frac{\pi}{2}\varphi})^{-1}\right)\\
&=\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha k\sin\frac{\pi}{2}(1-\varphi)}{\rho+\alpha k\cos\frac{\pi}{2}(1-\varphi)}\right)\\
&\geq\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{\frac{\ell}{\lambda}(1-B)+p(A-B)}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)=\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (2.8) and (2.9), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}} = -ia\ (a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(I_p^{m+1}(n,\lambda,\ell)f(z_0))'}{I_p^{m+1}(n,\lambda,\ell)g(z_0)}+\gamma\right)\right]\\
&\leq-\frac{\pi}{2}\alpha-\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{\frac{\ell}{\lambda}(1-B)+p(A-B)}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)\\
&=-\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (2.8) and (2.9), respectively, which contradicts the assumption. Therefore we complete the proof of Theorem 2.5. □

Taking $A = 1,\ B = 0$ and $\delta = 1$ in Theorem 2, we have the following corollary.

Corollary 2.6. *Let $f(z) \in \Sigma_{p,n}$. If*

$$-\operatorname{Re}\left\{\frac{z(I_p^{m+1}(n,\lambda,\ell)f(z))'}{I_p^{m+1}(n,\lambda,\ell)g(z)}\right\}>\gamma\quad(0\leq\gamma<p)$$

for some $g \in \Sigma_{p,n}$ satisfying the condition

$$\left|\frac{z(I_p^{m+1}(n,\lambda,\ell)g(z))'}{I_p^{m+1}(n,\lambda,\ell)g(z)}+p\right|<p,$$

then

$$-\operatorname{Re}\left\{\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)g(z)}\right\}>\gamma\quad(0\le\gamma<p).$$

Taking $A=1,\ B=0$ and $g(z)=\frac{1}{z^p}$ in Theorem 2.5, we have the following corollary.

Corollary 2.7. *Let $f(z)\in\Sigma_{p,n}$. If*

$$\left|\arg\left[-z^{p+1}(I_p^{m+1}(n,\lambda,\ell)f(z))'-\gamma\right]\right|<\frac{\pi}{2}\delta\quad(0\le\gamma<p;0<\delta\le1),$$

then

$$\left|\arg\left[-z^{p+1}(I_p^{m}(n,\lambda,\ell)f(z))'-\gamma\right]\right|<\frac{\pi}{2}\delta\ .$$

Taking $m=0$ and $\delta=1$ in Corollary 2, we have the following corollary.

Corollary 2.8. *Let $f(z)\in\Sigma_{p,n}$. If*

$$-\operatorname{Re}\left\{z^{p+1}\left[\frac{\lambda}{\ell}zf''(z)+\left(1+\frac{\lambda}{\ell}+\frac{\lambda}{\ell}p\right)f'(z)\right]\right\}>\gamma\quad(0\le\gamma<p)\ ,$$

then

$$-\operatorname{Re}\left\{z^{p+1}f'(z)\right\}>\gamma\ .$$

By the same techniques as in the proof of Theorem 2.5, we obtain

Theorem 2.9. *Let $f(z)\in\Sigma_{p,n}$. Choose λ and ℓ such that*

$$\frac{\lambda}{\ell}\ge\frac{p(A-B)}{1+B}\qquad(-1<B<A\le1;\ p\in\mathbb{N})\ .$$

If

$$\left|\arg\left(\frac{z(I_p^{m+1}(n,\lambda,\ell)f(z))'}{I_p^{m+1}(n,\lambda,\ell)g(z)}+\gamma\right)\right|<\frac{\pi}{2}\delta\ \ (\gamma>p;0<\delta\le1)$$

for some $g\in\Sigma^*_{p,n}[\ell,\lambda,m+1;A,B]$, then

$$\left|\arg\left(\frac{z(I_p^{m}(n,\lambda,\ell)f(z))'}{I_p^{m}(n,\lambda,\ell)g(z)}+\gamma\right)\right|<\frac{\pi}{2}\alpha\ ,$$

where α $(0<\alpha\le1)$ is the solution of the equation given by (2.8).

Theorem 2.10. *Let h be convex univalent in U with $h(0) = 1$ and* $\operatorname{Re} h$ *be bounded in U. Let $F_{\nu,p}(f)(z)$ be the integral operator defined by (1.9). If $f \in \Sigma_{p,n}$ satisfies the condition*

$$-\frac{z(I_p^m(n,\lambda,\ell)f(z))^{'}}{pI_p^{m+1}(n,\lambda,\ell)f(z)} \prec h(z),$$

then

$$-\frac{z(I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z))^{'}}{pI_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z)} \prec h(z)$$

for $\max_{z\in U} \operatorname{Re} h(z) < \frac{\nu+p}{p}$ *(provided $I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z) \neq 0$ in U).*

Proof. Let

$$q(z) = -\frac{z(I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z))^{'}}{pI_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z)}.$$

Then, by using (1.10), we have

$$pq(z) - (\nu + p) = -\nu \frac{I_p^m(n,\lambda,\ell)f(z)}{I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z)} \ . \tag{2.12}$$

Taking logarithmic derivatives in both sides of (2.12) and multiplying by z, we get

$$\frac{zq^{'}(z)}{-pq(z) + (\nu + p)} + q(z) = -\frac{z(I_p^m(n,\lambda,\ell)f(z))^{'}}{pI_p^m(n,\lambda,\ell)f(z)} \prec h(z).$$

Therefore, by using Lemma 2.1, we have

$$-\frac{z(I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z))^{'}}{pI_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z)} \prec h(z) \ ,$$

for $\max_{z\in U} \operatorname{Re} h(z) < \dfrac{\nu+p}{p}$ (provided $I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z) \neq 0$ in U). This completes the proof of Theorem 2.10. □

Theorem 2.11. *Let $f(z) \in \Sigma_{p,n}$ and choose a positive number ν such that $\nu \geq \frac{1+A}{1+B} - p$, where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If*

$$\left|\arg\left(-\frac{z(I_p^m(n,\lambda,\ell)f(z))^{'}}{I_p^m(n,\lambda,\ell)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \qquad (0 \leq \gamma < p; 0 < \delta \leq 1)$$

for some $g \in \Sigma^*_{p,n}[\ell,\lambda,m;A,B]$, *then*

$$\left|\arg\left(-\frac{z(I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z))'}{I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z)}-\gamma\right)\right| < \frac{\pi}{2}\alpha \ ,$$

where $F_{\nu,p}(f)(z)$ *is the integral operator given by (1.9),*

$$G_{\nu,p}(g)(z)=\frac{\nu}{z^{\nu+p}}\int_0^z t^{\nu+p-1}g(t)dt \quad (\nu>0) \ , \tag{2.13}$$

and α $(0<\alpha\le 1)$ *is the solution of the equation*

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B,\nu))}{\frac{(\nu+p)(1-B)+p(A-B)}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B,\nu))}\right\} \tag{2.14}$$

where

$$t(A,B,\nu)=\frac{2}{\pi}\sin^{-1}\left[\frac{p(A-B)}{(\nu+p)(1-B^2)-p(1-AB)}\right] \ . \tag{2.15}$$

Proof. Let

$$q(z)=-\frac{1}{p-\gamma}\left(\frac{z(I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z))'}{I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z)}+\gamma\right) \qquad (z\in U) \ .$$

Since $g \in \Sigma^*_{p,n}[\ell,\lambda,m;A,B]$, from Theorem 2.10, $G_{\nu,p}(g)(z) \in \Sigma^*_{p,n}[\ell,\lambda,m;A,B]$. Using (1.10), we have

$$(p-\gamma)q(z)I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z)-(\nu+p)I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z)$$

$$=-\nu I_p^m(n,\lambda,\ell)f(z)-\gamma I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z) \ .$$

Then, by a simple calculation, we have

$$(p-\gamma)\left\{zq'(z)+q(z)\left[-r(z)+\nu+p\right]\right\}+\gamma\left[-r(z)+\nu+p\right]$$

$$=-\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z)} \ ,$$

where

$$r(z)=-\frac{z(I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z))'}{I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z)} \ .$$

Hence, we have

$$q(z)+\frac{zq^{'}(z)}{-r(z)+\nu+p}=-\frac{1}{p-\gamma}\left(\frac{z(I_p^m(n,\lambda,\ell)f(z))^{'}}{I_p^m(n,\lambda,\ell)g(z)}+\gamma\right).$$

The remaining part of the proof is similar to that of Theorem 2.5 and so we omit it. □

Taking $m=0,\ A=1,\ B=0$ and $\delta=1$ in Theorem 2.11, we obtain the following result.

Corollary 2.12. *Let $\nu>0$ and $f(z)\in\Sigma_{p,n}$. If*

$$-\operatorname{Re}\left\{\frac{zf^{'}(z)}{g(z)}\right\}>\gamma\quad(0\le\gamma<p)$$

for some $g\in\Sigma_{p,n}$ satisfying the condition

$$\left|\frac{zg^{'}(z)}{g(z)}+p\right|<p\ ,$$

then

$$-\operatorname{Re}\left\{\frac{zF^{'}_{\nu,p}(f)(z)}{G_{\nu,p}(g)(z)}\right\}>\gamma\quad(0\le\gamma<p)\ ,$$

where $F_{\nu,p}(f)(z)$ and $G_{\nu,p}(g)(z)$ are given by (1.9) and (2.13), respectively.

Taking $m=0,\ B\to A$ and $g(z)=\frac{1}{z^p}$ in Theorem 2.11, we obtain the following corollary.

Corollary 2.13. *Let $\nu>0$ and $f(z)\in\Sigma_{p,n}$. If*

$$\left|\arg(-z^{p+1}f^{'}(z)-\gamma)\right|<\frac{\pi}{2}\delta\qquad(0\le\gamma<p;0<\delta\le1)$$

then

$$\left|\arg(-z^{p+1}F^{'}_{\nu,p}(f)(z)-\gamma)\right|<\frac{\pi}{2}\alpha\ ,$$

where $F_{\nu,p}(f)(z)$ is the integral operator given by (1.9) and $\alpha(0<\alpha\le1)$ is the solution of the equation

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{\nu+p}\right).$$

By using the same method as in proving Theorem 2.11, we have

Theorem 2.14. *Let $f(z) \in \Sigma_{p,n}$ and choose a positive number ν such that $\nu \geq \frac{1+A}{1+B} - p$ where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If*

$$\left|\arg\left(\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)g(z)} + \gamma\right)\right| < \frac{\pi}{2}\delta \quad (\gamma > p; 0 < \delta \leq 1)$$

for some $g \in \Sigma_{p,n}^*[\ell,\lambda,m;A,B]$, then

$$\left|\arg\left(\frac{z(I_p^m(n,\lambda,\ell)F_{\nu,p}(f)(z))'}{I_p^m(n,\lambda,\ell)G_{\nu,p}(g)(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha\ ,$$

where $F_{\nu,p}(f)(z)$ and $G_{\nu,p}(g)(z)$ are given by (1.9) and (2.13), respectively, and α $(0 < \alpha \leq 1)$ is the solution of the equation given by (2.14).

Finally, we derive the following theorem.

Theorem 2.15. *Let $f(z) \in \Sigma_{p,n}$. Choose λ and ℓ such that $\frac{\ell}{\lambda} \geq \frac{p(A-B)}{1+B}$, where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If*

$$\left|\arg\left(-\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < p; 0 < \delta \leq 1)$$

for some $g \in \Sigma_{p,n}^*[\ell,\lambda,m;A,B]$, then

$$\left|\arg\left(-\frac{z(I_p^{m+1}(n,\lambda,\ell)F_{\nu,p}(f)(z))'}{I_p^{m+1}(n,\lambda,\ell)G_{\nu,p}(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta\ ,$$

where $F_{\nu,p}(f)(z)$ and $G_{\nu,p}(g)(z)$ are given by (1.9) and (2.13), respectively, with $\nu = \dfrac{\ell}{\lambda}$.

Proof. From (1.9) and (1.10) with $\nu = \dfrac{\ell}{\lambda}$, we have $I_p^m(n,\lambda,\ell)f(z) = I_p^{m+1}(n,\lambda,\ell)F_{\nu,p}(f)(z)$. Therefore,

$$\frac{z(I_p^m(n,\lambda,\ell)f(z))'}{I_p^m(n,\lambda,\ell)g(z)} = \frac{z(I_p^{m+1}(n,\lambda,\ell)F_{\nu,p}(f)(z))'}{I_p^{m+1}(n,\lambda,\ell)G_{\nu,p}(g)(z)}$$

and the result follows. □

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