THE APPLICATION OF FUZZY NUMBERS IN CAPITAL BUDGETING PROBLEMS

Somrote Komolavanij
Department of Industrial Engineering
Sirindhron International Institute of Technology, Thammasat University

ABSTRACT

Most cash flow analysis problems deal with uncertain data. Practically, if enough information is obtained, cash flow analysis can be performed with the help of probabilistic theory. Another alternative when the uncertainty of data presents is the utility theory. Unfortunately, it is often impossible to obtain enough information to set up a good probability distribution or good utility function. For ease of computation, the uncertainty of the data is usually ignored and a deterministic solution is found. It is believed that the resulting analysis may be more useful and accurate if fuzzy data is properly analyzed through the use of fuzzy set theory techniques. By using fuzzy set theory techniques, less information is required to set up the distribution of the uncertainty information. In this study, some uncertain components in the total cash flow model were assumed to be fuzzy numbers. Finally, mutually exclusive project's selection by using fuzzy NPV is proposed.

Keywords Fuzzy numbers, Net present value, Capital budgeting.

INTRODUCTION

In early 1960's, Zadeh (1965) introduced the theory of fuzzy sets to deal with imprecise, vague, or uncertain situations. Since then, fuzzy set theory has been applied in many areas of study such as engineering, science, and economics. In industrial engineering, fuzzy sets theory has also had many applications such as Engineering Economics, Operations Research, Human Factors, Inventory Control, etc, The advantage of fuzzy sets theory is that it can deal with the problem of an uncertain situations that are ever present in the real world. Exact decisions are often required even though only uncertain data is available for analysis. From previous studies in cash flow analysis with fuzzy set theory, Chiu and Park (1994) proposed models where cash flow and discount rate for each year were specified as triangular fuzzy numbers (TFN). The approximated form of the present worth fuzzy function was suggested and a method for comparing mutually exclusive fuzzy projects by using weighted method was also included. Leung (1980) studied project selection with fuzzy procedures. he constructed a criterion function which had components of worth, cost and risk in the function and he assumed those three components to be linguistic variables. Values of the criterion function imply which alternative project was the most suitable for all required specifications. Ward studied fuzzy discounted cash flow analysis by computing deterministic (crisp) values for cash flows and then transforming the results into symmetrical triangular fuzzy numbers (TFN). He also defined a fuzzy present worth and fuzzy internal rate of return of the fuzzy cash flows. Buckley (1987) studied fuzzy present value and fuzzy future value when assuming a cash amount, interest rate, and period of time were assumed to be fuzzy. He also discussed the methods of comparing fuzzy net cash flows in order to rank fuzzy investment alternatives from the best to the worst. Actually, cash flows are fuzzy because of the fuzzy nature of the components in the model. Therefore, this study will extend the previous work of Chiu and Park by using total cash flow models proposed by Stevens (1992) and assuming some components be triangular fuzzy numbrs (TFN) with a linear relationship. However, if we assume all components in the cash flow model to be fuzzy variables, a result will be too fuzzy and useless. Therefore, only the critical components will be assumed to be fuzzy numbers.

The organization of this study will be as follows. Firstly, the introduction of fuzzy number and the deterministic total cash flow model are presented as a basic idea of initial calculation of cash flows. Secondly, the fuzzy components in the model are discussed. By applying triangular fuzzy numbers (TFN) to all vague components, the deterministic model has to be rearranged and the calculation of cash flows and NPV by some fuzzy data is introduced. Finally, mutually exclusive project evaluation by fuzzy weight method is proposed.

A FUZZY NUMBER

The fuzzy number is actually an uncertain number. An uncertainty of a number occurs when a real number is combined to a linguistic variable. The linguistic variables are such as "approximately," "around," or "much larger than," etc. Someone, for example, may say the budget of the investment cost of the new project is "around 5.0 million dollars." This "around 5.0 million dollars" term is a fuzzy number. Therefore, the boundary of this set may be between 4.5 to 5.0 million dollars. One of the fuzzy set theory properties is the membership value of an element. The membership value will be an index to indentify whether an element is in the fuzzy set or not. The membership values typically range between 0 to 1. The closer the membership value is to one, the more the element is in the set. Suppose there are three alternatives for a new project. The costs of investment of alternatives A, B, C are 4.9, 5.5 and 6.0 million dollars, respectively. The membership values for being in the set of "around 5.0 million dollars" may be 1.0 for project A, 0.2 for project B and 0 for project C which mean investment of A is totally in the set of "around 5.0 million dollar", investment of B is partially in the set and investment of C is totally not in the set.

Actually, there are many types of fuzzy numbers, depending on types of membership functions as discussed in detail by Kandel (1988). The triangular fuzzy number (TFN) is the simplest form of the fuzzy numbers and will be used in this study. Figure 1 shows a TFN.

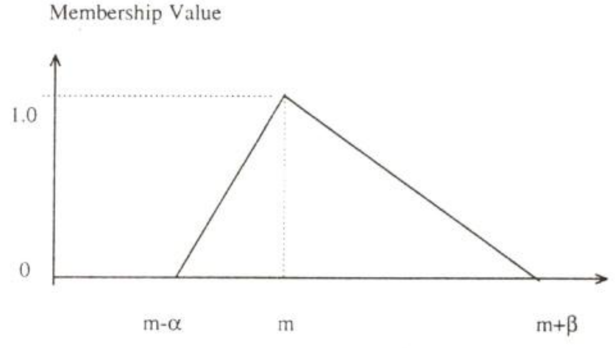

Figure 1 A triangular fuzzy number (TFN)

There are many ways to represent a fuzzy number. In this study, only two forms of a fuzzy number would be suggested.

First form : $(m-\alpha, m, m+\beta)$

By the fisrt form, the fuzzy number will be represented by three numbers. For example, let **A** be a TFN and **A** = (400, 500, 550). These three numbers show the minimum value, the most possible value, and the maximum value of the fuzzy number, respectively. By writing a fuzzy number in this format, the interval of the fuzzy number **A** will easily be investigated.

Second form : $[\alpha, m, \beta]$

A fuzzy number is also represented by another three numbers in the second form, but in different meaning. For example, let **B** be a fuzzy number, and **B** = [100, 500, 50]. This form will show the value deviating from the most possible value of **B** (m) on the left hand-side (α), and right hand-side (β). This form of fuzzy number is more suitable when performing operations between fuzzy numbers.

A real number

Any real number could be considered as a fuzzy number. It is the fuzzy number with zero deviation from the most possible value. For example, if tax amount (T) is assumed to be a fixed rate at 40%, T will be equal to (0.40, 0.40, 0.40) or [0, 0.40, 0]. By this assumption, real numbers and fuzzy numbers can perform any operation together.

OPERATIONS OF TRIANGULAR FUZZY NUMBERS

Based on Bonissone, the basic operations of the fuzzy number are summarized in Table 1. The operations of TFN will be applied during the cash flows calculation.

Assume **M**, and **N** are TFN and

$$\mathbf{M} = [\alpha, m, \beta]$$
$$\mathbf{N} = [\gamma, n, \sigma].$$

Table 1 Summary of triangular fuzzy number operations

Operations		Conditions
Addition	$\mathbf{M+N} = [\alpha+\gamma,\ m+n,\ \beta+\sigma]$	For all m, and n.
Subtraction	$\mathbf{M-N} = [\alpha+\sigma,\ m-n,\ \beta+\gamma]$	For all m, and n.
Multiplication	$\mathbf{M*N} = [m\gamma+n\sigma,\ mn,\ m\sigma+n\beta]$	m>0, n>0
	$\mathbf{M*N} = [n\alpha-m\sigma,\ nm,\ n\beta-m\gamma]$	m<0, n>0
	$\mathbf{M*N} = [-n\beta-m\sigma,\ nm,\ -n\alpha-m\gamma]$	m<0, n<0
Division	$\mathbf{M/N} = [(m\alpha+n\sigma)/n^2,\ m/n,\ (m\gamma+n\beta)/n^2]$	m>0, n>0
	$\mathbf{M/N} = [(n\alpha-m\gamma)/n^2,\ m/n,\ (n\beta-m\sigma)/n^2]$	m<0, n>0
	$\mathbf{M/N} = [(-n\beta+m\gamma)/n^2,\ m/n,\ (-n\alpha+m\sigma)/n^2]$	m<0, n<0

TOTAL CASH FLOW MODEL

Cash flows can be obtained by many models. The common one is total cash flow model as the following equations [Stevens, 1992].

$$X_j = (G_j - C_j) - (G_j - C_j - D_j)T - K_j + L_j \pm W_j + V_j \quad ....(1)$$

$$NPV = \sum_{j=0}^{n} \frac{X_j}{(1+i)^j} \quad ....(2)$$

where,

X_j = Net total cash flow at the end of year j
G_j = Total gross income at the end of year j
C_j = The total operating cost of year j
W_j = Net increase or decrease in working capital in year j
V_j = Investment tax credit in year j
D_j = Tax depreciation amount in year j
K_j = Total capital expenditure in year j
T = Tax rate (40%)
L_j = Salvage value received in year j
i = Discount rate (MARR)
NPV = Net present value

FUZZY COMPONENTS IN THE TOTAL CASH FLOW MODEL

With uncertain data, some components in the total cash flow model should be assumed to be fuzzy. In this section, the fuzziness of components are discussed.

TOTAL ANNUAL COST (C_j)

Total annual cost (C_j) should always be assumed as a fuzzy quantity because of the nature of the variable. It is very hard to obtain exact values of cost of a project before it is done. Therefore, it is very sensible that this variable should be fuzzy. The total cost (C_j) can be assumed to be either a symmetric or non-symmetric triangular fuzzy number, based upon estimated values by experts. A non-symmetric triangular might be more realistic because a cost of a project usually increases rather decreases over time. The symmetric and non-symmetric triangular fuzzy number for the total cost (C_j) is depicted in Figures 2 and 3.

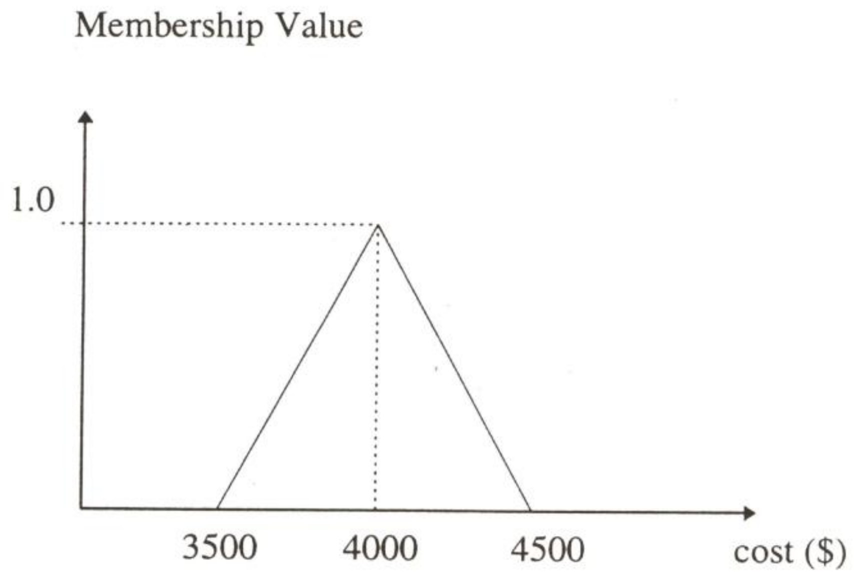

Figure 2 A symmetric triangular fuzzy of total cost (C_j) at (500, 4000, 500)

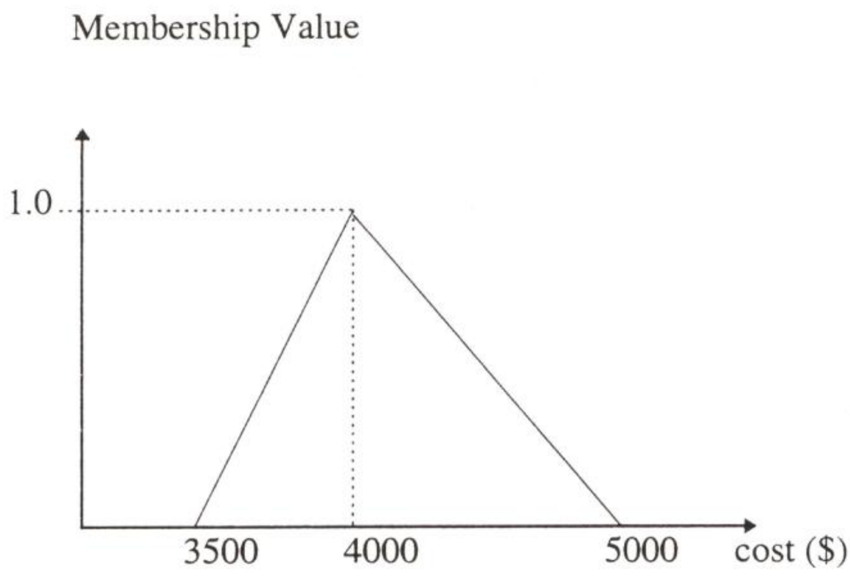

Figure 3 A non-symmetric triangular fuzzy total cost (C_j) at (500, 4000, 1000)

TOTAL GROSS INCOME (G_j)

For similar reasons to those explained for total annual cost, total gross income of a project is hard to be represented by a deterministic number. Therefore, it should be assumed to be a fuzzy number.

TOTAL CAPITAL EXPENDITURE (K_j)

Total capital expenditure can be either deterministic or fuzzy quantity depending upon the information available at the time of study for a particular project. The capital expenditure can be a deterministic quantity, if and only if a firm quote for the particular project is proposed. Otherwise, it should be assumed as a fuzzy quantity.

TAX DEPRECIATION AMOUNT (D_j)

Depreciation, in this study, is calculated by straight-line method and the salvage value is assumed to be zero. As known, the depreciation is usually dependent to capital expenditure. Therefore, If capital expenditure (K_j) is a fuzzy quantity, depreciation should be also fuzzy.

After we know which components should be fuzzy number ($a-\alpha$, a, $a+\beta$), experts or financial analysts can give values of a, α, and β for K_j, G_j, C_j, and D_j by giving the most likely possible values of those parameters (a- value) and intervals of fuzziness (α, β- values).

For ease of calculation, some the assumptions should be stated for equation (1) as follows.

1. W_j, V_j, and L_j equal to zero.
2. The incremental tax rate (T), interest rate (i), and project's life are real number.
3. G_j, C_j, D_j, and K_j are fuzzy numbers.

Thus, the total cash flow model can be rewritten as Equation (3). Each term in equation (3) is independent.

$$X_j = (1-T)G_j-(1-T)C_j + D_jT-K_j \quad .....(3)$$

With (3) and the operations of fuzzy numbers shown in Table 1, the value of cash flow for each year can easily be obtained. Finally, the fuzzy Net Present Value can be calculated by (4), (5).

$$PX_j = (X_j)/(1+i)^j \quad .....(4)$$

where,

PX_j = A present value of a cash flow (X_j)

i = Discount rate ($MARR_{total}$)

j = Life of a project

Now, we have to combine the PX_j terms together by

$$NPV = \sum_{j=0}^{n} P X_j \quad .....(5)$$

wehre,

NPV = The net present value of a project

Again, by using fuzzy operations in Table 1 and Equations (3), (4) (5), fuzzy cash flows, fuzzy equivalence values at present time and fuzzy Net Present Value of the project can be obtained. In fact, after two fuzzy numbers perform multiplication or division operations, the result is a fuzzy number with a non-linear relation. The study of Chiu and Park (1994), Kaufmann and Gupta (1988) show that the deviation between an exact (non-linear relation) NPV and approximated linear NPV is not significant and can be neglected. Therefore, NPV is always an approximated net present value of the project.

In a real world problem, a financial expert might give different percentages of fuzziness of the components. For example, A pessimistic financial expert might believed that the annual cost of a project tends to increase rather than decrease, so the expert might recommend that annual cost (C_j) of the project should have 5% and 10% of fuzziness of β, and α, respectively, and the annual gross income (G_j) of this project should have 10% and 5% of fuzziness for α, and β, respectively.

COMPARISON OF MUTUALLY EXCLUSIVE PROJECTS

One of the most common questions in a capital budgeting problem is that management have to choose the most preferable investment among a set of mutually exclusive projects. For the deterministic model, the management usually select the project with the largest NPV. However, the result of the fuzzy total cash flow model is the fuzzy NPV. This can be done by comparing a fuzzy net present values of the project and finding the most desirable one. There are many methods proposed for comparing fuzzy net present values and ranking projects. A summary of those methods are shown in the study of Chiu and Park (1994). Some methods need to be solved graphically while some methods may give an inaccuracy result. Therefore, the Fuzzy weighted method is proposed in this section which can get an effective result and can be done easily.

FUZZY WEIGHTED METHOD

A fuzzy number can be represented by three numbers ($m-\alpha$, m, $m+\beta$) which are the worst possible value fuzzy NPV of a project ($m-\alpha$), the most likely possible value (m), and the best possible value ($m+\beta$). The most likely possible value had the highest possibility of occurrence, while other values within a range $m-\alpha$ to $m+\beta$ have less possibility to occur. Fuzzy weight method uses this fact to set up the index to compare fuzzy NPV of projects. The value of the index is the summation of two terms, most possible value (m) and average value of the best and the worst possible value, with different weight as shown in Equation (8).

$$Index = w_1 [(m-\alpha) + (m+\beta)] /2 + w_2 (m). \quad .....(8)$$

Since m is believed to be the most possible value of NPV of the project, its weight (w_2) is assumed to be one. For another term which is less important, Chiu & Park suggested that the weight of this term should be between 0.1 to 0.3. However, the exact value of the weight of term $[(m-\alpha) + (m+\beta)] /2$ is uncertain. For the fuzzy weighted method, the weight is assumed to be fuzzy information with range between 0 to 1, and the most likely possible value is 0.5. For that reason, w_1 can be represented by $w_1 = (0.5, 1.0, 0.5)$. Equation (8) becomes

$$Index = \text{Most possible value } [w_1[m-\alpha)+(m+\beta)]/2+(m)]]. \quad .....(9)$$

The best alternative is the one with the highest index. The next example will show how to range four mutually exclusive projects and show the results of other ranking methods.

EXAMPLE comparison of Four Mutually Exclusive Projects

For comparison proposes, an example proposed by Chiu & Park (pp. 131-133) is used here. Suppose we have a set of four mutually exclusive projects with fuzzy information. The Net Present Value (NPV) of each of these project are as A = (2350, 2725, 2850), B = (2250, 2650, 2800), C = (2325, 2600, 2900), D = (2200, 2425, 2725).

With this data, the NPV of the four alternatives can be depicted in Figure 4. It is quite hard to conclude which project is the best for investment because there are many areas of overlapping of the fuzzy NPV. Therefore, we use the fuzzy weighted method, as defined by Equation (9), to determine which alternative is the best. The values of the indexes and comparison of the results by different methods are summarized in Table 2.

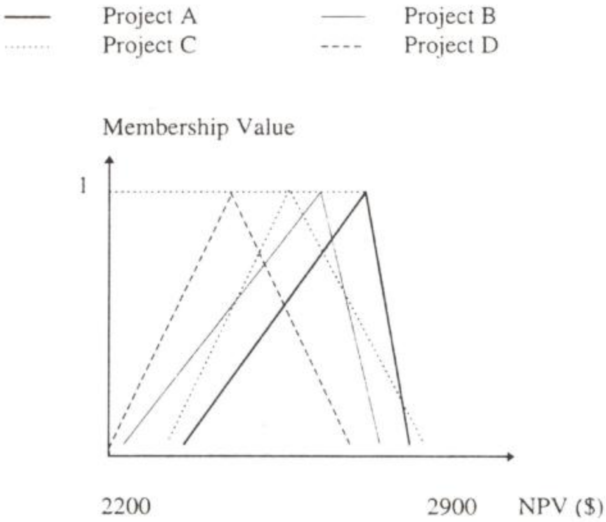

Figure 4 Fuzzy NPV of project A, B, C, D

Table 2 Comparing of fuzzy mutually exclusive projects by different methods

Method	**A**	**B**	**C**	**D**	**Result**
Chang	660417	705833	749896	643125	C > B > A > D
C&P	2914	2832	2868	2693	A > C > B > D
K&G	2663	2588	2606	2444	A > C > B > D
Jain	.79	.70	.70	0	A > D ≅ C > D
D&P	1	.85	.81	.55	A > B > C > D
Fuzzy weighted	4025	3912.5	3906.25	3656.25	A > B > C > D

The result shows that A is the best alternative and D is the worst, except when using Chang's method. The reason is because the Chang's method gives higher weight on a difference between maximium and minimium values. Therefore, an alternative with a large value of that width will have a large index and might be the best project as shown in this example which is not true. Different results are shown for alternatives B and C. Kaufmann and Gupta's method (K&D) and Chui and Park's method (C&P) conclude that C is better that B, and D&P and fuzzy weight method conclude B is better that C while the Jain method concludes the two alternatives are tied. There are reasons to believe the conclusion of D&P and fuzzy weight method is superior. First, the most possible value of B is greater than that of C and second between 90 - 100% of possibility of being a NPV, B always gives the better value of NPV.

CONCLUSION

A capital budgeting analysis usually has a problem in uncertainty of data. To cope with this problem, the fuzzy sets theory is recommended by representing triangular fuzzy number (TFN) for vague data. In this study, the deterministic total cash flow model is used as a basis model. Parameters, in the model, which usually are uncertain, are initial investment amount (K_j), total annual gross income (G_j), total annual cost (C_j), and depreciation amount (D_j). By letting financial experts estimate the values of those parameters, the triangular fuzzy data can be obtained. However, the initial investment amount (K_j) can be replaced by a deterministic number only if we have a firm quote from bidding. If K_j is a deterministic number, depreciation amount will also be deterministic, since the depreciation amount (D_j) is always depend upon the initial investment amount. Total annual gross income and total annual cost should always be represented by fuzzy numbers due to the difficulty of obtaining accurate estimations for these parameters. The fuzzy total cash flow model is represented after we know which parameters should be fuzzy. With the help of operations of fuzzy numbers and total cash flow model, annual cash flows of a project can be obtained. Also, through the use of fuzzy number operation, the cash flows can be converted to a net present value (NPV).

A mutually exclusive projects evaluation are also included in this study. For a mutually exclusive projects' evaluation, the Fuzzy weighted method has been proposed. Since NPV becomes fuzzy, the most possible value of NPV alone is not a good indicator for mutually exclusive projects' evaluation. Therefore, two indicators: the most possible value of NPV and an average between the worst and the best possible value of NPV will be considered as an index. The two indicators are multiplied by fuzzy weighted factor because an exact weight is unknown. By a fuzzy weighted method, the fuzzy indexes for each project is obtained. The project with the highest fuzzy index value is considered to be the best alternative. By this way, mutually exclusive project can be ranked from the best to the worst. Other evaluation indexes proposed by previous works are also shown in this research for comparison.

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