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Game domination numbers of a disjoint union of chains and cycles of complete graphs

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Abstract: The domination game played on a graph G consists of two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen, at least one additional vertex is dominated. Playing a vertex will make all vertices in its closed neighborhood dominated. The game ends when all vertices are dominated i.e. the chosen vertices form a dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The game domination number is the total number of chosen vertices after the game ends when Dominator and Staller play the game by using optimal strategies.

In this paper, we determine the game domination numbers of a disjoint union of chains and cycles of complete graphs together with optimal strategies for Dominator and Staller.

Keywords: domination game, game domination number, disjoint union of chains and cycles of complete graphs

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1 Introduction

A set S of vertices of a graph G is a *dominating set* if every vertex not in S is adjacent to some vertex of S. The *domination number* of a graph G is the number of vertices in a minimum dominating set for G, denoted by $\gamma(G)$.

There are many game variations of domination [1, 2, 3, 4, 10]. In this paper we study the domination game introduced in 2010 by Brešar, Klavžar and Rall [4], where the original idea of the game is attributed to Henning (2003, personal communication). The domination game played on a graph G consists of two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen, at least one additional vertex is dominated. Playing a vertex will make all vertices in its closed neighborhood dominated. The game ends when all vertices are dominated i.e. the chosen vertices form a dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The *game domination number* is the size of the final dominating set when both players play optimally; it is denoted by $\gamma_g(G)$ when Dominator starts the game and by $\gamma_g'(G)$ when Staller starts the game.

Among many results, Brešar, Klavžar and Rall [4] gave a bound of game domination number in terms of domination number: for any graph G, $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$. They also studied the difference between the two types of game domination numbers of a graph. Later, Kinnersley, West and Zamani [11] improved upon this result and showed that the difference is at most 1 i.e. for any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.

In real life, domination can be used to optimize resource allocation. The game version can be viewed as a form of negotiation. For example, in an apartment building the owner and residents want to install WiFi routers so that all areas have WiFi coverage. The owner who pays for the installation cost wishes to minimize the number of routers while the residents who want strong signal wish to maximize the numbers of routers. For fairness both parties can play the domination game to pick locations for router installation where the owner acts as Dominator and the residents together act as Staller.

For a graph G and a subset of vertices $S \subseteq V(G)$, we denote by $G|S$ a *partially dominated graph* where the vertices of S are already dominated initially. Note that as the game progresses the graph becomes a partially dominated graph with fewer undominated vertices. We denote the open neighborhood of a vertex v of a graph

G by $N_G(v)$ and its closed neighborhood by $N_G[v]$. We simply write $N(v)$ and $N[v]$ if the graph is understood. A vertex u of a partially dominated graph G is *saturated* if every vertex in $N[u]$ is dominated. The *residual graph* of G is the graph obtained from G by removing all saturated vertices and all edges joining dominated vertices. Since removing such vertices and edges does not affect the game, the game domination numbers of a graph and its residual graph are the same.

Domination game played on various families of graphs have been studied. In 2013, Zamani [11, 14] determined the game domination numbers of paths and cycles, and Brešar and Klavžar [5] proved a lower bound of the game domination number of a tree in terms of its order and maximum degree. In 2015, Bujtás [7] proved a lower bound of the game domination number of a certain families of forests. Dorbec, Košmrlj, and Renault [9] showed how the game domination number of the union of two no-minus graphs corresponds to the game domination numbers of the initial graphs. This result led to another proof of the game domination numbers of paths and cycles [12], and the game domination numbers of a graph constructed from 1-sum of paths [8]. In 2018, Onphaeng, Raksasakcha, and Worawannotai [13] showed the game domination numbers of a disjoint union of paths and cycles.

In this paper, we determine the game domination numbers of a disjoint union of chains and cycles of complete graphs together with optimal strategies for Dominator and Staller. Our proof is based on the following observation. When the domination game is played on a disjoint union of chains and cycles of complete graphs, at any stage of the game, the residual graph is a disjoint union of cycles of complete graphs and partially-dominated chains of complete graphs. In other words, the type of the residual graph does not change during the game. Therefore, if we can find an optimal first move, we have an optimal strategy for the whole game.

2 Preliminaries

In this section, we give some definitions for describing the game domination numbers of a disjoint union of chains and cycles of complete graphs and we also give useful lemmas for proving our results.

Definition 2.1. A graph G is said to be a *complete graph* if every vertex in G is joined to each other by exactly one edge. We denote complete graphs on n vertices by K_n.

In this paper, we only consider complete graphs with at least 3 vertices.

Definition 2.2. A graph G is called a *chain of complete graphs* $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$ if G can be obtained from $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$ by identifying a vertex in K_{n_i} and a vertex in $K_{n_{i+1}}$ for $1 \leq i \leq k-1$ (a vertex can be identified at most once).

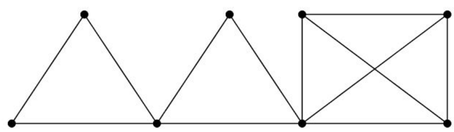

Figure 1: A chain of K_3, K_3, K_4

Definition 2.3. A *cycle of complete graphs* $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$ $(k \geq 3)$ is the graphs obtained from the chain of $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$ by identifying a vertex in $V(K_{n_1}) \setminus V(K_{n_2})$ and a vertex in $V(K_{n_k}) \setminus V(K_{n_{k-1}})$.

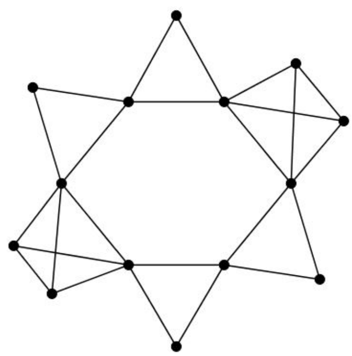

Figure 2: A cycle of $K_3, K_3, K_4, K_3, K_3, K_4$

Definition 2.4. A graph is *CC* if each of its component is either a chain of complete graphs or a cycle of complete graphs.

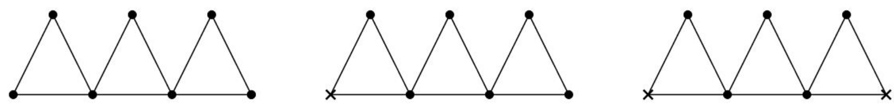

Figure 3: M_3, M_3' and M_3'' of K_3's

For convenience, we define some notations for the components of CC graphs and their residual graphs.

Definition 2.5. Let M_m denote a chain of m complete graphs. Let M_m' denote a partially-dominated chain of m complete graphs where one vertex in $V(K_{n_1})\setminus V(K_{n_2})$ is dominated. Let M_m'' denote a partially-dominated chain of m complete graphs where a vertex in $V(K_{n_1})\setminus V(K_{n_2})$ and a vertex in $V(K_{n_m})\setminus V(K_{n_{m-1}})$ are dominated. Let N_n denote a cycle of n complete graphs.

Definition 2.6. For $i\in\{0,1,2,3\}$, M_m is said to be in *class* $[i]$ if $m\equiv i \pmod 4$, where m is a positive integer, and N_n is said to be in *class* (i) if $n\equiv i \pmod 4$, where $n\geq 3$. Moreover M_m is said to be in *class* $[m]_*$ if $m\in\{1,2\}$ and in *class* $[i]_>$ if $m\equiv i \pmod 4$ and $m\geq 3$.

Definition 2.7. For a CC graph G, let $a(G)$ and $b(G)$ denote the numbers of components of G that are in $[2]\cup[3]$ and (2), respectively.

When a graph is a disjoint union of graphs $G_1, G_2, ..., G_n$, we will simply write $G=G_1+G_2+...+G_n$.

Definition 2.8. Let $G=M_{m_1}+M_{m_2}+...+M_{m_k}+N_{n_1}+N_{n_2}+...+N_{n_l}$. Define $\theta(G)=\sum_{i=1}^{k}(m_i-\lfloor\frac{m_i}{4}\rfloor)+\sum_{i=1}^{l}(n_i-\lfloor\frac{n_i+2}{4}\rfloor)$.

When there are two vertices in a graph whose closed neighborhoods are the same, we can remove one of the vertices or mark it dominated without changing the game domination number of the graph.

Lemma 2.9. [6, Proposition 1.4] *Let G be a graph and let u,v be two distinct vertices of G. If $N[u]=N[v]$, then $\gamma_g(G\setminus\{u\})=\gamma_g(G)=\gamma_g(G|\{u\})$ and $\gamma_g'(G\setminus\{u\})=\gamma_g'(G)=\gamma_g'(G|\{u\})$.*

From the lemma above and since we only consider complete graphs with at least 3 vertices, the game domination numbers of a chain of m complete graphs are the same as those of a chain of m K_3's and the game domination numbers of a cycle of n complete graphs are the same as those of a cycle of n K_3's. Moreover, Lemma 2.9 yields the following important result.

Lemma 2.10. *For any positive integer* m, $\gamma_g(M_m) = \gamma_g(M'_m) = \gamma_g(M''_m)$ *and* $\gamma'_g(M_m) = \gamma'_g(M'_m) = \gamma'_g(M''_m)$.

A fundamental tool for comparing choices of moves called Continuation Principle is given below.

Theorem 2.11. [11, Lemma 2.1 (Continuation Principle)] *Let* G *be a (partially-dominated) graph and let* A *and* B *be subsets of* $V(G)$. *Let* $G|A$ *and* $G|B$ *be the partially-dominated graphs in which the sets* A *and* B *have already been dominated, respectively. If* $B \subseteq A$, *then* $\gamma_g(G|A) \leq \gamma_g(G|B)$ *and* $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.

The continuation principle allows us to make the following assumption which we will use throughout the paper.

Assumption 2.12. A Dominator's move always dominates two consecutive complete graphs (if available) and a Staller's move always dominates exactly one complete graph.

3 Main results

In this section, we find the game domination numbers of a disjoint union of chains and cycles of complete graphs together with optimal strategies for both players.

Theorem 3.1. *Let* G *be a* CC *graph. Let* $\theta = \theta(G)$, $a = a(G)$ *and* $b = b(G)$. *Then*

$$\gamma_g(G) = \theta + \left\lfloor \frac{b-a}{2} \right\rfloor$$

and

$$\gamma'_g(G) = \theta + \left\lceil \frac{b-a}{2} \right\rceil.$$

Moreover, an optimal strategy for each player is as follows.

A Dominator's optimal strategy: For each turn, Dominator always plays on a connected component that is not in $[1]_*$ *or* (3) *if possible. When Dominator* *plays on a chain component, he plays to make the residual graph of this component contains a component from class* [0] *or* [2]*.*

A Staller's optimal strategy: For each turn, Staller always plays on a connected component that is not in $[1]_*$ *or* (1) *if possible. When Staller plays on a chain component (except* K_3*), he plays to make the residual graph of this component contains a component from class* [1] *or* [3]*.*

Proof. Let $G = M_{m_1} + M_{m_2} + ... + M_{m_k} + N_{n_1} + N_{n_2} + ... + N_{n_l}$. By Lemma 2.9 we can assume that each component of G is either a chain or a cycle of K_3's. We prove the results by induction on $m_1 + m_2 + ... + m_k + n_1 + n_2 + ... + n_l$. One can check that the theorem holds for any CC graph with $\sum_{i=1}^{k} m_i + \sum_{i=1}^{l} n_i \le 4$. Assume that $\sum_{i=1}^{k} m_i + \sum_{i=1}^{l} n_i \ge 5$. First, we determine the value of $\gamma_g(G)$. To do this, we find Dominator's optimal first move by comparing all his valid first moves on a Dominator-start game.

Let $\tilde{G}$ be the residual graph of G after Dominator plays his first move on G. Then $\gamma_g(G) \le 1 + \gamma_g'(\tilde{G})$ with equality if Dominator plays his first move optimally. We divide our arguments based on the choice of Dominator's first move. In each case, we count the number of moves of the game with specified Dominator's first move and the remaining moves are played optimally by both players. After Dominator makes his first move, the component in G on which he plays will either be

1. reduced to nothing in $\tilde{G}$ if Dominator plays his first move on a component of G that is M_1 or M_2,

2. reduced to one component in $\tilde{G}$ if Dominator plays his first move on a component of G that is a cycle of complete graphs, or his first move dominates the first two complete graphs or the last two complete graphs of a chain of complete graphs, or

3. reduced to two components in $\tilde{G}$ if his first move does not dominate the first two complete graphs nor the last two complete graphs of a chain of complete graphs of G with at least four complete graphs.

Table 1 shows the values of $1 + \gamma_g'(\tilde{G})$ for all residual graphs $\tilde{G}$ obtained from Dominator making first move on G. The first column of the table shows the classes of components on which Dominator plays his first move. The second column shows the classes of residual graphs of the components that were played on. The third to fifth columns show the changes in values of parameters $g \in \{\theta, a, b\}$ where $\Delta g = g(\tilde{G}) - g(G)$. The last column shows the values of $1 + \gamma'_g(\tilde{G})$.

1st move	Residual	$\Delta\theta$	Δa	Δb	$1 + \gamma'_g(\tilde{G})$
[0]	[2] [0], [2]	-1	$+1$	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
	[1], [1]	-1	0	0	$\theta + \lceil \frac{b-a}{2} \rceil$
	[3], [3]	0	$+2$	0	$\theta + \lceil \frac{b-a}{2} \rceil$
$[1]_*$	–	-1	0	0	$\theta + \lceil \frac{b-a}{2} \rceil$
$[1]_>$	[3] [0], [3]	-1	$+1$	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
	[1], [2]	-1	$+1$	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
$[2]_*$	–	-2	-1	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
$[2]_>$	[0] [0], [0]	-2	-1	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
	[1], [3]	-1	0	0	$\theta + \lceil \frac{b-a}{2} \rceil$
	[2], [2]	-1	$+1$	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
[3]	[1] [0], [1]	-2	-1	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
	[2], [3]	-1	$+1$	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
(0)	[2]	-1	$+1$	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
(1)	[3]	-1	$+1$	0	$\theta + \lceil \frac{b-a-1}{2} \rceil$
(2)	[0]	-1	0	-1	$\theta + \lceil \frac{b-a-1}{2} \rceil$
(3)	[1]	-1	0	0	$\theta + \lceil \frac{b-a}{2} \rceil$

Table 1: Effect of Dominator's first moves on a CC graph.

Now, we show how to obtain the entries in Table 1 by considering how Dominator makes his first move. Let R be the residual graph of the component that Dominator starts on.

Case 1 : Dominator starts on M_{m_j} where $m_j \equiv 0 \pmod 4$.

Case 1.1 : R contains exactly one component and it is in [2] or R contains exactly two components and one is in [0] and the other is in [2].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is

$\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 1.2 : R contains exactly two components and both are in [1].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

Case 1.3 : R contains exactly two components and both are in [3].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a + 2$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta + \left\lceil \frac{b-(a+2)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

Case 2 : Dominator starts on M_1.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

Case 3 : Dominator starts on M_{m_j} where $m_j \equiv 1 \pmod 4$ and $m_j \geq 5$.

Case 3.1 : R contains exactly one component and it is in [3] or R contains exactly two components and one is in [0] and the other is in [3].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 3.2 : R contains exactly two components and one is in [1] and the other is in [2].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 4 : Dominator starts on M_2.

Then $\theta(\tilde{G}) = \theta - 2$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 2 + \left\lceil \frac{b-(a-1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 5 : Dominator starts on M_{m_j} where $m_j \equiv 2 \pmod 4$ and $m_j \geq 6$.

Case 5.1 : R contains exactly one component and it is in [0] or R contains exactly two components and one is in [0] and the other is in [0].

Then $\theta(\tilde{G}) = \theta - 2$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 2 + \left\lceil \frac{b-(a-1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 5.2 : R contains exactly two components and one is in [1] and the other is in [3].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

Case 5.3 : R contains exactly two components and both are in [2].

Then $\theta(\tilde{G}) = \theta - 1,\ a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 6 : Dominator starts on M_{m_j} where $m_j \equiv 3 \pmod 4$.

Case 6.1 : R contains exactly one component and it is in $[1]$ or R contains exactly two components and one is in $[0]$ and the other is in $[1]$.

Then $\theta(\tilde{G}) = \theta - 2,\ a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 2 + \left\lceil \frac{b-(a-1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 6.2 : R contains exactly two components and one is in $[2]$ and the other is in $[3]$.

Then $\theta(\tilde{G}) = \theta - 1,\ a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 7 : Dominator starts on N_{n_j} where $n_j \equiv 0 \pmod 4$. Then R contains exactly one component and it is in $[2]$.

Then $\theta(\tilde{G}) = \theta - 1,\ a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 8 : Dominator starts on N_{n_j} where $n_j \equiv 1 \pmod 4$. Then R contains exactly one component and it is in $[3]$.

Then $\theta(\tilde{G}) = \theta - 1,\ a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 9 :Dominator starts on N_{n_j} where $n_j \equiv 2 \pmod 4$. Then R contains exactly one component and it is in $[0]$.

Then $\theta(\tilde{G}) = \theta - 1,\ a(\tilde{G}) = a$ and $b(\tilde{G}) = b - 1$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-1-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 10 : Dominator starts on N_{n_j} where $n_j \equiv 3 \pmod 4$. Then R contains exactly one component and it is in $[1]$.

Then $\theta(\tilde{G}) = \theta - 1,\ a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

From 10 cases above, we get that $\gamma_g(G) = \min(1 + \gamma_g'(\tilde{G})) = \theta + \left\lceil \frac{b-a-1}{2} \right\rceil = \theta + \left\lfloor \frac{b-a}{2} \right\rfloor$.

Next, we determine the value of $\gamma_g'(G)$. To do this, we find Staller's optimal first move by comparing all his valid first moves on a Staller-start game.

Let $\tilde{G}$ be the residual graph of G after Staller plays his first move on G. Then $\gamma_g'(G) \geq 1 + \gamma_g(\tilde{G})$ with equality if Staller plays his first move optimally. We divide our arguments based on the choice of Staller's first move. In each case, we count the number of moves of the game with specified Staller's first move and the remaining moves are played optimally by both players. After Staller makes his first move, the component in G on which he plays will either be

1. reduced to nothing in $\tilde{G}$ if Staller plays his first move on a component of G that is M_1,
2. reduced to one component in $\tilde{G}$ if Staller plays his first move on a component of G that is a cycle of complete graphs, or his first move dominates the first complete graph or the last complete graph of a chain of complete graphs, or
3. reduced to two components in $\tilde{G}$ if his first move does not dominate the first complete graph nor the last complete graph of a chain of complete graphs of G with at least three complete graphs.

Table 2 shows the values of $1+\gamma_g(\tilde{G})$ for all residual graphs $\tilde{G}$ obtained from Staller making first move on G. The first column of the table shows the classes of components on which Staller plays his first move. The second column shows the classes of residual graphs of the components that were played on. The third to fifth columns show the changes in values of parameters $g \in \{\theta, a, b\}$ where $\Delta g = g(\tilde{G}) - g(G)$. The last column shows the values of $1+\gamma_g(\tilde{G})$.

Now, we show how to obtain the entries in Table 2 by considering how Staller makes his first move. Let R be the residual graph of the component that Dominator starts on.

Case 1 : Staller starts on M_{m_j} where $m_j \equiv 0 \pmod 4$.

Case 1.1 : R contains exactly one component and it is in $[3]$ or R contains exactly two components and one is in $[0]$ and the other is in $[3]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

Case 1.2 : R contains exactly two components and one is in $[1]$ and the other is in $[2]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

1st move	Residual	$\Delta\theta$	Δa	Δb	$1+\gamma_g(\tilde{G})$
$[0]$	$[3]$ $[0],[3]$	0	$+1$	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
	$[1],[2]$	0	$+1$	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
$[1]_*$	$-$	-1	0	0	$\theta+\lfloor\frac{b-a}{2}\rfloor$
$[1]_>$	$[0]$ $[0],[0]$	-1	0	0	$\theta+\lfloor\frac{b-a}{2}\rfloor$
	$[1],[3]$	0	$+1$	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
	$[2],[2]$	0	$+2$	0	$\theta+\lfloor\frac{b-a}{2}\rfloor$
$[2]_*$	$[1]_*$	-1	-1	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
$[2]_>$	$[1]$ $[0],[1]$	-1	-1	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
	$[2],[3]$	0	$+1$	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
$[3]$	$[2]$ $[0],[2]$	-1	0	0	$\theta+\lfloor\frac{b-a}{2}\rfloor$
	$[1],[1]$	-1	-1	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
	$[3],[3]$	0	$+1$	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
(0)	$[3]$	0	$+1$	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
(1)	$[0]$	-1	0	0	$\theta+\lfloor\frac{b-a}{2}\rfloor$
(2)	$[1]$	0	0	-1	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$
(3)	$[2]$	0	$+1$	0	$\theta+\lfloor\frac{b-a+1}{2}\rfloor$

Table 2: Effect of Staller's first moves on a CC graph.

Case 2 : Staller starts on M_1.

Then $\theta(\tilde{G})=\theta-1$, $a(\tilde{G})=a$ and $b(\tilde{G})=b$. By the induction hypothesis, we have $\gamma_g(\tilde{G})=\theta-1+\lfloor\frac{b-a}{2}\rfloor$. Thus the number of moves in this case is $\theta+\lfloor\frac{b-a}{2}\rfloor$.

Case 3 : Staller starts on M_{m_j} where $m_j\equiv 1 \pmod 4$ and $m_j\geq 5$.

Case 3.1 : R contains exactly one component and it is in $[0]$ or R contains exactly two components and both are in $[0]$.

Then $\theta(\tilde{G})=\theta-1$, $a(\tilde{G})=a$ and $b(\tilde{G})=b$. By the induction hypothesis, we have $\gamma_g(\tilde{G})=\theta-1+\lfloor\frac{b-a}{2}\rfloor$. Thus the number of moves in this case is $\theta+\lfloor\frac{b-a}{2}\rfloor$.

Case 3.2 : R contains exactly two components and one is in $[1]$ and the other is in $[3]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 3.3 : R contains exactly two components and both are in $[2]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+2$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+2)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a}{2} \rfloor$.

Case 4 : Staller starts on M_2.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a-1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-(a-1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 5 : Staller starts on M_{m_j} where $m_j \equiv 2 \pmod 4$.

Case 5.1 : R contains exactly one component and it is in $[1]$ or R contains exactly two components and one is in $[0]$ and the other is in $[1]$.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a-1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-(a-1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 5.2 : R contains exactly two components and one is in $[2]$ and the other is in $[3]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 6 : Staller starts on M_{m_j} where $m_j \equiv 3 \pmod 4$.

Case 6.1 : R contains exactly one component and it is in $[2]$ or R contains exactly two components and one is in $[0]$ and the other is in $[2]$.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \lfloor \frac{b-a}{2} \rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a}{2} \rfloor$.

Case 6.2 : R contains exactly two components and both are in $[1]$.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a-1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-(a-1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 6.3 : R contains exactly two components and both are in $[3]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 7 : Staller starts on N_{n_j} where $n_j \equiv 0 \pmod 4$. Then R contains exactly one component and it is in $[3]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 8 : Staller starts on N_{n_j} where $n_j \equiv 1 \pmod 4$. Then R contains exactly one component and it is in $[0]$.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-a}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a}{2} \right\rfloor$.

Case 9 : Staller starts on N_{n_j} where $n_j \equiv 2 \pmod 4$. Then R contains exactly one component and it is in $[1]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b - 1$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-1-a}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

Case 10 : Staller starts on N_{n_j} where $n_j \equiv 3 \pmod 4$. Then R contains exactly one component and it is in $[2]$.

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

From 10 cases above, we get that $\gamma'_g(G) = \max(1 + \gamma_g(\tilde{G})) = \theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor = \theta + \left\lceil \frac{b-a}{2} \right\rceil$.

□

For a predicate $\mathcal{P}$, let $[\mathcal{P}]$ equals 1 if $\mathcal{P}$ is true; otherwise $\mathcal{P} = 0$.

Corollary 3.2. *Let G be a chain of m complete graphs. Then $\gamma_g(G) = m - \left\lfloor \frac{m}{4} \right\rfloor - [m \equiv 2, 3 \pmod 4]$ and $\gamma'_g(G) = m - \left\lfloor \frac{m}{4} \right\rfloor$.*

Corollary 3.3. *Let G be a cycle of n complete graphs. Then $\gamma_g(G) = n - \left\lfloor \frac{n+2}{4} \right\rfloor$ and $\gamma'_g(G) = n - \left\lfloor \frac{n+2}{4} \right\rfloor + [n \equiv 2 \pmod 4]$.*

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