Analysis of Cracking Localization Using the Smeared Crack Approach

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Abstract

In consideration of cracking localization, it is more suitable to have an energy expression written in terms of discrete irreversible variables, which will allow the variations of the energy with respect to the irreversible variables to be considered easily. This implies that the discrete crack approach should be more appropriate for this kind of analysis than the smeared crack approach. However, the discrete crack approach may not be the best choice for problems with many cracks, which are unavoidable for the analysis of the cracking localization. To avoid the drawbacks in both approaches, a special treatment on the smeared crack approach to allow the consideration of the cracking localization is developed. To this end, discrete irreversible variables related to crack strains are introduced, and the cracking localization is investigated, based on these discrete irreversible variables. The results obtained show promising capability of the method in analyzing problems with the cracking localization.

1. Introduction

Cracking localization prior to the failure plays a very important role in the fracture behavior of quasi-brittle materials, such as concrete. In order to capture the real ultimate capacity of such materials in structures, consideration of the cracking localization cannot generally be neglected. However, the analysis of the cracking localization is very expensive. Because of this reason, many researchers avoid the consideration of the cracking localization. This can be done by either allowing many cracks to open or grow without the consideration of the localization [1, 2, 3, 4, 5] or assuming the locations of the localized cracks [5, 6]. The first approach is not realistic and can lead to very inaccurate results. When compared with having one or a few localized cracks, having many cracks without localization allows different amounts of energy to dissipate from the domain. Thus, the obtained results will be different as well. Only in some cases where the stress gradients of the problems are very large and the stress criteria for crack initiation are used, can the localized solution possibly be obtained from this approach [1, 2, 4]. When the stress gradient is very high, it is numerically possible that major cracks will finally prevail and the other cracks will undergo the elastic unloading. The second approach, which assumes the locations of the localized cracks prior to the analysis, may also possibly yield reasonable results in some cases. These include cases where the assumed locations of the localized cracks are reasonably correct, such as bending problems of concrete beams with long notches [6]. The others are cases where the required solutions, such as the ultimate loads, are not sensitive to the locations of the localized cracks [5]. Nevertheless, this second approach is not appropriate for general cases since the locations of the localized cracks may not be easily predicted or the required solution may be sensitive to the locations of the cracks.

In the analysis of the cracking localization, consideration of stability and bifurcation of equilibrium states is one of the tasks to be done. Many researchers have considered the stability and bifurcation of the equilibrium states by investigating the definiteness of the stiffness matrices (Hessian Matrices) [7, 8, 9]. When the matrix is positive-definite, the equilibrium is stable. The same theory can be applied to the analysis of the cracking localization. Nevertheless, to consider the stability and bifurcation of irreversible processes such as cracking, the stationary condition of the energy of the system with respect to irreversible parameters has to be examined [10, 11, 12]. This requires expression of the energy in terms of the irreversible parameters. For crack problems, the irreversible parameters can be the crack opening displacement variables in the discrete crack approach or the crack strain variables in the smeared crack approach. In the discrete crack approach, the crack opening displacement variables are usually discretized along crack paths and treated as the degrees of freedom in the analysis. The energy of the system is expressed in terms of these degrees of freedom. Computing the first and second variations of the energy with respect to the crack opening displacement degrees of freedom can be done easily. The stability and bifurcation of the equilibrated solutions can be considered by employing just the ordinary calculus [12]. On the contrary, if the smeared crack approach is employed, the energy of the system will be expressed in terms of the irreversible crack strain variables, which are not discretized variables. These crack strain variables are functions of position. To compute the first and second variations of the energy with respect to these crack strain functions, complex mathematics involving the calculus of variations must be employed.

This fact implies that the discrete crack approach in the finite element method may be more suitable for the cracking localization analysis than the smeared crack approach. Nevertheless, the discrete crack approach may not perform best when there are many cracks. In this aspect, the smeared crack approach is more appropriate.

To avoid the drawbacks in both methods, in this study, a special treatment on the smeared crack finite element analysis is proposed. The proposed treatment will make it possible to consider the cracking localization by using the smeared crack models. In the proposed method, discrete irreversible variables related to the crack strains are introduced in the smeared crack models. These discrete variables will allow the consideration of the stability and bifurcation of the equilibrated solution to be done by considering the variations of the energy with respect to the proposed discrete variables. The proposed scheme will not be used to obtain the stiffness equation that is used to obtain the equilibrium paths. The original smeared crack models will be still used for that purpose. The proposed method will be used only for the investigation of the stability and bifurcation.

2. Cracking Localization

Consider a system of a deformable body with cracks where the energy is dissipated. Following Nguyen [10] and Brocca [12], we define the total energy of the body as

$$\Pi(u_i,\alpha_j) = \Pi^M(u_k,\alpha_l) + \Pi^D(\alpha_m) \qquad (1)$$

where $\Pi^M(u_k,\alpha_l)$ is the mechanical potential energy and $\Pi^D(\alpha_m)$ is the dissipated energy. The arguments of the functions, u_i $(i = 1,\ldots,N)$ and α_i $(i = 1,\ldots,K)$, represent the reversible variables and irreversible variables, respectively. Here, N is the number of the reversible variables and K is the number of the irreversible variables.

Applying the stationary conditions to Eq. (1), we have

$$\frac{\partial \Pi}{\partial u_i} = 0 \qquad i = 1, \ldots, N, \tag{2a}$$

$$\frac{\partial \Pi}{\partial \alpha_j} = 0 \qquad j = 1, \ldots, K. \tag{2b}$$

From Eq. (2), the equilibrated solution can be obtained. Employing the obtained solution, we can express the reversible parameters in terms of the irreversible parameters, i.e., $u_i = u_i(\alpha_j)$. Therefore, we can express the total energy in Eq. (1) as a function of only the irreversible parameters, i.e.,

$$\Pi^*(\alpha_l) = \Pi^{*M}(\alpha_j) + \Pi^D(\alpha_m) \tag{3}$$

where $\Pi^*(\alpha_l) = \Pi(u_i(\alpha_k), \alpha_j)$ and $\Pi^{*M}(\alpha_j) = \Pi^M(u_k(\alpha_m), \alpha_l)$.

The signs of the eigenvalues of the Hessian Matrix $\left[\frac{\partial^2 \Pi^*}{\partial \alpha_i \partial \alpha_j}\right]$ are used to check the stability of the equilibrated solution obtained from Eq. (2). If all the eigenvalues are positive, the equilibrated solution is stable and there is no bifurcation. Otherwise, the solution is unstable and the bifurcation, which leads to the localization, occurs.

3. Smeared Crack Finite Element Analysis for Cracking Localization

The fundamental scheme of the smeared crack models is the decomposition of the total strain increment $\Delta\boldsymbol{\epsilon}$ into a strain increment of the intact solid between the cracks $\Delta\boldsymbol{\epsilon}^o$ and the crack strain increment $\Delta\boldsymbol{\epsilon}^{cr}$, i.e., [1, 13, 14]

$$\Delta\boldsymbol{\epsilon} \;=\; \Delta\boldsymbol{\epsilon}^o + \Delta\boldsymbol{\epsilon}^{cr}. \tag{4}$$

The strain increment vectors in the above equation are in the global coordinate system. It will be helpful to consider the strain increments also in a local coordinate system, which aligns with the crack. Based on the local coordinate system, a local crack strain increment vector in two-dimensional cases is written as

$$\Delta\hat{\boldsymbol{\epsilon}}^{cr} = \left(\Delta\hat{\varepsilon}_{nn}^{cr} \quad \Delta\hat{\gamma}_{nt}^{cr}\right)^T \tag{5}$$

where $\Delta\hat{\varepsilon}_{nn}^{cr}$ and $\Delta\hat{\gamma}_{nt}^{cr}$ are the mode I normal crack strain increment and the mode II shear crack strain increment, respectively. The relationship between the global crack strain increment $\Delta\boldsymbol{\epsilon}^{cr}$ and the local crack strain increment $\Delta\hat{\boldsymbol{\epsilon}}^{cr}$ is written as

$$\Delta\boldsymbol{\epsilon}^{cr} = \mathbf{T}\Delta\hat{\boldsymbol{\epsilon}}^{cr} \tag{6}$$

where $\mathbf{T}$ is the transformation matrix between the global and local coordinate systems defined as

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ \sin^2\theta & \sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{7}$$

where θ is the angle between the normal of the crack and the global x-axis. In the local coordinate system, we consider the local traction increment across the crack, i.e.,

$$\Delta\hat{\mathbf{t}}^{cr} = \left(\Delta\hat{t}_n^{cr} \quad \Delta\hat{t}_t^{cr}\right)^T \tag{8}$$

where $\Delta\hat{t}_n^{cr}$ denotes the mode I normal traction increment and $\Delta\hat{t}_t^{cr}$ denotes the mode II shear traction increment. By using the transformation matrix $\mathbf{T}$, the relationship between the traction increment $\Delta\hat{\mathbf{t}}^{cr}$ and the global stress increment $\Delta\boldsymbol{\sigma}$ is expressed as

$$\Delta\hat{\mathbf{t}}^{cr} = \mathbf{T}^T\Delta\boldsymbol{\sigma}. \tag{9}$$

The constitutive models for the material between the cracks and for the smeared cracks must be specified. For the material between the cracks, we have

$$\Delta\boldsymbol{\sigma} = \mathbf{D}^o\Delta\boldsymbol{\epsilon}^o \tag{10}$$

where $\mathbf{D}^o$ is the constitutive matrix for the material between the cracks. For the cracks, we have the local traction-crack strain relationship, i.e.,

$$\Delta\hat{\mathbf{t}}^{cr} = \hat{\mathbf{D}}^{cr}\Delta\hat{\boldsymbol{\epsilon}}^{cr} \tag{11}$$

where $\hat{\mathbf{D}}^{cr}$ is the crack constitutive matrix incorporating mixed-mode properties of the cracks.

By using Eqs. (4)-(11), the incremental stress-strain relationship for the cracked material is obtained as

$$\Delta\boldsymbol{\sigma} = \left(\mathbf{D}^o - \mathbf{D}^o\mathbf{T}\left[\hat{\mathbf{D}}^{cr} + \mathbf{T}^T\mathbf{D}^o\mathbf{T}\right]^{-1}\mathbf{T}^T\mathbf{D}^o\right)\Delta\boldsymbol{\epsilon}. \tag{12}$$

In order to discuss the cracking localization, we follow the concept of the localization explained in the previous section. To begin with, we consider the total energy increment for the domain of interest V, i.e.,

$$\Delta\Pi = \left[\frac{1}{2}\int_V \Delta\boldsymbol{\epsilon}^{o^T}\Delta\boldsymbol{\sigma}dV - \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dV - \int_S \Delta\mathbf{u}^T\Delta\mathbf{t}dS\right] + \left[\frac{1}{2}\int_V \Delta\hat{\boldsymbol{\epsilon}}^{cr^T}\Delta\hat{\mathbf{t}}^{cr}dV\right] \tag{13}$$

where the first and second pairs of the brackets represent the mechanical potential energy increment and the dissipated energy increment, respectively [10, 12]. Here, $\Delta\mathbf{t}$ and $\Delta\mathbf{f}$ denote the surface traction increment vector and the body force increment vector, respectively. In addition, $\Delta\mathbf{u}$ denotes the total displacement increment vector.

From Eqs. (4), (10), (11), and (13) and the inverse relationship of Eq. (6), i.e.,

$$\Delta\hat{\boldsymbol{\epsilon}}^{cr} = \hat{\mathbf{T}}\Delta\boldsymbol{\epsilon}^{cr} \tag{14}$$

where

$$\hat{\mathbf{T}} = \begin{bmatrix} \cos^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}^T, \tag{15}$$

we obtain

$$\Delta\Pi = \left[\frac{1}{2}\int_V (\Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}^{cr})^T\mathbf{D}^o(\Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}^{cr})^T dV - \int_V \Delta\mathbf{u}^T\Delta\mathbf{f}dV - \int_S \Delta\mathbf{u}^T\Delta\mathbf{t}dS\right] + \left[\frac{1}{2}\int_V \Delta\boldsymbol{\epsilon}^{cr^T}\mathbf{D}^{cr}\Delta\boldsymbol{\epsilon}^{cr}dV\right] \tag{16}$$

in which

$$\mathbf{D}^{cr} = \hat{\mathbf{T}}^T\hat{\mathbf{D}}^{cr}\hat{\mathbf{T}}. \tag{17}$$

Here, we introduce a crack displacement increment vector $\Delta\mathbf{u}^{cr}$ defined as

$$\Delta\mathbf{u} = \Delta\mathbf{u}^o + \Delta\mathbf{u}^{cr} \tag{18}$$

where the strain increments computed from $\Delta\mathbf{u}$, $\Delta\mathbf{u}^o$ and $\Delta\mathbf{u}^{cr}$ are $\Delta\boldsymbol{\epsilon}$, $\Delta\boldsymbol{\epsilon}^o$ and $\Delta\boldsymbol{\epsilon}^{cr}$, respectively.

Consider the i^{th} element in the finite element analysis. The element is assumed to be a cracked element. Interpolate these three displacement increments from nodal quantities, i.e.,

$$\begin{aligned} &\Delta^i\mathbf{u} = \mathbf{N}\Delta^i\mathbf{U}, \quad \Delta^i\mathbf{u}^o = \mathbf{N}\Delta^i\mathbf{U}^o, \\ &\Delta^i\mathbf{u}^{cr} = \mathbf{N}\Delta^i\mathbf{U}^{cr}, \quad \Delta^i\mathbf{U} = \Delta^i\mathbf{U}^o + \Delta^i\mathbf{U}^{cr} \end{aligned} \tag{19}$$

in which $\Delta^i\mathbf{U}$, $\Delta^i\mathbf{U}^o$ and $\Delta^i\mathbf{U}^{cr}$ are the nodal quantities of $\Delta\mathbf{u}$, $\Delta\mathbf{u}^o$ and $\Delta\mathbf{u}^{cr}$, respectively. Here, $\mathbf{N}$ is the shape function matrix. Note that the superscript i for the i^{th} element is used in the equations because the nodal crack displacement increments of the same node for different elements can be different. This is natural because, in the smeared crack approach, cracking in each element is completely

6

independent of each other. Therefore, the continuity of the crack displacement increment between elements is not required and must not be enforced. On the contrary, the total displacement increment $\Delta\mathbf{u}$ must be continuous across elements. Therefore, the superscript i representing the element number is not actually necessary for the nodal values of the total displacement increment. Similar to the crack displacement increment, the displacement increment related to the strain increment of the uncracked solid $\Delta^i\mathbf{u}^o$ is not continuous across elements' boundaries; therefore, the superscript i is required.

Computing strains from Eq. (18), we obtain Eq. (4), i.e.,

$$\Delta^i\boldsymbol{\epsilon} = \Delta^i\boldsymbol{\epsilon}^o + \Delta^i\boldsymbol{\epsilon}^{cr} \tag{20}$$

where

$$\Delta^i\boldsymbol{\epsilon} = \mathbf{B}\Delta^i\mathbf{U}\,, \tag{21a}$$
$$\Delta^i\boldsymbol{\epsilon}^o = \mathbf{B}\Delta^i\mathbf{U}^o\,, \tag{21b}$$
$$\Delta^i\boldsymbol{\epsilon}^{cr} = \mathbf{B}\Delta^i\mathbf{U}^{cr}\,. \tag{21c}$$

Substituting Eq. (21) into Eq. (16) for the i^{th} element gives

$$\begin{aligned}\Delta\Pi = &\frac{1}{2}\Delta\mathbf{U}^T\int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV\Delta\mathbf{U}\\ &-\frac{1}{2}\Delta\mathbf{U}^T\int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV\Delta^i\mathbf{U}^{cr}\\ &-\frac{1}{2}\Delta^i\mathbf{U}^{cr^T}\int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV\Delta\mathbf{U}\\ &+\frac{1}{2}\Delta^i\mathbf{U}^{cr^T}\int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV\Delta^i\mathbf{U}^{cr}\\ &+\frac{1}{2}\Delta^i\mathbf{U}^{cr^T}\int_V\mathbf{B}^T\mathbf{D}^{cr}\mathbf{B}dV\Delta^i\mathbf{U}^{cr}\\ &-\Delta\mathbf{U}^T\int_V\mathbf{N}^T\Delta\mathbf{f}dV-\Delta\mathbf{U}^T\int_S\mathbf{N}^T\Delta\mathbf{t}dS.\end{aligned} \tag{22}$$

Next, we apply the stationary condition $\delta(\Delta\Pi)=0$, and assume that both $\mathbf{D}^o$ and $\mathbf{D}^{cr}$ are symmetric. Since $\delta(\Delta\mathbf{U}^T)$ and $\delta(\Delta^i\mathbf{U}^{cr^T})$ are arbitrary, we obtain the element stiffness equation for the i^{th} element, i.e.,

$$\begin{bmatrix}\int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV & -\int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV\\ -\int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV & \int_V\mathbf{B}^T\mathbf{D}^o\mathbf{B}dV+\int_V\mathbf{B}^T\mathbf{D}^{cr}\mathbf{B}dV\end{bmatrix}\begin{Bmatrix}\Delta\mathbf{U}\\ \Delta^i\mathbf{U}^{cr}\end{Bmatrix} = \begin{Bmatrix}\int_V\mathbf{N}^T\Delta\mathbf{f}dV+\int_S\mathbf{N}^T\Delta\mathbf{t}dS\\ \mathbf{0}\end{Bmatrix}. \tag{23}$$

After assembling all elements and applying prescribed displacements and forces, we arrange the global stiffness equation as

$$\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}\begin{Bmatrix}\Delta\mathbf{U}\\ \Delta\mathbf{U}^{cr}\end{Bmatrix} = \begin{Bmatrix}\Delta\mathbf{R}_1\\ \Delta\mathbf{R}_2\end{Bmatrix}. \tag{24}$$

The static condensation is used to remove the nodal total displacement increment from the obtained global matrix equation. Therefore, the equation can be written in the following form, i.e.,

$$\mathbf{K}^{cr}\Delta\mathbf{U}^{cr} = \Delta\mathbf{R}^{cr}\,. \tag{25}$$

It must be noted that Eq. (25) is a singular equation because $\Delta\mathbf{U}^{cr}$ contains the rigid-body crack displacement increments, i.e., for two-dimensional cases, two rigid translations and one rigid rotation. These three rigid-body crack displacement increments can be found in all cracked elements. To avoid them, constraints to remove them from all elements must be applied to the equation. In this study, the following constraints are employed at the center of each element without loss of generality, i.e.,

$$\begin{aligned}&\Delta u^{cr}(\xi=0,\eta=0)=0\\ &\Delta v^{cr}(\xi=0,\eta=0)=0\\ &\frac{\partial v^{cr}(\xi=0,\eta=0)}{\partial x}=0\end{aligned} \tag{26}$$

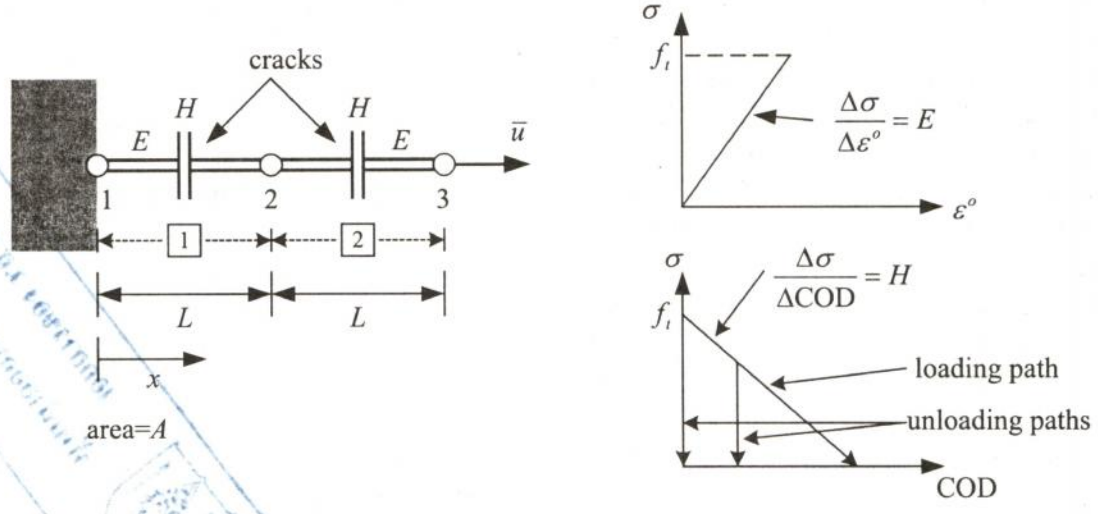

Fig. 1 Uniaxial problem using two 1-D bar elements

where the global $x-y$ and natural $\xi-\eta$ coordinate systems are used in the equation. Here, Δu^{cr} and Δv^{cr} are the incremental crack displacements in *x*- and *y*-directions, respectively.

Eq. (25), after applying the constraints, can be expressed as

$$\hat{\mathbf{K}}^{cr}\Delta\hat{\mathbf{U}}^{cr} = \Delta\hat{\mathbf{R}}^{cr}. \quad (27)$$

The stability condition is obtained by checking the eigenvalues of $\hat{\mathbf{K}}^{cr}$. If all the eigenvalues are positive, it means that the equilibrium path is stable with respect to the current crack pattern and there is no bifurcation. On the contrary, if some of the eigenvalues are negative, the equilibrium path is not stable with respect to the current crack pattern and a stable crack pattern must be found.

4. Results and Discussion

In order to illustrate the advantage of the method in the analysis of the cracking localization, a simple one-dimensional uniaxial problem shown in Fig. 1 is considered. Application of the proposed method to problems in two- and three-dimensional domains is just straightforward. Nevertheless, two- and three-dimensional problems are not used here because illustrative analytical results cannot be easily obtained from them. As shown in Fig. 1, the bar has one fixed support at one end. At the other end, controlled displacement $\overline{u}$ is applied. The length of the bar is $2L$ and the area is A. The material is assumed to be elastic with Young's modulus equal to E. The bar is discretized into two elements, each of which has the length of L. Each element can accommodate one crack. The characteristic length or crack band width of each crack, in this case, is equal to the length of the element. The conventional linear shape function is used for the displacement interpolation.

It is assumed that there is no crack at the beginning. The controlled displacement is then increased until the stress of the bar reaches the tensile strength f_t. By the strength criterion, both elements are cracked. The cracks follow the constitutive law for cracks. For opening cracks, a linear relationship between the transmitted tensile stress and the crack opening displacement (COD) with the slope $\dfrac{\Delta\sigma}{\Delta\mathrm{COD}}$ equal to H is assumed. For unloading cracks, a vertical unloading path with a constant COD equal to the existing COD is applied (see Fig. 1).

Consider an incremental step after the initiation of the cracks. Assembling all element stiffness equations given by Eq. (23), we can write the global stiffness equation. After applying the prescribed boundary conditions, we obtain

$$\frac{A}{L}\begin{bmatrix} 2E & E & -E & -E & E \\ E & E+\widetilde{H} & -(E+\widetilde{H}) & 0 & 0 \\ -E & -(E+\widetilde{H}) & E+\widetilde{H} & 0 & 0 \\ -E & 0 & 0 & E+\widetilde{H} & -(E+\widetilde{H}) \\ E & 0 & 0 & -(E+\widetilde{H}) & E+\widetilde{H} \end{bmatrix}\begin{Bmatrix} \Delta U_2 \\ \Delta^1 U_1^{cr} \\ \Delta^1 U_2^{cr} \\ \Delta^2 U_2^{cr} \\ \Delta^2 U_3^{cr} \end{Bmatrix} = \frac{A}{L}\begin{Bmatrix} E\Delta\bar{u} \\ 0 \\ 0 \\ -E\Delta\bar{u} \\ E\Delta\bar{u} \end{Bmatrix} \tag{28}$$

where $\widetilde{H} = HL$. Here, ΔU_i represents the nodal displacement increment of the node i. Moreover, $\Delta^i U_j^{cr}$ represents the nodal crack displacement increment of the node j and, at the same time, of the element i.

Using the static condensation to remove ΔU_2, we get

$$\frac{A}{L}\begin{bmatrix} \frac{E+2\widetilde{H}}{2} & -\frac{E+2\widetilde{H}}{2} & \frac{E}{2} & -\frac{E}{2} \\ -\frac{E+2\widetilde{H}}{2} & \frac{E+2\widetilde{H}}{2} & -\frac{E}{2} & \frac{E}{2} \\ \frac{E}{2} & -\frac{E}{2} & \frac{E+2\widetilde{H}}{2} & -\frac{E+2\widetilde{H}}{2} \\ -\frac{E}{2} & \frac{E}{2} & -\frac{E+2\widetilde{H}}{2} & \frac{E+2\widetilde{H}}{2} \end{bmatrix}\begin{Bmatrix} \Delta^1 U_1^{cr} \\ \Delta^1 U_2^{cr} \\ \Delta^2 U_2^{cr} \\ \Delta^2 U_3^{cr} \end{Bmatrix} = \frac{A}{L}\begin{Bmatrix} -\frac{E\Delta\bar{u}}{2} \\ \frac{E\Delta\bar{u}}{2} \\ -\frac{E\Delta\bar{u}}{2} \\ \frac{E\Delta\bar{u}}{2} \end{Bmatrix}. \tag{29}$$

The above equation is singular due to the rigid-body crack displacement increments in the two elements. For one-dimensional problems, the crack displacement increment at the center of each element is set to zero, i.e.,

$$\begin{aligned} \Delta^1 u^{cr}(\xi = 0) &= \frac{1}{2}(\Delta^1 U_1^{cr} + \Delta^1 U_2^{cr}) = 0, \\ \Delta^2 u^{cr}(\xi = 0) &= \frac{1}{2}(\Delta^2 U_2^{cr} + \Delta^2 U_3^{cr}) = 0, \end{aligned} \tag{30}$$

which leads to

$$\frac{A}{L}\begin{bmatrix} 2(E+2\widetilde{H}) & 2E \\ 2E & 2(E+2\widetilde{H}) \end{bmatrix}\begin{Bmatrix} \Delta^1 U_1^{cr} \\ \Delta^2 U_1^{cr} \end{Bmatrix} = \frac{A}{L}\begin{Bmatrix} -E\Delta\bar{u} \\ -E\Delta\bar{u} \end{Bmatrix}. \tag{31}$$

Note that, in applying the constraints to Eq. (29), not only the row but also the column operations must be performed to the stiffness matrix so as to obtain the symmetric matrix in Eq. (31). Actually, the constraints may be directly applied to each element before assembling the element stiffness equations.

The eigenvalues of the obtained stiffness matrix are $\frac{4A\widetilde{H}}{L}$ and $\frac{4A(E+\widetilde{H})}{L}$. Both eigenvalues are positive only when $\widetilde{H}$>0. This means that the crack pattern having two cracks opening at the same time is unstable unless hardening behavior occurs at the cracks.

If we assume that the crack in the element 2 undergoes the elastic unloading, the global stiffness equation will contain only one cracked element. Employing the same process of applying the prescribed boundary conditions and using the static condensation for this case, we obtain

$$\frac{A}{L}\left[2(E+2\widetilde{H})\right]\left\{\Delta^1 U_1^{cr}\right\} = \frac{A}{L}\left\{-E\Delta\bar{u}\right\} \tag{32}$$

The eigenvalue of the stiffness matrix is $\frac{2A(E+2\widetilde{H})}{L}$ which is positive when $\widetilde{H} > -\frac{E}{2}$. Assuming that the crack in the element 1 undergoes the elastic unloading will yield the same conclusion.

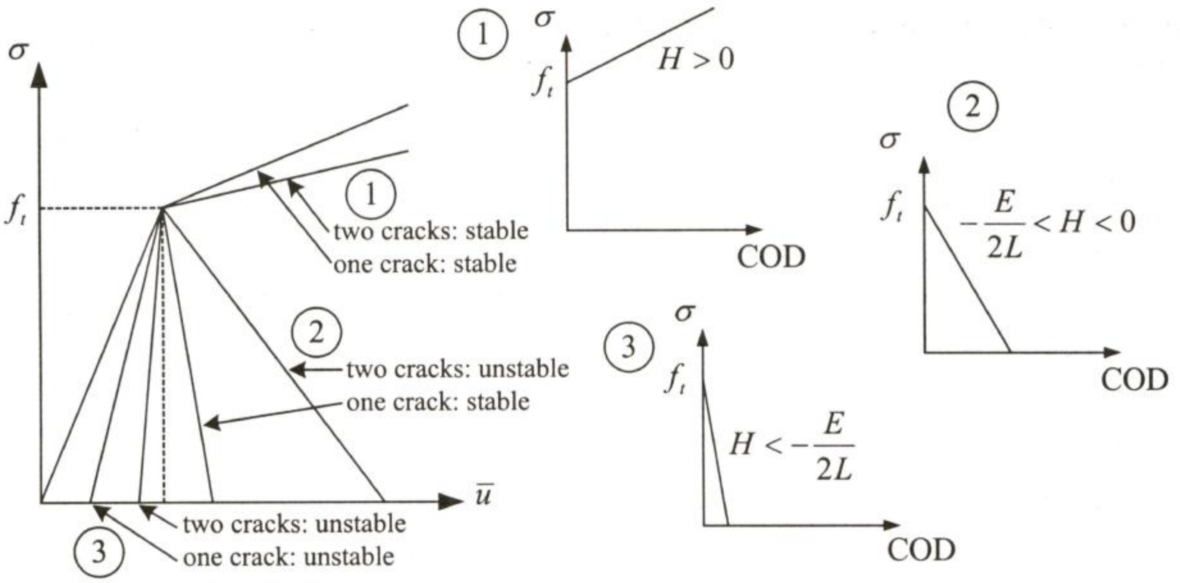

Fig. 2 Responses of the uniaxial problem using two 1-D bar elements

In summary, immediately after the two elements are cracked due to the strength criterion employed, the equilibrium path is unstable and bifurcation occurs unless both cracks exhibit hardening behavior, i.e., when $\widetilde{H}>0$. In reality, cracks will exhibit softening behavior. Therefore, the two cracks cannot continue to open at the same time. If one of the cracks undergoes the elastic unloading, the stable equilibrium path can be observed as long as $\widetilde{H} > -\frac{E}{2}$. As shown in Fig. 2, which summarizes all the possible results, the cases where $\widetilde{H} < -\frac{E}{2}$ represent the responses with snapback behavior. Under the displacement control, the snapback responses are always unstable.

It can be seen from the results that the proposed method allows the consideration of the cracking localization to be done even when the smeared crack approach is used. As mentioned before, there is no intention to use the stiffness equation obtained from the proposed method in the analysis to obtain the unknown displacements. For that purpose, the original smeared crack approach is much more appropriate and will be used. The proposed scheme is used only for the investigation of the stability of the crack patterns.

The reason that the smeared crack approach is selected for the analysis of the localization is that the analysis of this kind involves problems with many cracks. To permit the investigation of the stability of the crack patterns with the smeared crack models, the discrete irreversible variables are introduced to the models. As it is seen from the derivation and the results, the introduced irreversible variables, which are the crack displacement variables, allow the energy of the system to be expressed as a function of the discrete irreversible variables. Therefore, the stationary condition of the energy with respect to these discrete irreversible variables can be done easily. Consequently, we can use the smeared crack models both for computing the unknown displacements and for checking the stability of the crack patterns.

Nevertheless, there are still many more problems, related to the analysis of the cracking localization, to be solved. For complex localization problems, such as the four-point bending problem, there are chances that there will be many stable crack patterns occurring at the same time. Because of the complicated crack patterns, it is not easy to single out the correct solutions. The complete and efficient analysis methods still have to be developed.

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References

[1] J. G. Rots and R. de Borst, "Analysis of Mixed-Mode Fracture in Concrete", Journal of Engineering Mechanics, Vol. 113, No. 11, 1987, pp. 1739-1758.

[2] J. G. Rots, "Stress Rotation and Stress Locking in Smeared Analysis of Separation", Fracture Toughness and Fracture Energy: Test Methods for Concrete and Rock (Eds: H. Mihashi, H. Takahashi and F. Wittmann), Rotterdam, Balkema, 1989, pp. 367-382.

[3] E. N. Dvorkin and A. P. Assanelli, "2D Finite Elements with Displacement Interpolated Embedded Localization Lines: The Analysis of Fracture in Frictional Materials", Computer Methods in Applied Mechanics and Engineering, Vol. 90, 1991, pp. 829-844.

[4] M. Jirasek and T. Zimmermann, "Rotating Crack Model with Transition to Scalar Damage", Journal of Engineering Mechanics, 1998, Vol. 124, No. 3, pp. 277-284.

[5] N. Shirai, "JCI Round Robin Analysis of Size Effect in Concrete Structures", Size Effect in Concrete Structures (Eds: H. Mihashi, H. Okamura and Z. P. Bazant), London, E&FN Spon, 1994, pp. 295-322.

[6] A. Carpinteri, "Softening and Snap-Back Instability in Cohesive Solids" International Journal for Numerical Methods in Engineering, 1989, Vol. 28, pp. 1521-1537.

[7] E. Riks, "An Incremental Approach to the Solution of Snapping and Buckling Problems", International Journal of Solids Structures, 1979, Vol. 15, pp. 529-551.

[8] R. de Borst, "Computation of Post-Bifurcation and Post-Failure Behavior of Strain-Softening Solids", Computers and Structures, 1987, Vol. 25, No. 2, pp. 211-224.

[9] S. Valente, "Bifurcation Phenomena in Cohesive Crack Propagation", Computers and Structures, 1992, Vol. 44, No. 1/2, pp. 55-62.

[10] Q. S. Nguyen, "Bifurcation and Post-Bifurcation Analysis in Plasticity and Brittle Fracture", Journal of the Mechanics and Physics of Solids, 1987, Vol. 35, No. 3, pp. 303-324.

[11] H. Horii, "Micromechanics and Localization in Concrete and Rock", Fracture and Damage of Concrete and Rock (Ed: H. P. Rossmanith), London, E&FN Spon, 1993, pp. 54-65.

[12] M. Brocca, Analysis of Cracking Localization and Crack Growth Based on Thermomechanical Theory of Localization (Master's Degree Thesis), Tokyo, University of Tokyo, 1997.

[13] Z. P. Bazant and B. H. Oh, "Crack Band Theory for Fracture of Concrete", Materiaux et Constructions, 1983, Vol. 16, No. 93, pp. 155-177.

[14] S. P. Shah, S. E. Swartz and C. Ouyang, Fracture Mechanics of Concrete: Applications of Fracture Mechanics to Concrete, Rock, and Other Quasi-Brittle Materials, New York, John Wiley & Sons, 1995.