Finite Element Analysis of Problems Involving Material Nonlinearity by the Equivalent Inclusion Method

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Abstract

In this study, a new finite element analysis method for problems involving material nonlinearity is proposed. The proposed analysis method is derived from the Equivalent Inclusion Method. In the proposed method, the material undergoing nonlinear behavior is changed into an equivalent material with strains called eigenstrains. For convenience, the material in the linear region is selected as the equivalent material. Doing analysis this way will keep the constitutive relationship unchanged. Therefore, the global stiffness remains the same for all computational steps. The task in each step is then reduced into finding the varying eigenstrains. The equation for finding the eigenstrains can be much smaller than the global stiffness equation. Therefore, the computational time can be greatly reduced if the nonlinear region is small compared with the whole domain.

1. Introduction

The finite element analysis involving material nonlinearity generally consumes long computational time. It is because, in this kind of analysis, the constitutive relationship of the material changes along the computational steps. The change in each step will lead to a completely new global stiffness equation that has to be solved again. In some cases where the proportion of the material undergoing nonlinear behavior is small compared with the whole domain, it will be advantageous if we are able to utilize solutions from the previous step in the current step. The Equivalent Inclusion Method [1, 2] may be the answer. In this method, the inhomogeneous material will be changed to an equivalent homogeneous material with a kind of residual strain called eigenstrain. For example, if we apply the method to elastic-plastic analysis, the material with plastic regions will be changed into the original elastic material with eigenstrains. Since the constitutive relationship can be kept unchanged in this method, the global stiffness will not change. Therefore, the computation is changed from solving a new global stiffness equation to solving an equation for eigenstrains. If the proportion of the nonlinear material is small, the equation for computing eigenstrains will be small and the computational time is expected to decrease significantly.

The Equivalent Inclusion Method has been applied to solve many problems that contain inhomogeneities (see, for example, [3-5]). However, the applications are usually limited to inhomogeneities in two- or three-dimensional infinite solid domains. In most cases, the elliptic (2-D) or ellipsoidal (3-D) inhomogeneities will be considered because the analytical solutions for these cases can be obtained with less mathematical difficulty. Nevertheless, Yamaguchi et al. [6] applied the Equivalent Inclusion Method in the elastic-plastic analysis of frame structures. They introduced eigenmoment and eigenshear and formulated an analysis method based on these eigenmoment and eigenshear. By finding the Green's function associated with the eigenmoment numerically, the analysis of the

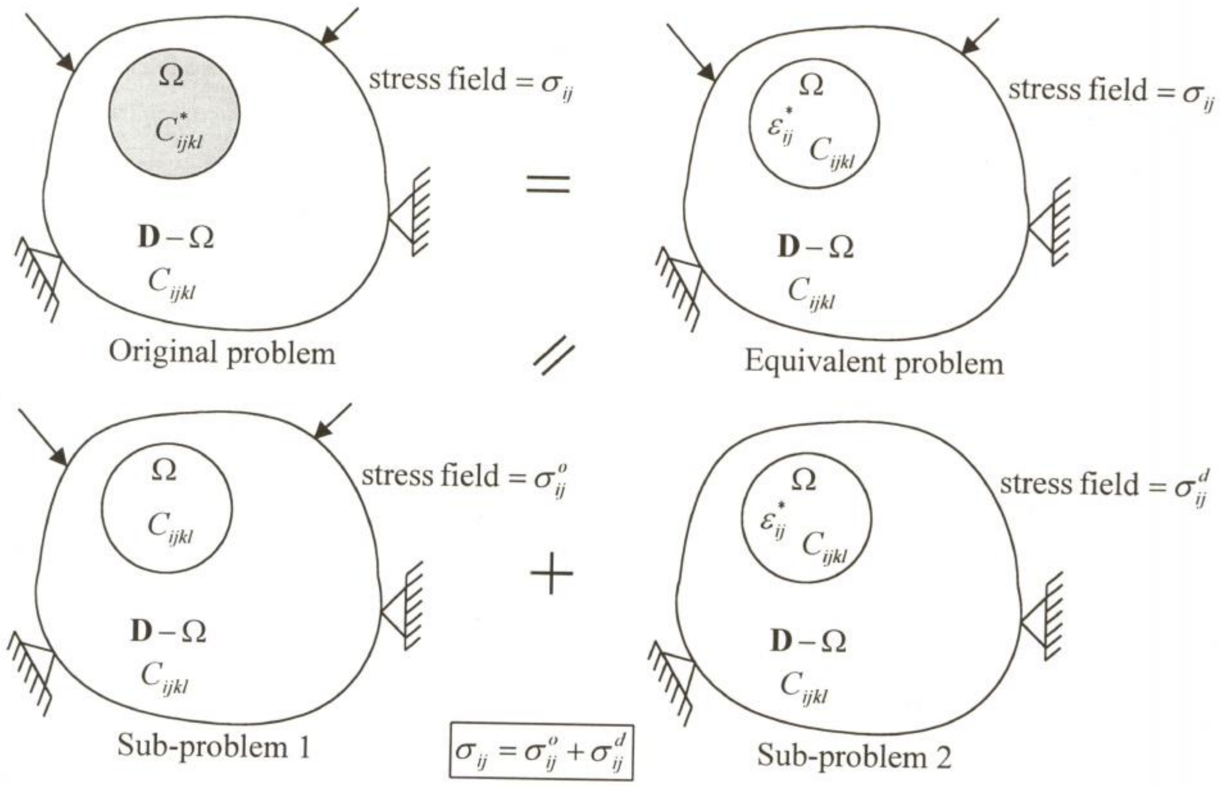

Fig. 1 Equivalent problem and its decomposition

frame structures could be carried out. Inspired by their work, this paper presents a finite element analysis method for general solid problems having nonlinear materials based on the Equivalent Inclusion Method. Although only two-dimensional problems are discussed, the idea can be generally extended to three-dimensional problems. The validity of the proposed method is shown and the advantage over the conventional method is also presented.

2. Equivalent Inclusion Method

Consider a body **D** with a sub-domain Ω having different material properties from those of the remaining matrix shown in Fig. 1. The material properties of the matrix and the sub-domain Ω are denoted by C_{ijkl} and C^*_{ijkl}, respectively. It is assumed that the body is under the stress-free condition before the application of loads. A stress field caused by applied loads is denoted by $\sigma_{ij}(\mathbf{x})$ where **x** is a position vector. This stress field can be thought of as the superposition of a stress field when a homogeneous material is assumed and the stress disturbance due to the presence of the inhomogeneity. Eshelby [1, 2] first pointed out that the stress disturbance due to inhomogeneities could be simulated by stresses caused by inclusions with appropriate applied strains. These strains are generally called eigenstrains. Therefore, instead of solving the original inhomogeneous problem, an equivalent problem, where the material is considered homogeneous, can be considered. In the equivalent problem, the inhomogeneity is replaced by an inclusion that has the same material properties as those of the matrix but contains eigenstrains as shown in Fig. 1. These eigenstrains, if appropriately selected, will result in a stress field that simulates the stress disturbance due to the inhomogeneity. Since, in this method, the original problem with inhomogeneities is changed into an equivalent homogeneous problem containing inclusions with eigenstrains, the method is called "Equivalent Inclusion Method."

Following Mura [2], we denote the stress field and the corresponding strain field, when a homogeneous material is assumed, by $\sigma_{ij}^{o}(\mathbf{x})$ and $\varepsilon_{ij}^{o}(\mathbf{x})$, respectively. In addition, the stress disturbance and strain disturbance are denoted by $\sigma_{ij}^{d}(\mathbf{x})$ and $\varepsilon_{ij}^{d}(\mathbf{x})$, respectively. According to the Hooke's law, we have

$$\begin{aligned} \sigma_{ij}^{o}+\sigma_{ij}^{d} &= C_{ijkl}^{*}\left(\varepsilon_{kl}^{o}+\varepsilon_{kl}^{d}\right) \quad \text{in } \Omega \\ \sigma_{ij}^{o}+\sigma_{ij}^{d} &= C_{ijkl}\left(\varepsilon_{kl}^{o}+\varepsilon_{kl}^{d}\right) \quad \text{in } \mathbf{D}-\Omega. \end{aligned} \tag{1}$$

Consider the equivalent problem. The problem is decomposed into 2 sub-problems (see Fig. 1). The first sub-problem corresponds to a problem that considers no inhomogeneity and no eigenstrains in the domain. The relationship between the stress field $\sigma_{ij}^{o}(\mathbf{x})$ and the strain field $\varepsilon_{ij}^{o}(\mathbf{x})$ in this sub-problem can be expressed as

$$\sigma_{ij}^{o} = C_{ijkl}\varepsilon_{kl}^{o}. \tag{2}$$

The second sub-problem represents the stress disturbance field $\sigma_{ij}^{d}(\mathbf{x})$ and the strain disturbance field $\varepsilon_{ij}^{d}(\mathbf{x})$ due to the inhomogeneity. In the second sub-problem, no applied loads are considered and the eigenstrain field $\varepsilon_{ij}^{*}(\mathbf{x})$ is introduced in the sub-domain Ω. The applied eigenstrains result in the stress field $\sigma_{ij}^{d}(\mathbf{x})$ and the strain field $\varepsilon_{ij}^{d}(\mathbf{x})$ in the domain. The relationship between the stress disturbance and the strain disturbance can be written as

$$\begin{aligned} \sigma_{ij}^{d} &= C_{ijkl}(\varepsilon_{kl}^{d}-\varepsilon_{kl}^{*}) \quad \text{in } \Omega \\ \sigma_{ij}^{d} &= C_{ijkl}\varepsilon_{kl}^{d} \quad \text{in } \mathbf{D}-\Omega. \end{aligned} \tag{3}$$

From Eqs. (2) and (3), we get

$$\begin{aligned} \sigma_{ij}^{o}+\sigma_{ij}^{d} &= C_{ijkl}\left(\varepsilon_{kl}^{o}+\varepsilon_{kl}^{d}-\varepsilon_{kl}^{*}\right) \text{ in } \Omega \\ \sigma_{ij}^{o}+\sigma_{ij}^{d} &= C_{ijkl}\left(\varepsilon_{kl}^{o}+\varepsilon_{kl}^{d}\right) \quad \text{in } \mathbf{D}-\Omega. \end{aligned} \tag{4}$$

The stress fields in the original problem and in the equivalent problem must be equal. Therefore, from Eqs. (1) and (4), we have

$$C_{ijkl}^{*}\left(\varepsilon_{kl}^{o}+\varepsilon_{kl}^{d}\right) = C_{ijkl}\left(\varepsilon_{kl}^{o}+\varepsilon_{kl}^{d}-\varepsilon_{kl}^{*}\right) \text{ in } \Omega. \tag{5}$$

It is possible to express the stain disturbance $\varepsilon_{ij}^{d}(\mathbf{x})$ in terms of the applied eigenstrain $\varepsilon_{ij}^{*}(\mathbf{x})$, i.e.,

$$\varepsilon_{ij}^{d}(\mathbf{x}) = \int_{\Omega} s_{ijkl}(\mathbf{x},\mathbf{x}')\varepsilon_{kl}^{*}(\mathbf{x}')d\mathbf{x}' \tag{6}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are position vectors and $s_{ijkl}(\mathbf{x},\mathbf{x}')$ is a function which can be obtained by the Green's function method [2]. Therefore, we have

$$\begin{aligned} &C_{ijkl}^{*}\left(\varepsilon_{kl}^{o}(\mathbf{x})+\int_{\Omega} s_{ijkl}(\mathbf{x},\mathbf{x}')\varepsilon_{mn}^{*}(\mathbf{x}')d\mathbf{x}'\right) = \\ &C_{ijkl}\left(\varepsilon_{kl}^{o}(\mathbf{x})+\int_{\Omega} s_{ijkl}(\mathbf{x},\mathbf{x}')\varepsilon_{mn}^{*}(\mathbf{x}')d\mathbf{x}'-\varepsilon_{kl}^{*}(\mathbf{x})\right) \\ &\text{in } \Omega \end{aligned} \tag{7}$$

where the eigenstrain ε_{ij}^{*} can be determined for a given ε_{ij}^{o}. After obtaining ε_{ij}^{*}, the stress field $\sigma_{ij}^{o}+\sigma_{ij}^{d}$ and strain field $\varepsilon_{ijl}^{o}+\varepsilon_{ij}^{d}$ can be subsequently determined. Note that the last equation is valid everywhere in Ω.

3. Finite element analysis of problems with inhomogeneities using the Equivalent Inclusion Method

Consider a simple two-dimensional finite element problem shown in Fig. 2. The type of the elements used in this analysis is the four-noded quadrilateral element. In the problem, there are some elements whose material

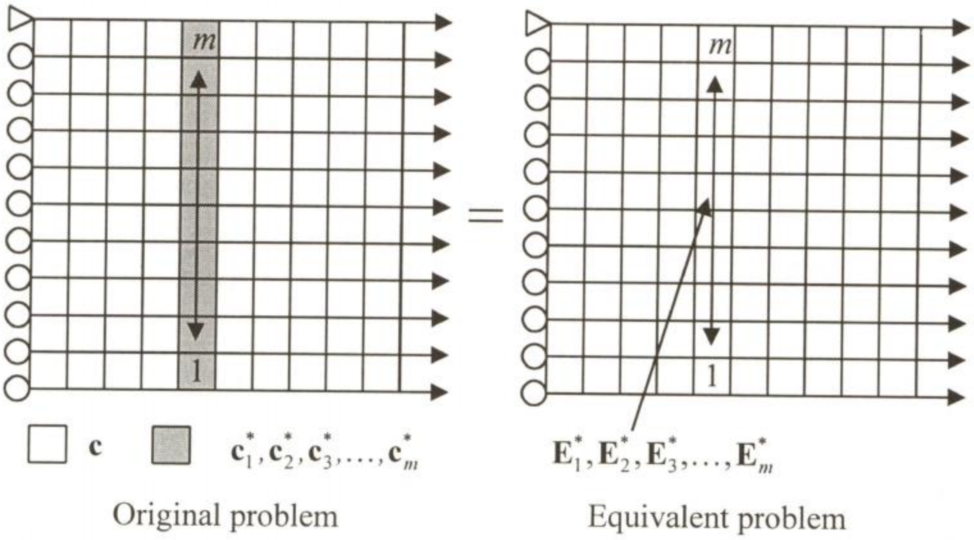

Fig. 2 Equivalent Inclusion Method in finite element analysis

properties are different from the rest of the domain. These elements are distinguished by the shaded area. To solve the problem, we consider an equivalent problem, which is also shown in Fig. 2. In the equivalent problem, the material is considered homogeneous. The inhomogeneity elements are replaced by elements with the material of the matrix and, in these elements, eigenstrains are introduced. The first task is to determine these eigenstrains. Knowing the eigenstrains will subsequently lead to the complete solution of the problem. To this end, Eq. (7) will be employed in each inhomogeneity element. Before that, consider Eq. (6), which is the relationship between the disturbance strain $\varepsilon_{ij}^d(\mathbf{x})$ and the eigenstrain $\varepsilon_{ij}^*(\mathbf{x})$. Assume in the equation that both strains are constant in each element and use the values of the strains evaluated at the centers of the elements. Therefore, Eq. (6) can be rewritten in matrix form as

$$\mathbf{E}^d = \mathbf{S}\mathbf{E}^* \tag{8}$$

where $\mathbf{E}^*$ denotes the eigenstrains at the centers of the inhomogeneity elements, i.e.,

$$\mathbf{E}^* = \begin{Bmatrix} \mathbf{E}_1^* \\ \mathbf{E}_2^* \\ \vdots \\ \mathbf{E}_m^* \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1^* \\ \varepsilon_2^* \\ \vdots \\ \varepsilon_{3m}^* \end{Bmatrix} \tag{9}$$

in which $\mathbf{E}_i^*$ represents an eigenstrain vector $\begin{bmatrix} \varepsilon_x^* & \varepsilon_y^* & \gamma_{xy}^* \end{bmatrix}^T$ evaluated at the center of the i^{th} inhomogeneity element. It is assumed in this example that there are totally m inhomogeneity elements. Here, ε_i^* represents each component of the vector $\mathbf{E}^*$. The vector $\mathbf{E}^d$, written in the same way as $\mathbf{E}^*$, represents the strain disturbance components at the centers of the inhomogeneity elements. Note that both $\mathbf{E}^*$ and $\mathbf{E}^d$ are $3m \times 1$ vectors.

To solve the equivalent problem, the problem is decomposed into 2 sub-problems as discussed earlier (see Fig. 1). Components of the matrix $\mathbf{S}$ in Eq. (8) can be obtained numerically by solving the second sub-problem with different unit eigenstrain components. Note that, in the second sub-problem, there are no applied loads and only eigenstrains are considered. For example, we first set

$$\mathbf{E}^* = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{10}$$

in the second sub-problem and solve the problem by FEM. From the analysis, the displacement disturbance u_{ij}^d at all nodes and the strain disturbance ε_{ij}^d at the centers of all elements are obtained. Therefore, we have the first column of the matrix **S**, i.e.,

$$\mathbf{S} = \begin{bmatrix} S_{11} & \times & \cdots & \times \\ S_{21} & \times & \cdots & \times \\ \vdots & \vdots & \ddots & \vdots \\ S_{m1} & \times & \ldots & \times \end{bmatrix}. \tag{11}$$

Next, we set

$$\mathbf{E}^* = \begin{Bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{Bmatrix} \tag{12}$$

and obtain the second column of the matrix **S**. The analysis is repeated with different unit components of the eigenstrains until the complete matrix **S** is obtained. This means that the total number of the analysis will be $3m$ times.

Solving the first sub-problem (see Fig. 1) which is the problem without inhomogeneities by FEM, we obtain the displacements u_i^o at all nodes and also the strains ε_{ij}^o at the centers of all elements. After that, the strains ε_{ij}^o in the inhomogeneity elements are arranged in matrix form in the same way as the eigenstrain vector E^*, i.e.,

$$\mathbf{E}^o = \begin{Bmatrix} \mathbf{E}_1^o \\ \mathbf{E}_2^o \\ \vdots \\ \mathbf{E}_m^o \end{Bmatrix} \tag{13}$$

where E_i^o represents a strain vector $\begin{bmatrix} \varepsilon_x^o & \varepsilon_y^o & \gamma_{xy}^o \end{bmatrix}^T$ evaluated at the center of the i^{th} inhomogeneity element.

Next, construct a $3m \times 3m$ constitutive matrix for the inhomogeneity elements defined as

$$\mathbf{C}^* = \begin{bmatrix} \mathbf{c}_1^* & & \mathbf{0} \\ & \mathbf{c}_2^* & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{c}_m^* \end{bmatrix} \tag{14}$$

where the diagonal term $\mathbf{c}_i^*$ is the ordinary 3×3 constitutive matrix for the i^{th} inhomogeneity element. Also, construct a $3m \times 3m$ constitutive matrix for the inhomogeneity elements when the homogeneous material is assumed, i.e.,

$$\mathbf{C} = \begin{bmatrix} \mathbf{c} & & \mathbf{0} \\ & \mathbf{c} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{c} \end{bmatrix}. \tag{15}$$

The matrix **C** is created by replacing all diagonal terms $\mathbf{c}_i^*$ in the matrix $\mathbf{C}^*$ by the matrix **c** that is the ordinary 3×3 constitutive matrix for the matrix material.

Then, Eq. (7) can be written as

$$\mathbf{C}^*\left(\mathbf{E}^o + \mathbf{S}\mathbf{E}^*\right) = \mathbf{C}\left(\mathbf{E}^o + \mathbf{S}\mathbf{E}^* - \mathbf{E}^*\right) \tag{16}$$

which yields the solution for the eigenstrains, i.e.,

$$\mathbf{E}^* = \left[\left(\mathbf{C} - \mathbf{C}^*\right)\mathbf{S} - \mathbf{C}\right]^{-1}\left(\mathbf{C}^* - \mathbf{C}\right)\mathbf{E}^o. \tag{17}$$

During the analysis to obtain the matrix **S**, the displacement disturbance u_{ij}^d at all nodes and the strain disturbance ε_{ij}^d in all elements are obtained for different unit eigenstrains. Therefore, the nodal displacement vector **U** can be expressed as

$$\mathbf{U} = \mathbf{U}^o + \sum_{i=1}^{3m} \mathbf{U}_i^d \varepsilon_i^* \qquad (18)$$

where $\mathbf{U}^o$ represents the nodal displacement vector obtained from the first sub-problem and $\mathbf{U}_i^d$ is the nodal displacement disturbance vector obtained when $\varepsilon_i^* = 1$ [see Eq. (9)] is applied to the second sub-problem.

Moreover, the strains in an arbitrary I^{th} element can be computed from

$$\mathbf{E}_I = \mathbf{E}_I^o + \sum_{i=1}^{3m} \left(\mathbf{E}_I^d\right)_i \varepsilon_i^* \qquad (19)$$

where $\mathbf{E}_I$ represents the element strain vector $\begin{bmatrix}\varepsilon_x & \varepsilon_y & \gamma_{xy}\end{bmatrix}^T$. Here, $\mathbf{E}_I^o$ denotes the element strain vector obtained from the first sub-problem and $\left(\mathbf{E}_I^d\right)_i$ denotes the element strain disturbance when $\varepsilon_i^* = 1$ [see Eq. (9)] is applied in the second sub-problem. Note that all element strain vectors represent values at the centers of the elements.

It is clear that big computational effort is necessary for solving this sample problem using the Equivalent Inclusion Method. The problem must be analyzed one time to obtain the solution when the homogeneous material is assumed and another $3m$ times to obtain the matrix **S**. Furthermore, the matrix equation for the eigenstrains must be solved before the final solution can be obtained. In this case, the proposed solution method, of course, cannot compete with one step calculation in the conventional method. However, the advantage of the proposed method is expected in the analysis of problems with material nonlinearity where many computational steps are necessary.

4. Finite element analysis of problems with material nonlinearity using the Equivalent Inclusion Method

Before considering problems with material nonlinearity, the validity of the method will be shown. In the proposed method, the eigenstrains are assumed constant within the elements. To verify that the assumption does not significantly affect the accuracy of the method, a problem that exhibits high stress gradient is selected for the investigation. Here,

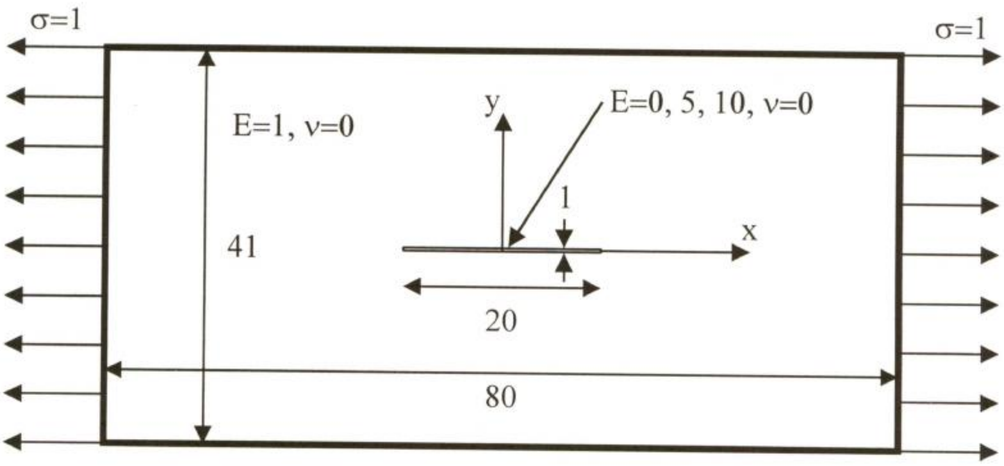

Fig. 3 Plane stress problem with inhomogeneity

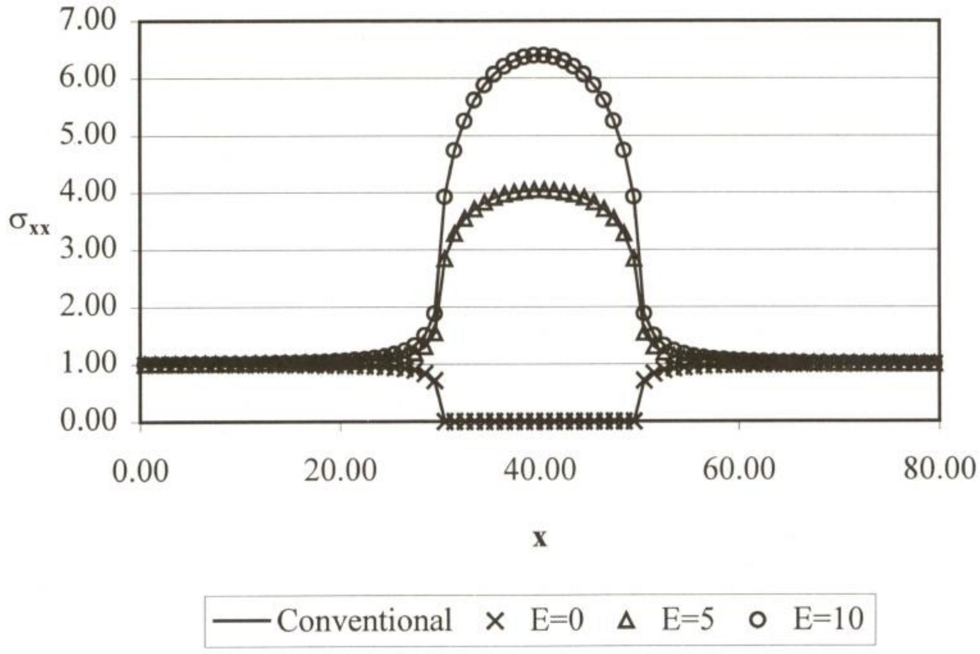

Fig. 4 Results from the conventional method and the proposed method

the method is used to solve a plane stress problem shown in Fig. 3. In the problem, a unit thickness plate with the dimension of 80x41 is subjected to uniform applied stress $\sigma = 1$ at both ends. Assume the Young's modulus $E = 1.0$ and Poisson's ratio $\nu = 0$ everywhere in the domain except in a small region at the center of the plate (see Fig. 3). In the small region, the Poisson's ratio is set to be 0 and various values of the Young's modulus are used. The problem is solved by the conventional method and the proposed method.

In the analysis, 80 elements and 41 elements are used along horizontal and vertical directions, respectively, and there are 20 elements in the inhomogeneity. The results are shown in Fig. 4 for the Young's modulus in the small region equal to 0, 5 and 10. In the figure, the normal stress σ_{xx} along a line $y = 0$ is plotted. The solid lines represent the results from the conventional method. From the comparison, a very good agreement between the results from the proposed method and the conventional method can be observed. Therefore, the validity of the proposed analysis method is confirmed.

Next, the method is used to analyze problems with material nonlinearity. Consider again a uniaxial problem shown in Fig. 2. Assume the dimensions of the domain to be 10x10 with unit thickness. Also assume the Young's modulus $E = 1.0$ and $\nu = 0$ for every element at the initial state. After loading, the Young's modulus for the material in the shaded area (see Fig. 2), which occupies 10% of the total area of the domain, is assumed to follow the piecewise-linear constitutive law given in Fig. 5. The mesh used in the analysis is also shown in Fig. 2. There are totally 10x10 elements and there are 10 elements with the nonlinear material.

The problem is solved incrementally by the conventional method and the proposed method. The analysis is stopped when the uniaxial normal stress is equal to 0.5, which is the end of the constitutive curve. The machine used in the analysis is a SUN ULTRA 1 workstation with the Solaris 2.5 operating system. The main purpose of solving this problem is to compare the computational time used by both methods. The number of steps used in the analysis is varied from 100 to 1000.

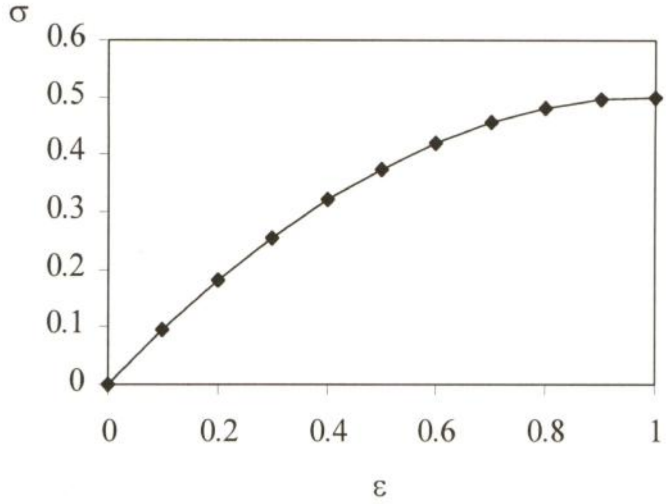

Fig. 5 Uniaxial stress-strain relationship

As for the accuracy, it is found that the incremental computations by the conventional and proposed methods yield the same exact solution. The exact solution can be obtained because the numbers of steps are selected so that the constitutive law, which is given in piecewise-linear fashion with the strain interval of 0.1, can be exactly satisfied. After completing the analysis with 10% nonlinear material, a new problem with 20% nonlinear material is solved. The problem with 20% nonlinear material is made by setting 10 more elements to be the nonlinear elements. The added nonlinear elements are selected across the section in such a way that the uniaxial condition is assured. Fig. 6 shows the results obtained from the proposed method. The graphs are the relationships between the applied load and the displacement of the surface where the load is applied for the cases with 10% and 20% nonlinear material. As mentioned earlier, the obtained results are exactly equal to the exact solutions.

The computational time for the analysis

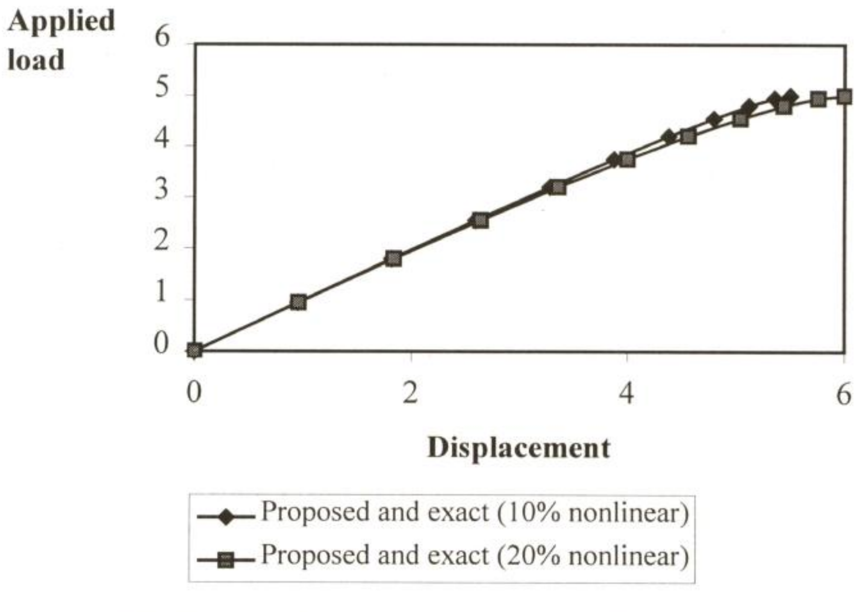

Fig. 6 Applied load and displacement relationships of the uniaxial problem

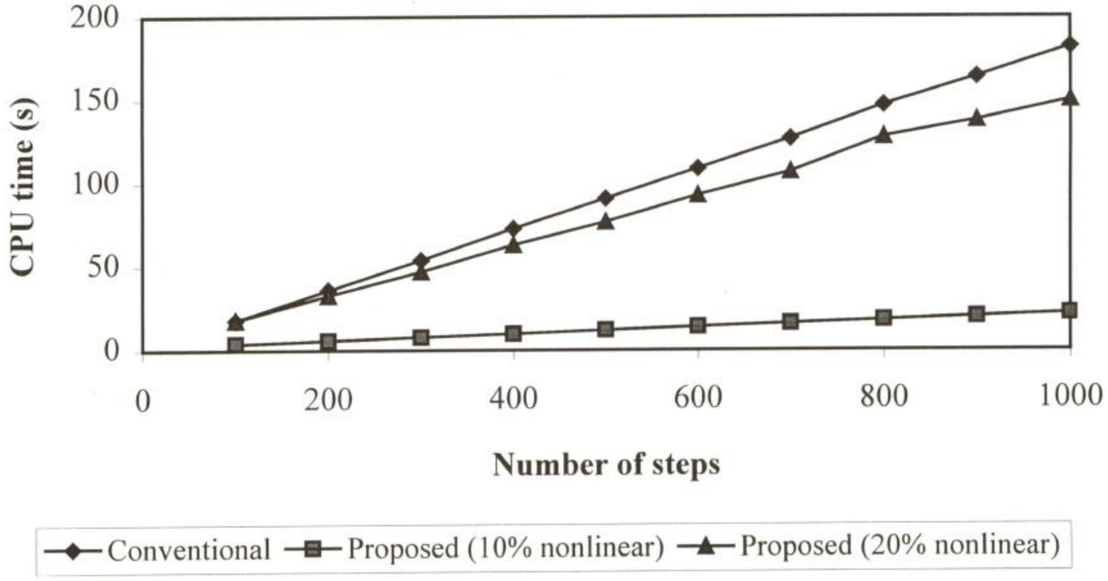

Fig. 7 Comparison of computational time

is shown in Fig. 7. From the figure, it is clear that the proposed analysis method is very much faster than the conventional one when the ratio of the material undergoing nonlinear behavior is small compared with the whole domain. This advantage is reduced when the proportion of the nonlinear material increases. This is expected because the larger number of nonlinear elements will result in the larger matrix equation for eigenstrains [Eq. (7)], which is to be solved in every step. Besides that, the larger number of nonlinear elements means the larger matrix **S** in Eq. (8). Therefore, the evaluation of the components of the matrix **S** at the beginning of the analysis will take longer time. Nevertheless, a big advantage over the conventional method can be seen if the ratio between the number of the nonlinear elements and the total number of the elements is less than 20%. Furthermore, the advantage becomes even stronger when the larger numbers of steps are used.

5. Conclusions

This work presents a new finite element analysis method for general solid problems with materials undergoing nonlinear behavior. For simplicity, only two-dimensional problems are discussed. However, the idea can be generally extended to three-dimensional problems. The proposed analysis method is based on the Equivalent Inclusion Method. In this method, the nonlinear material is changed into an equivalent material with eigenstrains. Since the constitutive relationship is kept unchanged, the computation in each step is reduced into solving for the unknown eigenstrains. If the proportion of the nonlinear material is small compared with the whole domain, the matrix equation for eigenstrains will be small. Therefore, the computational time can be reduced. The results obtained from the proposed method show very good agreement with the conventional method, and much shorter computational time is observed if the proportion of the material undergoing nonlinear behavior is small.

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