Flow Field Computation at River Mouth using Taylor-Galerkin Finite Element Method

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Abstract

It is well known that Galerkin finite element method for convective transport flow is often corrupted by node to node oscillations which leads to unstable solutions. To solve this problem, the present study employs forward-time Tayler series expansions up to second-order time derivative. These time derivative terms are evaluated from the governing equation of shallow water theory. Then, the normal Galerkin finite element method is applied to compute the flow variables. This results in a good agreement with the experimental data and no oscillations are also observed.

1 Introduction

Due to the increase in industrial and navigation activities related to river mouth, severe sedimentation and erosion, and also pollutant releases do occur. The analysis of their possible input on river environments is of great importance for engineers and scientists. Mathematical modeling of these phenomena bacame a necessary tool. Numerical solution of this kind contains two distinct modes of transport: advection and diffusion. Diffusive transport can be computed quite accurately using a variety of finite difference and finite element schemes. However, it has been proven more difficult to attain comparable accuracy in numerical computation of advection.

It is well known that Galerkin finite element solutions to advection problem are often corrupted by spurious node-to-node oscillations or 'wiggles'(Donea, 1984). To preclude such kind of oscillations, various procedures have been employed over the year in finite difference methods. The most popular is being the use of 'upwind differencing' on the convective term (Richtmyer and Morton, 1967). This has been extended to finite element methods (Christie, 1976 and Heinrich, 1977). The weighting functions are modified and are applied to all terms in the equation. This leads to the formulation, namely, Petrov-Galerkin weighted residual method (Griffiths, 1978).

The advantage of Petrov-Galerkin weighted residual is the use of upwind weighting function in eliminating spurious oscillations. The upwind finite element concept is analogous to the backward (upwind) difference approach used in finite difference schemes. However, Donea (1984) found that difficulties in applying the Petrov-Galerkin approach are the use of special weighting functions and also the determination of free parameter to maximize the accuracy. Therefore, he proposed the Taylor-Galerkin (TG) scheme for solving a problem of the scalar convection equation in one and more space dimensions. In his paper, the TG scheme was found to exhibit particularly high phase-accuracy with minimum damping compared to the Petrov-Galerkin methods.

Hawken et al. (1990) applied the TG scheme for viscous incompressible flow while Tamaddon-Jahromi et al. (1992) used the TG scheme for the non-newtonian flow. Both papers provided firm evidence of the effectiveness of the TG scheme.

This paper presents the application of the Talor-Galerkin based algorithm for solving the flow fields at river mouth. The method employs forward Taylor series expansions including time derivative of second-order which is evaluated from the shallow water equations. Then, the conventional Galerkin finite element method is applied. Comparisons are made with the experimental data.

2 Method of study

On assuming that vertical acceleration of water is small compared to the gravitational acceleration, the usual shallow water equations (with depth-averaged velocities) can be written as (Kowalik and Murty, 1993)(see also Fig. 1)

$$\frac{\partial H}{\partial t}+\frac{\partial M}{\partial x_1}+\frac{\partial N}{\partial x_2}=0 \qquad (1)$$

$$\frac{\partial M}{\partial t}+\frac{\partial (M^2/H)}{\partial x_1}+\frac{\partial (MN/H)}{\partial x_2}+\frac{gn^2\sqrt{M^2+N^2}.M}{H^{7/3}}+gH\frac{\partial (H-h)}{\partial x_1}=0 \qquad (2)$$

$$\frac{\partial N}{\partial t}+\frac{\partial (MN/H)}{\partial x_1}+\frac{\partial (N^2/H)}{\partial x_2}+\frac{gn^2\sqrt{M^2+N^2}.N}{H^{7/3}}+gH\frac{\partial (H-h)}{\partial x_2}=0 \qquad (3)$$

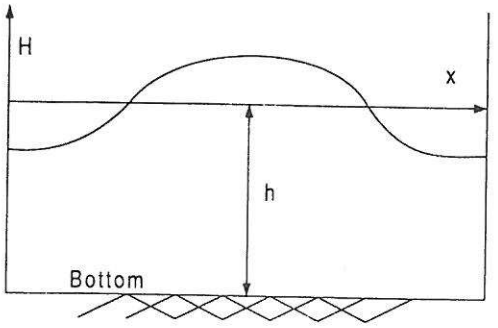


Fig. 1 Definition sketch

where H = water level, M and N = discharge fluxes in x_1 and x_2 directions, n = Manning roughness, h = mean depth, g = gravitational acceleration.

Equations (1), (2), and (3) can be written in term p as

$$\frac{\partial p}{\partial t}+\frac{\partial F_i(p)}{\partial x_i}=S(p) \qquad (4)$$

$$A_i=\frac{\partial F_i(p)}{\partial p} \qquad (5)$$

$$C=\frac{\partial S(p)}{\partial p} \qquad (6)$$

where p, F_i, S, A_i, C are expressed in form of matrix as follows:

$$p=[H \quad M \quad N]^T \qquad (7)$$

$$F_1=\begin{bmatrix} M & \frac{M^2}{H}+\frac{H^2}{2} & \frac{MN}{H}\end{bmatrix}^T \qquad (8)$$

$$F_2=\begin{bmatrix} N & \frac{MN}{H} & \frac{N^2}{H}+\frac{H^2}{2}\end{bmatrix}^T \qquad (9)$$

$$S=\begin{bmatrix} 0 & gH\frac{\partial h}{\partial x_1}-\frac{gn^2}{H^{7/3}}M\sqrt{M^2+N^2} & gH\frac{\partial h}{\partial x2}-\frac{gn^2}{H^{7/3}}N\sqrt{M^2+N^2}\end{bmatrix}^T \qquad (10)$$

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ -\frac{M^2}{H^2}+gH & \frac{2M}{H} & 0 \\ -\frac{MN}{H^2} & \frac{N}{H} & \frac{M}{H}\end{bmatrix} \qquad (11)$$

$$A_2=\begin{bmatrix} 0 & 1 & 0 \\ -\frac{MN}{H^2} & \frac{N}{H} & \frac{M}{H} \\ -\frac{N^2}{H^2}+gH & 0 & \frac{2N}{H}\end{bmatrix} \qquad (12)$$

$$C=\begin{bmatrix} 0 \\ g\frac{\partial h}{\partial x1}+\frac{7}{3}\frac{gn^2}{H^{10/3}}M\sqrt{M^2+N^2} \\ g\frac{\partial h}{\partial x2}+\frac{7}{3}\frac{gn^2}{H^{10/3}}N\sqrt{M^2+N^2}\end{bmatrix}$$

$$0$$

$$-\frac{gn^2M^2}{H^{7/3}\sqrt{M^2+N^2}}-\frac{gn^2}{H^{7/3}}\sqrt{M^2+N^2}$$

$$-\frac{gn^2}{H^{7/3}}\frac{MN}{\sqrt{M^2+N^2}}$$

$$
\begin{matrix} 0 \\ -\frac{gn^2}{H^{7/3}}\frac{MN}{\sqrt{M^2+N^2}} \\ -\frac{gn^2}{H^{7/3}}\frac{N^2}{\sqrt{M^2+N^2}}-\frac{gn^2}{H^{7/3}}\sqrt{M^2+N^2} \end{matrix} \Bigg] \quad (13)
$$

A forward-time Taylor expansion, including the second time derivative, is expressed as:

$$
p^{n+1} = p^n + \Delta t \frac{\partial p}{\partial t}|^n + \frac{(\Delta t)^2}{2}\frac{\partial^2 p}{\partial t^2}|^n + \dots\dots\dots \quad (14)
$$

where superscripts $(\ )^{n+1}$ and $(\ )^n$ represent variables at the next and last time steps, respectively. The partial derivative terms, $\partial p/\partial t$ and $\partial^2 p/\partial t^2$ in Eq. (13) are evaluated from Eqs. (4) to (6) as:

$$
\frac{\partial p}{\partial t}|^n = \left(S - \frac{\partial F_i}{\partial x_i}\right)|^n \quad (15)
$$

$$
\begin{aligned}
\frac{\partial^2 p}{\partial t^2} &= \frac{\partial}{\partial t}\left(S - \frac{\partial F_i}{\partial x_i}\right)|^n \\
&= \frac{\partial S}{\partial t}|^n - \frac{\partial}{\partial x_i}\left(\frac{\partial F_i}{\partial t}\right)|^n \\
&= \frac{\partial S}{\partial p}\frac{\partial p}{\partial t}|^n - \frac{\partial}{\partial x_i}\left(\frac{\partial F_i}{\partial x_i}\frac{\partial p}{\partial t}\right)|^n \\
&= \frac{\partial S}{\partial p}\left(S - \frac{\partial F_i}{\partial x_i}\right)|^n - \frac{\partial}{\partial x_i}\left\{\frac{\partial F_i}{\partial p}\left(S - \frac{\partial F_i}{\partial x_i}\right)\right\}|^n \\
&= C\left(S - \frac{\partial F_i}{\partial x_i}\right)|^n - \frac{\partial}{\partial x_i}\left\{A_i\left(S - \frac{\partial F_i}{\partial x_i}\right)\right\}|^n \quad (16)
\end{aligned}
$$

By inserting Eqs. (15) and (16) into Eq. (14), the result is

$$
p^{n+1} = p^n + \Delta t\left(S - \frac{\partial F_i}{\partial x_i}\right)|^n + \frac{(\Delta t)^2}{2}\left\{C\left(S - \frac{\partial F_i}{\partial x_i}\right)\right\}|^n - \frac{\partial}{\partial x_i}\left\{A_i\left(S - \frac{\partial F_i}{\partial x_i}\right)\right\}|^n \quad (17)
$$

The application of Galerkin finite element (Zienkiewicz, 1971) to Eq. (17) leads to

$$
\begin{aligned}
&\phi^0_{ij}(p^{n+1} - p^n) = \Delta t(\phi^0_{1j}S^n_j - \phi^1_{1,j}F^n_{1,j} - \phi^2_{1,j}F^n_{2,j}) \\
&+ (\Delta t)^2/2(\phi^3_{ijk}C^n_jS^n_k - \phi^4_{ijk}C^n_jF^n_{1k} - \phi^5_{ijk}C^n_jF^n_{2k}) \\
&+ \phi^6_{ijk}A^n_{1j}S^n_k + \phi^7_{ijk}A^n_{2j}S^n_k - \phi^8_{ijk}A^n_{1j}F^n_{1k} \\
&- \phi^9_{ijk}A^n_{1j}F^n_{2k} - \phi^{10}_{ijk}A^n_{2j}F^n_{1k} - \phi^{11}_{ijk}A^n_{2j}F^n_{2k} \\
&- \phi^{12}_{ijk}A^n_{1j}S^n_k l_{x1} - \phi^{13}_{ijk}A^n_{2j}S^n_k l_{x2} \\
&+ \phi^{14}_{ijk}A^n_{1j}F^n_{1k}l_{x1} + \phi^{15}_{ijk}A^n_{1j}F^n_{2k}l_{x2} \\
&+ \phi^{16}_{ijk}A^n_{2j}F^n_{1k}l_{x1} + \phi^{17}_{ijk}A^n_{2j}F^n_{2k}l_{x2} \quad (18)
\end{aligned}
$$

where $\phi^0_{ij} = \iint \phi_i \phi_j dA$ $\phi^1_{ij} = \iint \phi_i \phi_{j,1} dA$ $\phi^2_{ij} = \iint \phi_i \phi_{j,2} dA$

$\phi^3_{ij} = \iint \phi_i \phi_j \phi_k dA$ $\phi^4_{ij} = \iint \phi_i \phi_j \phi_{k,1} dA$, $\phi^5_{ij} = \iint \phi_i \phi_j \phi_{k,2} dA$

$\phi^6_{ij} = \iint \phi_{i,1} \phi_j \phi_k dA$ $\phi^7_{ij} = \iint \phi_{i,2} \phi_j \phi_k dA$ $\phi^8_{ij} = \iint \phi_{i,1} \phi_j \phi_{k,1} dA$

$\phi^9_{ij} = \iint \phi_{i,1} \phi_j \phi_{k,2} dA$ $\phi^{10}_{ij} = \iint \phi_{i,2} \phi_j \phi_{k,1} dA$ $\phi^{11}_{ij} = \iint \phi_{i,2} \phi_j \phi_{k,2} dA$

$\phi^{12}_{ij} = \iint \phi_i \phi_j \phi_k ds$ $\phi^{13}_{ij} = \iint \phi_i \phi_j \phi_k ds$ $\phi^{14}_{ij} = \iint \phi_i \phi_j \phi_{k,1} ds$

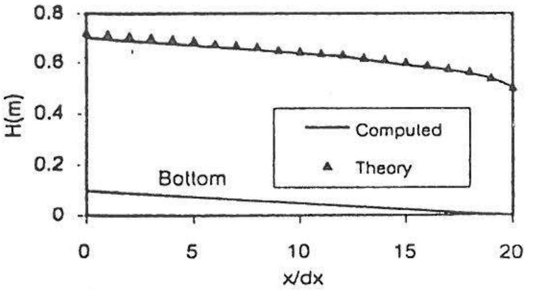
$\phi^{15}_{ij} = \iint \phi_i \phi_j \phi_{k,2} ds$ $\phi^{16}_{ij} = \iint \phi_i \phi_j \phi_{k,1} ds$ $\phi^{17}_{ij} = \iint \phi_i \phi_j \phi_{k,2} ds$

Values of ϕ_i, ϕ_j and ϕ_k represent the Galerkin weighting functions. For the present paper, the linear triangular basis function are used. Variables with subscript $(\ )_{,1}$ and $(\ )_{,2}$ mean the differentiation with respect to x_1 and x_2. (lx_1, lx_2) are the unit normal vector to ds.

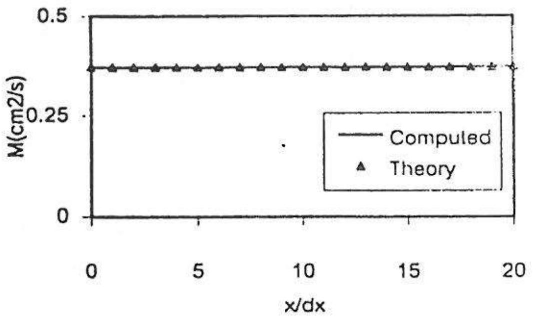
3 Results and discussions

To evaluate the validity of the present numerical model, the model accuracy has to be examined. This was done by comparisons with the theoretical solution for a simple 1-D case. Numerical example was performed on a straight channel of 20 m long and 3 m wide. The discharge was set = 1.1 m^3/s with a depth of 0.6525 m at the upstream boundary. The downstream boundary depth was set = 0.5 m with a slope of 0.005. The water surface profile of this type shows an M2-curve (Subramanya, 1998). Time step of 0.1 s was used and the computation was done for 250 s.

Figure 2 shows results of water level and the discharge flux. Excellent agreements between the theoretical and numerical results are found for both H and M. Deviations are found smaller than 0.1%. This indicates the good performance of the model used in the present study.



a) Water level



b) Discharge flux

Fig. 2 Numerical example for 1-D case

The model was then applied to simulate the flow field for a 2-D case such as river mouth. Experimental data for case study no. 3 of Yazawa(1990) was used for model comparison(see Fig. 3). The discharge was set equal to 1600 cm^3/s. Computational domain is also given in Fig. 3 where the smallest portion of the river mouth is 0.1 m in width. The domain contains 490 elements. The time step of 0.01 s was used due to small width of the river mouth. The boundary conditions are given in Table1.

Table 1 Boundary conditions

Variables	Values	Boundary
M	32 cm^2/s	S1
M	0	S2
N	0	S3
H	12.11 cm	S4
N	0.016	Bottom

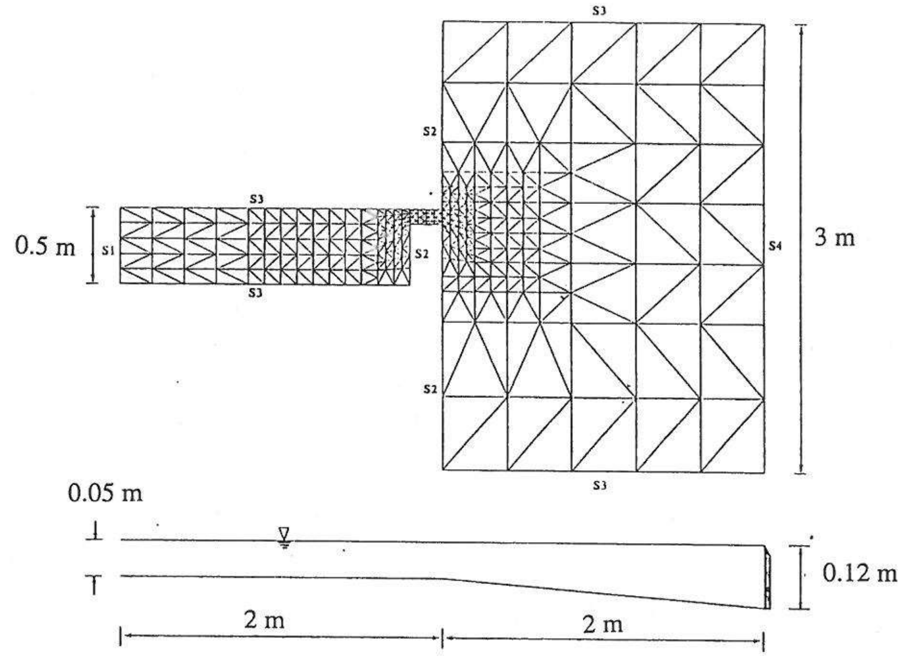


Fig. 3 Computation domain

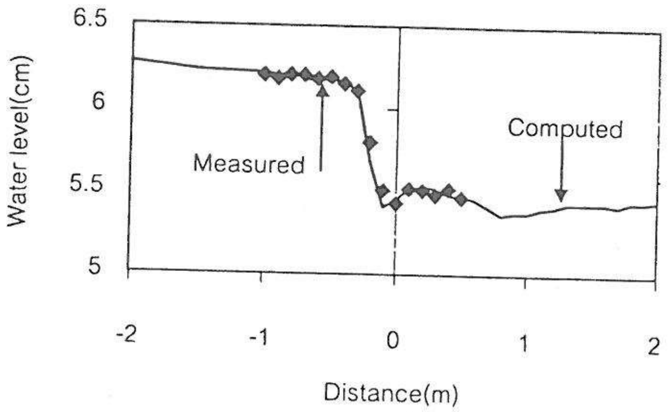


Fig. 4 Longitudinal profile of water level for 2-D case

Figure 4 shows the longitudinal profile of water level along the centerline of river mouth. The measured data are shown by marks while the computed results are shown by a solid line. It can be seen that the model can predict very well the water level upstream and downstream of the river mouth. The results show an abrupt drop of 0.7 cm before the river mouth location which can be simulated well by the model.

Figure 5 shows the longitudinal profile of velocity at the same location. It is observed that the increase in velocity occurs sharply at about 0.25 m upstream of the river mouth. This is also simulated well by the model. However, just downstream of the river mouth, the dropping and the fluctuation of velocities of the measured data cannot be predicted satisfactorily. This may be caused by error measurements in the experiments especially in the vicinity of sudden expansion region. The velocity field for the same case study is shown in Fig. 6. An asymmetrical vortex downstream of the river mouth can be represented well by the model compared to the measured data.

4 Conclusions

The computation of flow fields using the shallow water equation was simulated reasonably well by the Taylor-Galerkin finite element method. This method applies the Taylor series expansions including the second time derivative. The model accuracy was firstly examined by comparisons with the theoretical 1-D solution. This was shown to be

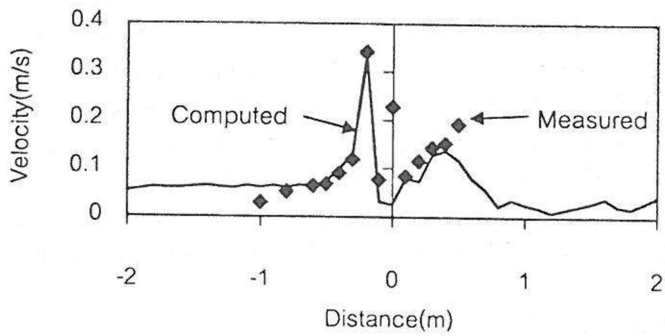


Fig. 5 Longitudinal profile of velocity

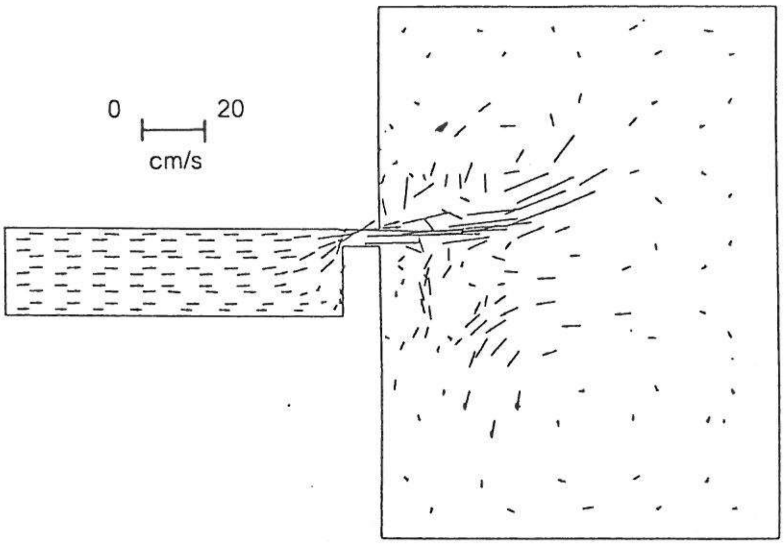


Fig. 6 Velocity field

excellent agreements. The model was, then, applied to simulate the flow fields of the river mouth configuration. Agreements were also found satisfactorily. Therefore, it can be concluded that the Taylor-Galerkin finite element method is suitable for the simulation of the flow of mixed convective-diffusive situations.

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