A 2D Euler-Bernoulli beam element for large displacement analysis

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Abstract

This paper presents a new 2D Euler-Bernoulli beam element for large displacement analysis using the total Lagrangian formulation. In large displacement analysis, the beam kinematics based on the total Lagrangian description is complicated. Consequently, it is not easy to find interpolation functions that are field consistent for beam elements developed in the total Lagrangian description. For this reason, existing total Lagrangian beam elements employ field-inconsistent interpolation functions, which results in degradation of element performances. In this study, a 2D Euler-Bernoulli beam element for large displacement analysis that is field consistent is developed. The field consistency is made possible by allowing the interpolations to be expressed as nonlinear functions of nodal degrees of freedom. The validity and efficiency of the proposed element are shown by solving several numerical examples found in the literature.

1. Introduction

Development of beam finite elements for large displacement analysis has been undertaken by many researchers [1-11]. There are essentially three main aspects in the development of this type of beam element. These aspects are the reference configuration, the beam kinematics and the approximation of the displacement fields. The choice of reference configuration depends on the choice of kinematic description. Three kinematic descriptions that are in current use are the total Lagrangian description, the updated Lagrangian description, and the co-rotational description. In this paper, a new 2D Euler-Bernoulli beam element for large displacement analysis with small strains derived based on the total Lagrangian description is proposed. The focus of the development of the proposed element is on creating interpolations that are field consistent even when the displacements are large. It is apparent that using field-inconsistent interpolations in beam elements results in degradation of element performances. This is due to the fact that, in general, some of the interpolated fields are interdependent. The forms of the interdependency are defined by the governing equations of the beam. As a result, the interdependent fields cannot be independently interpolated since the employed interpolations may not satisfy the relationships between these fields. Instead, interpolations of all interdependent fields must be considered together. Various interpolation functions have been used in linear and geometrically nonlinear beam elements. These interpolation functions are defined based on various types of nodal degree of freedom. In case of 3D linear beam elements, it is common that the axial and transverse displacements as well as the axial and transverse rotations are used as nodal degrees of freedom. Under the assumption of small displacements in linear beam analysis, the transverse rotation and the transverse displacement of the same bending plane are dependent. In linear Euler-Bernoulli beam elements, which do not consider shear deformation, the dependency between the transverse rotation and the transverse displacement of each bending plane is considered during the construction of the interpolation functions. As a result, field-consistent interpolations are obtained. On the contrary, in linear Timoshenko beam elements,

RECEIVED 7 November, 2005
ACCEPTED 7 March, 2006

which consider shear deformation, this field dependency is neglected as a result of independent approximations for the transverse rotation and the transverse displacement of the same bending plane. The field-inconsistent interpolations in linear Timoshenko beam elements result in a well-known phenomenon called shear locking where inaccurately stiff solutions are obtained. It is possible, however, to remove the field inconsistency in linear Timoshenko beam elements by using appropriate interpolation functions [12, 13].

In case of geometrically nonlinear beam elements with small strains that employ the total Lagrangian formulation, nodal degrees of freedom are described with respect to a fixed reference frame that is defined at the initial configuration. This fixed reference frame is normally defined by using coordinate axes that are aligned with the axis and the two orthogonal bending directions of the beam at its initial configuration. With the reference frame defined as such, the displacements and the rotations in the directions of the coordinate axes are still mostly used as nodal degrees of freedom. Due to large displacements, the longitudinal displacement, the transverse displacements and the transverse rotations become complicatedly dependent on each other. As a result, it becomes difficult to find a set of interpolation functions that are field consistent. One practical interpolation technique used in the total Lagrangian beam elements is to approximate displacements at a point from displacements of points on nodal cross sections that have the same projected position as the interested point [14-16]. This technique often employs standard polynomial interpolation functions. As a result, it becomes that the displacements are interpolated from the values of nodal displacements and the values of nonlinear functions of nodal rotations. This interpolation technique, unfortunately, does not yield field-consistent interpolations.

When the updated Lagrangian formulation is used in large displacement analysis with small strains, the problem of field inconsistency may be avoided. This is true when the displacements from the last reference configuration to the next configuration are small and appropriate interpolation functions are used. Accordingly, the Euler-Bernoulli interpolation functions for small displacement analysis are sometimes used in beam elements derived with the updated Lagrangian formulation without any modification [5, 17]. Note that, with the updated Lagrangian formulation, the problem of field inconsistency will arise every time the displacements from the last reference configuration to the next configuration are not small. The co-rotation formulation [3, 7, 8, 18, 19] can also avoid the problem of field inconsistency when it is used in large displacement analysis with small strains. It is because the basic concept of the formulation is to decompose the displacements into a portion of large rigid displacements and a portion of small displacements due to small deformation. The interpolation functions are consequently used for the small displacements. If appropriate interpolation functions are used, there will be no problem of field inconsistency.

This paper presents an approach to handle the problem of field inconsistency found in 2D Euler-Bernoulli beam elements for large displacements with small strains derived with the total Lagrangian formulation. The proposed beam element will remove this disadvantage of the total Lagrangian formulation. The derivation of the proposed beam element is straightforward and does not involve any complicated concepts or assumptions. The construction of the displacement interpolations is achieved simply by appropriately allowing the interpolations to be expressed as nonlinear functions of nodal degrees of freedom. The validity and efficiency of the proposed element are shown by using various numerical examples found in the literature.

2. Kinematics of an Euler-Bernoulli Beam with Large Displacements

Consider a section of a beam shown in Fig. 1. Due to the deformation of the beam, a particle at $P_0(X,Y)$ in the original configuration $\mathcal{C}_0$ moves to $P(x,y)$ in the current configuration $\mathcal{C}$. Under the assumptions

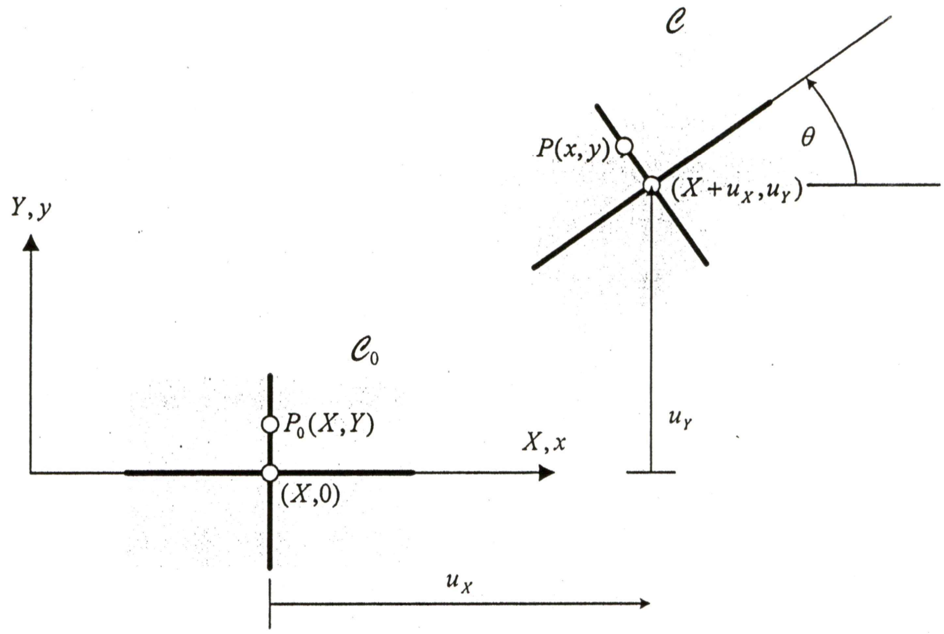

Fig. 1 Euler-Bernoulli beam with large displacements

of the Euler-Bernoulli beam, the following relations can be obtained, i.e.,

$$x = X + u_X - Y\sin\theta, \tag{1}$$

$$y = u_Y + Y\cos\theta, \tag{2}$$

where u_X and u_Y are the displacements of the projection of $P_0(X,Y)$ on the neutral axis in the XY system. In addition, θ represents the rotation of the cross-section.

Denote s as the arc length along the neutral axis in the current configuration. The following relations can be obtained, i.e.,

$$1+u_X' = s'\cos\theta, \qquad u_Y' = s'\sin\theta, \tag{3}$$

$$\cos\theta = \frac{1+u_X'}{\sqrt{(1+u_X')^2+(u_Y')^2}}, \tag{4}$$

$$\sin\theta = \frac{u_Y'}{\sqrt{(1+u_X')^2+(u_Y')^2}}, \tag{5}$$

$$\tan\theta = \frac{u_Y'}{1+u_X'}. \tag{6}$$

By using (1) and (2), the deformation gradient matrix is written as

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1+u_X' - Y\theta'\cos\theta & -\sin\theta \\ u_Y' - Y\theta'\sin\theta & \cos\theta \end{bmatrix}, \tag{7}$$

where primes denote the derivatives with respect to X. The deformation gradient matrix is used to compute the Green strain matrix as

$$\mathbf{E} = \begin{bmatrix} E_{XX} & E_{XY} \\ E_{YX} & E_{YY} \end{bmatrix} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right). \tag{8}$$

Introduce an orthogonal matrix $\mathbf{R}$ defined as

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{9}$$

Define a matrix $\bar{\mathbf{F}}$ as

$$\bar{\mathbf{F}} = \mathbf{R}\mathbf{F} = \begin{bmatrix} (1+u_X')\cos\theta + u_Y'\sin\theta - Y\theta' & 0 \\ 0 & 1 \end{bmatrix}. \tag{10}$$

Note that from (3)

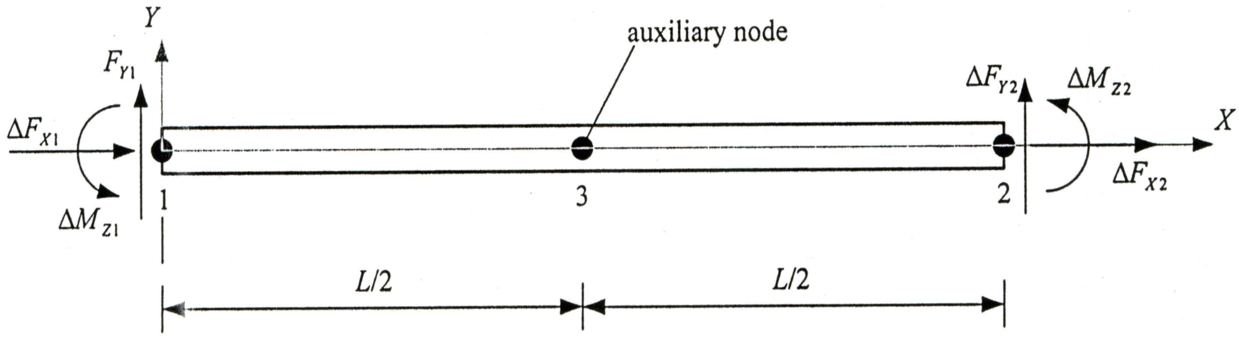

Fig. 2 Proposed beam element

$$-(1+u_X')\sin\theta + u_Y'\cos\theta = 0. \quad (11)$$

With help of (3), write $\bar{\mathbf{F}}$ as

$$\bar{\mathbf{F}} = \mathbf{I} + \mathbf{L}, \quad (12)$$

where

$$\mathbf{L} = \begin{bmatrix} (s'-1) - Y\theta' & 0 \\ 0 & 0 \end{bmatrix}. \quad (13)$$

Since $\mathbf{R}$ is orthogonal, the Green strain $\mathbf{E}$ can be rewritten as

$$\mathbf{E} = \frac{1}{2}\left(\bar{\mathbf{F}}^T\bar{\mathbf{F}} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T + \mathbf{L}^T\mathbf{L}\right) \approx \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \mathbf{L}. \quad (14)$$

The term $\mathbf{L}^T\mathbf{L}$ can be neglected because, under the Euler-Bernoulli beam theory that assumes small strains, $(s'-1)$ and $Y\theta'$ in $\mathbf{L}$ are small. Note that E_{XX} is the only non-zero strain component.

3. Displacement Interpolation

Consider a two-dimensional beam element of length L in the XY plane shown in Fig. 2. The element has two end nodes and one auxiliary node at its center. Define the displacement vector $\mathbf{u}$ as

$$\mathbf{u} = \begin{Bmatrix} u_X \\ u_Y \end{Bmatrix} = \mathbf{P}^T\mathbf{A}, \quad (15)$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & X & X^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & X & X^2 & X^3 \end{bmatrix}^T, \quad (16)$$

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & b_0 & b_1 & b_2 & b_3 \end{bmatrix}^T. \quad (17)$$

Here, $\mathbf{P}$ holds polynomial bases while $\mathbf{A}$ contains their coefficients.

Define the nodal displacement vector $\mathbf{U}$ and the nodal force vector $\mathbf{f}$ as

$$\mathbf{U} = \begin{bmatrix} u_{X1} & u_{Y1} & \theta_{Z1} & u_{X2} & u_{Y2} & \theta_{Z2} & u_{X3} \end{bmatrix}^T, \quad (18)$$

$$\mathbf{f} = \begin{bmatrix} F_{X1} & F_{Y1} & M_{Z1} & F_{X2} & F_{Y2} & M_{Z2} & 0 \end{bmatrix}^T. \quad (19)$$

The number subscripts represent node numbers. The last degree of freedom at the center of the element is introduced only to increase the order of the interpolation of the axial displacement. The applied force is not allowed at the center node. As a result, for the user, the element has only two nodes.

From (15), with help of (6), the degrees of freedom can be expressed as

$$u_{X1} = a_0, \quad (20)$$

$$u_{Y1} = b_0, \quad (21)$$

$$\theta_{Z1} = \tan^{-1}\left(\frac{u_Y'(0)}{1+u_X'(0)}\right) = \tan^{-1}\left(\frac{b_1}{1+a_1}\right), \quad (22)$$

$$u_{X2} = a_0 + a_1 L + a_2 L^2, \quad (23)$$

$$u_{Y2} = b_0 + b_1 L + b_2 L^2 + b_3 L^3, \quad (24)$$

$$\theta_{Z2} = \tan^{-1}\left(\frac{u_Y'(L)}{1+u_X'(L)}\right) = \tan^{-1}\left(\frac{b_1 + 2b_2 L + 3b_3 L^2}{1 + a_1 + 2a_2 L}\right), \tag{25}$$

$$u_{X3} = a_0 + a_1 \frac{L}{2} + a_2 \frac{L^2}{4}. \tag{26}$$

By using (20) to (26), the coefficients in (17) are expressed in terms of the nodal degrees of freedom as

$$a_0 = u_{X1}, \tag{27}$$

$$a_1 = -\frac{3u_{X1} + u_{X2} - 4u_{X3}}{L}, \tag{28}$$

$$a_2 = \frac{2(u_{X1} + u_{X2} - 2u_{X3})}{L^2}, \tag{29}$$

$$b_0 = u_{Y1}, \tag{30}$$

$$b_1 = \frac{1}{L}(L - 3u_{X1} - u_{X2} + 4u_{X3}) \tan\theta_{Z1}, \tag{31}$$

$$\begin{aligned} b_2 = -\frac{1}{L^2}\big[&3(u_{Y1} - u_{Y2}) \\ &+2(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\ &+(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}\big], \end{aligned} \tag{32}$$

$$\begin{aligned} b_3 = \frac{1}{L^3}\big[&2(u_{Y1} - u_{Y2}) \\ &+(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\ &+(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}\big]. \end{aligned} \tag{33}$$

Consequently, the displacements can be expressed as

$$\begin{aligned} u_X = u_{X1} &- \frac{X}{L}[3u_{X1} + u_{X2} - 4u_{X3}] \\ &+ \frac{X^2}{L^2}[2(u_{X1} + u_{X2} - 2u_{X3})], \end{aligned} \tag{34}$$

$$\begin{aligned} u_Y = u_{Y1} &+ \frac{X}{L}[(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1}] \\ &- \frac{X^2}{L^2}\big[3(u_{Y1} - u_{Y2}) \\ &\quad +2(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\ &\quad +(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}\big] \\ &+ \frac{X^3}{L^3}\big[2(u_{Y1} - u_{Y2}) \\ &\quad +(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\ &\quad +(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}\big]. \end{aligned} \tag{35}$$

In contrast to conventional displacement interpolations, it can be seen that the displacements are not linear functions of the degrees of freedom.

4. Derivation of the Nonlinear Stiffness Matrix Equation

Consider the virtual work equation for the total Lagrangian formulation written as

$$\begin{aligned} \int_0^L \int_A S_{XX} \delta E_{XX} dA dX &= \int_0^L \int_A E E_{XX} \delta E_{XX} dA dX \\ &= \sum_{I=1}^{2} F_{XI} \delta u_{XI} + \sum_{I=1}^{2} F_{YI} \delta u_{YI} + \sum_{I=1}^{2} M_{ZI} \delta\theta_{ZI} \\ &+ \int_0^L p_X \delta u_X dX + \int_0^L p_Y \delta u_Y dX. \end{aligned} \tag{36}$$

Here, S_{XX} is the XX component of the second Piola-Kirchhoff stresses and it is related to the Green strain E_{XX} through the Young's modulus E. The cross-sectional area of the beam is denoted as A. Moreover, p_X and p_Y denote the distributed axial and transverse loads, respectively.

From (14), the virtual Green strain can be expressed as

$$\begin{aligned} \delta E_{XX} &= \delta\left[(1 + u_X')\cos\theta + u_Y' \sin\theta - Y\theta' - 1\right] \\ &= \cos\theta \delta u_X' - (1 + u_X')\sin\theta \delta\theta \\ &\quad + \sin\theta \delta u_Y' + u_Y' \cos\theta \delta\theta - Y\delta\theta' \\ &= \cos\theta \delta u_X' + \sin\theta \delta u_Y' - Y\delta\theta'. \end{aligned} \tag{37}$$

Hence, the internal virtual work is expressed as

$$\begin{aligned}
&\int_0^L\int_A EE_{XX}\delta E_{XX}dAdX \\
&= \int_0^L\int_A E[s'-Y\theta'-1] \\
&\quad\times\left[\cos\theta\delta u'_X+\sin\theta\delta u'_Y-Y\delta\theta'\right]dAdX \\
&= EA\int_0^L s'\cos\theta\delta u'_X dX+EA\int_0^L s'\sin\theta\delta u'_Y dX \\
&-EA\int_0^L\cos\theta\delta u'_X dX-EA\int_0^L\sin\theta\delta u'_Y dX+EI\int_0^L\theta'\delta\theta' dX \\
&= EA\int_0^L(1+u'_X)\delta u'_X dX+EA\int_0^L u'_Y\delta u'_Y dX \\
&-EA\int_0^L\cos\theta\delta u'_X dX-EA\int_0^L\sin\theta\delta u'_Y dX+EI\int_0^L\theta'\delta\theta' dX.
\end{aligned} \tag{38}$$

By using (4) and (6), θ' is derived as

$$\theta'=\frac{(1+u'_X)u''_Y-u'_Y u''_X}{(1+u'_X)^2+(u'_Y)^2}. \tag{39}$$

The internal virtual work then becomes

$$\begin{aligned}
&\int_0^L\int_A EE_{XX}\delta E_{XX}dAdX \\
&= EA\int_0^L(1+u'_X)\delta u'_X dX+EA\int_0^L u'_Y\delta u'_Y dX \\
&-EA\int_0^L\frac{1+u'_X}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\delta u'_X dX \\
&-EA\int_0^L\frac{u'_Y}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\delta u'_Y dX \\
&+EI\int_0^L\frac{(1+u'_X)u''_Y-u'_Y u''_X}{(1+u'_X)^2+(u'_Y)^2}\delta\theta' dX.
\end{aligned} \tag{40}$$

Define vectors $\mathbf{N}_X$ and $\mathbf{N}_Y$ as

$$\mathbf{N}_X=\left[\frac{\partial u_X}{\partial u_{X1}}\;\frac{\partial u_X}{\partial u_{Y1}}\;\frac{\partial u_X}{\partial \theta_{Z1}}\;\frac{\partial u_X}{\partial u_{X2}}\;\frac{\partial u_X}{\partial u_{Y2}}\;\frac{\partial u_X}{\partial \theta_{Z2}}\;\frac{\partial u_X}{\partial u_{X3}}\right], \tag{41}$$

$$\mathbf{N}_Y=\left[\frac{\partial u_Y}{\partial u_{X1}}\;\frac{\partial u_Y}{\partial u_{Y1}}\;\frac{\partial u_Y}{\partial \theta_{Z1}}\;\frac{\partial u_Y}{\partial u_{X2}}\;\frac{\partial u_Y}{\partial u_{Y2}}\;\frac{\partial u_Y}{\partial \theta_{Z2}}\;\frac{\partial u_Y}{\partial u_{X3}}\right]. \tag{42}$$

The virtual translational displacements and their derivatives are then written as

$$\delta u_X=\mathbf{N}_X\delta\mathbf{U},\quad \delta u_Y=\mathbf{N}_Y\delta\mathbf{U}, \tag{43}$$

$$\delta u'_X=\mathbf{N}'_X\delta\mathbf{U},\quad \delta u'_Y=\mathbf{N}'_Y\delta\mathbf{U}. \tag{44}$$

Note that $\mathbf{N}_X, \mathbf{N}_Y$, $\mathbf{N}'_X$ and $\mathbf{N}'_Y$ can be derived from (34) and (35).

By using (4), (6) and (44), the virtual rotational displacement and its derivative can be derived as

$$\begin{aligned}
\delta\theta &= -\frac{u'_Y}{(1+u'_X)^2+(u'_Y)^2}\delta u'_X \\
&\quad+\frac{1+u'_X}{(1+u'_X)^2+(u'_Y)^2}\delta u'_Y \\
&=\left[-\frac{u'_Y}{(1+u'_X)^2+(u'_Y)^2}\mathbf{N}'_X\right. \\
&\quad\left.+\frac{1+u'_X}{(1+u'_X)^2+(u'_Y)^2}\mathbf{N}'_Y\right]\delta\mathbf{U} \\
&=\mathbf{N}_\theta\delta\mathbf{U},
\end{aligned} \tag{45}$$

$$\delta\theta'=\mathbf{N}'_\theta\delta\mathbf{U}. \tag{46}$$

Finally, by using (40), (44), and (46) in (36), the nonlinear stiffness equation is obtained as

$$\Phi(\mathbf{U})=\mathbf{f}, \tag{47}$$

where

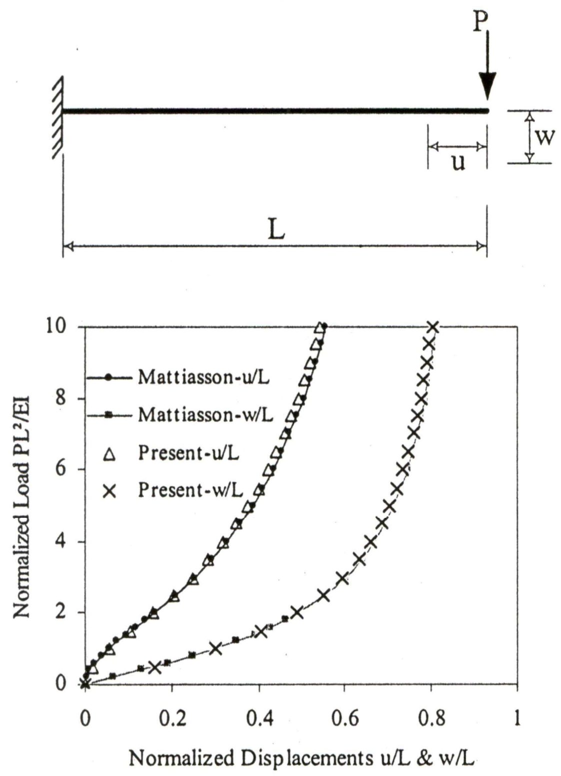

Fig. 3 Cantilever beam with an end point load

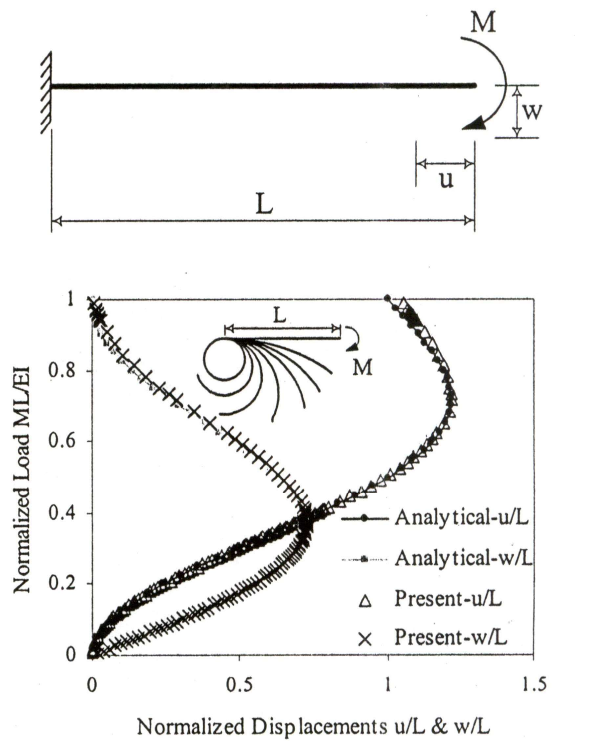

Fig. 4 Cantilever beam with an end moment

$$\begin{aligned}\Phi(\mathbf{U}) &= EA\int_0^L (1+u'_X)\mathbf{N}'^{\mathrm{T}}_X dX + EA\int_0^L u'_Y \mathbf{N}'^{\mathrm{T}}_Y dX \\ &-EA\int_0^L \frac{1+u'_X}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\mathbf{N}'^{\mathrm{T}}_X dX \\ &-EA\int_0^L \frac{u'_Y}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\mathbf{N}'^{\mathrm{T}}_Y dX \\ &+EI\int_0^L \frac{(1+u'_X)u''_Y - u'_Y u''_X}{(1+u'_X)^2+(u'_Y)^2}\mathbf{N}'^{\mathrm{T}}_\theta dX \\ &-\int_0^L \mathbf{N}_X^{\mathrm{T}} p_X dX - \int_0^L \mathbf{N}_Y^{\mathrm{T}} p_Y dX. \end{aligned} \quad (48)$$

Note that the derivatives of u_X and u_Y can be derived from (34) and (35).

5. Results

The following examples are used to illustrate the validity and efficiency of the proposed beam element. The results obtained from this study are compared with the existing results in the literature.

5.1 Cantilever Beam with an End Point Load

The first problem to be considered is a cantilever beam subjected to a concentrated load at the free end as shown in Fig. 3. In this problem, five elements are used to model the beam. The obtained results are compared with the elliptic integral solutions by Mattiasson [20] in Fig. 3. The results indicate good agreement between the present solutions and Mattiasson's solutions.

5.2 Cantilever Beam with an End Moment

A cantilever beam in this example is subjected to a concentrated end moment as shown in Fig. 4. This problem has been used by many researchers for testing nonlinear beam elements since the analytical solutions for the problem exist. Examples include works by Filho and Awruch [21] who solved the problem by using 160 eight-noded hexahedral isoparametric elements and by Shi and Voyiadjis [22] who used 10 four-noded plate elements. In this study, the problem is solved with five elements. The results obtained from

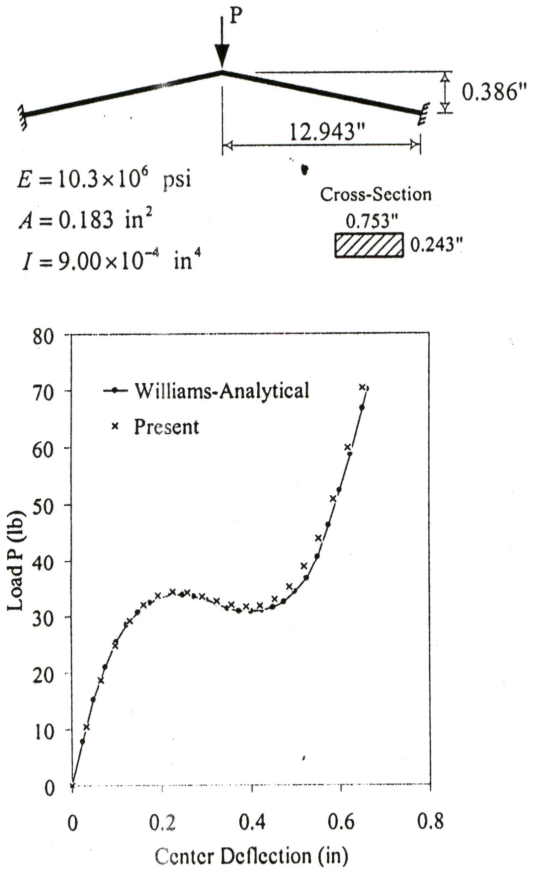

Fig. 5 Williams' Toggle

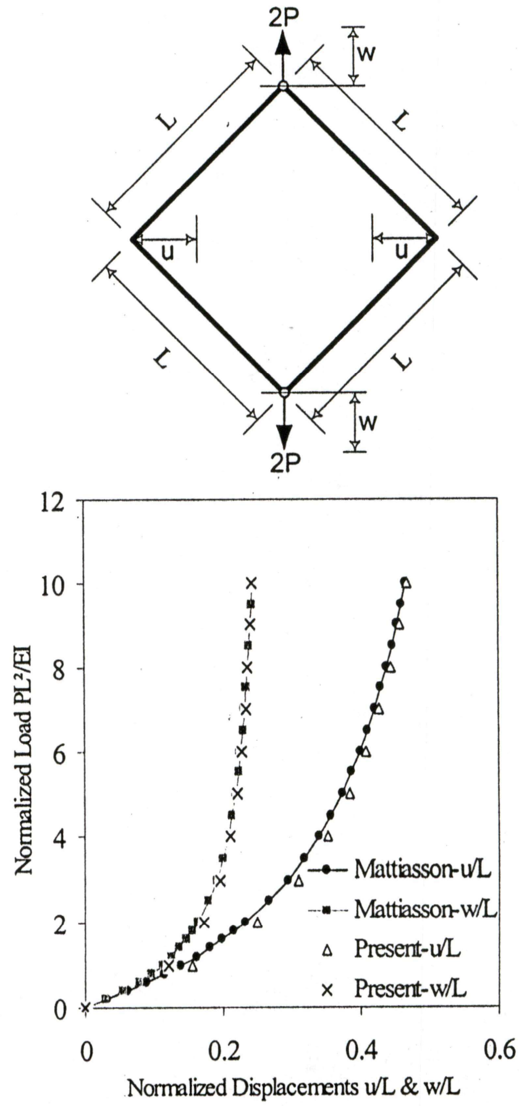

Fig. 6 Pinned-fixed square diamond frame in tension

this study are compared with the analytical solutions in Fig. 4. It can be seen from the comparison that good agreement between the present solutions and the analytical solutions is obtained. It can also be observed from the comparison that the discrepancies between the present and analytical solutions increase, though not significantly, only when the normalized load $\frac{ML}{EI}$ is very large.

5.3 Williams' Toggle Frame

A frame in Fig. 5 was first solved analytically and experimentally by Williams [23]. Only half of the frame can be analyzed since the problem is symmetric. The problem is considered as a good benchmark for testing new nonlinear beam elements and it has been investigated by many researchers [1, 24, 25]. Wood and Zienkiewicz [1] solved the problem by using five six-noded paralinear isoparametric elements per member. Yang and Chiou [25] employed five two-noded beam elements per member. In this study, the problem is analyzed by using two elements per member. The result obtained from this study is compared with the analytical solution by Williams [23] in Fig. 5. Good agreement between the present and analytical solutions is observed. In addition, it can be seen that, even though the solution of the problem is a snap-through response, the proposed beam element is able to trace the equilibrium path satisfactorily.

5.4 Pinned-Fixed Square Diamond Frame in Tension

A pinned-fixed square diamond frame is loaded in tension as shown in Fig. 6. Since the problem is symmetric, only half of the diamond is analyzed. The problem is analyzed by using two elements per member. The results are compared with those from Mattiasson [20] and good agreement is obtained.

6. Conclusion

In this study, a new 2D Euler-Bernoulli beam element for large displacement analysis with small strains using the total Lagrangian formulation is proposed. When the total Lagrangian formulation is used to formulate beam elements for large displacement analysis, it is virtually impossible to find a set of interpolation functions that are field consistent. This is because of the complex relationships between the relevant displacements. As a result, performances of total Lagrangian beam elements are unnecessarily degraded. To deal with the problem of field inconsistency, the relationships between the interdependent displacements must be considered and satisfied. In this study, this is done by allowing the interpolations to be expressed as nonlinear functions of nodal degrees of freedom. The proposed element, as a result, eliminates the aforesaid disadvantage of the total Lagrangian formulation. The comparison with the existing problems in the literature clearly shows the efficiency and validity of the proposed element.

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