Regular Elements on a Semigroup of the Pairs of Terms and Relational Terms of Type $(m;n)$

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***Abstract*—The concept of generalized hypersubstitution for algebraic systems was introduced by D. Phusanga and J. Joomwong which was extended from concept of generalized hypersubstitution for universal algebras. The concepts of the idempotent and regular elements are important role in semigroup theory. In this paper, we characterize the idempotent and regular elements of the set of all generalized hypersubstitutions for algebraic systems of type $(m;n)$ and give some structural properties.**

***Index Terms*—Algebraic System, Term, Relational Term, Generalized Hypersubstitution.**

I. Introduction

An *algebraic system* of type (τ,τ') is a triple $A:=(A;(f_i^A)_{i\in I},(\gamma_j^A)_{j\in J})$ consisting of a non-empty set A, a sequence $(f_i^A)_{i\in I}$ of operations on A indexed by the index set I where $f_i^A: A^{n_i} \to A$ is n_i - ary for $i\in I$ and a sequence $(\gamma_j^A)_{j\in J}$ of relations on A indexed by the index set J where $\gamma_j^A \subseteq A^{n_j}$ is n_j-ary relation for $j\in J$. The pair (τ,τ') with $\tau = (n_i)_{i\in I}$, $\tau' = (n_j)_{j\in J}$ of sequences of integers $n_i , n_j \in N^+$: $\mathrm{N}\setminus\{0\}$, is called the type of the algebraic system $\underline{A}$ (see [1]). The algebraic systems are related to the concepts of terms and formulas. To define terms [2] and formulas [3], [4] we need variables, operation symbols, logical connectives, and relational symbols as follows.

Let $1 \le n\in N^+$, let $X_n = \{x_1,x_2,...,x_n\}$ be a finite set of variables, and let $X:= \bigcup_{1\le n} Xn = \{x_1,...,x_n,...\}$ be countably infinite. Then the set $W_\tau(X_n)$ of all n-ary terms of type τ is defined in the usual way by the following conditions:

(i) Every $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, ..., t_{ni}$ are n -ary terms of type τ and if f_i is an n_i-ary operation symbol of type τ, then $f_i(t_1, ..., t_{ni})$ is an n-ary term of type τ.

Let $W_\tau(X) := \bigcup_{n\ge1} W_\tau(X_n)$ be the set of all terms of type τ. Let $1 \le n\in N^+$. An *n-ary formula* of type (τ,τ') is defined in the following inductive way:

(i) If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary formula of type (τ,τ'). All variables in $t_1 \approx t_2$ are free.

(ii) If $t_1, ..., t_{nj}$ are n-ary terms of type τ and if γ_j is an n_j-ary relational symbol, then $\gamma_j(t_1, ..., t_{nj})$ is an n-ary formula of type (τ,τ'). All variables in such a formula are free.

(iii) If F is an n-ary formula of type (τ,τ'), then $\neg F$ is an n-ary formula of type (τ,τ'). All free variables in F are also free in $\neg F$. All bound variables in F are also bound in $\neg F$.

(iv) If F_1 and F_2 are n-ary formulas of type (τ,τ') such that variables occurring simultaneously in both formulas are free in each of them, then $F_1 \vee F_2$ is an n-ary formula of type (τ,τ'). Variables that are free in at least one of the formulas F_1 or F_2 are also free in $F_1 \vee F_2$. Variables that are bound in either F_1 or F_2 are also bound in $F_1 \vee F_2$.

(v) If F is an n-ary formula of type (τ,τ') and $x_i \in X_n$ occurs freely in F, then $\exists x_i\,(F)$ is an n-ary formula of type (τ,τ'). The variable x_i is bound in the formula $\exists x_i(F)$ and all other free or bound variables in F are of the same nature in $\exists x_i(F)$.

Let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary formulas of type (τ,τ') and let $\mathcal{F}(\tau,\tau')(W_\tau(X)):= \bigcup_{n\ge1} \mathcal{F}(\tau,\tau')(W_\tau(X_n))$ be the set of all formulas of type (τ,τ'). A formula having the form $\gamma_j(t_1, ..., t_{nj})$, we will call a relational term of type (τ,τ'). Let $\gamma\mathcal{F}(\tau,\tau')(W_\tau(X))$ be the set of all relational terms of type (τ,τ') [1].

First, we recall the concept of a generalized superposition of terms [5]. Let $n \in N^+$. The operation S^n: $W_\tau(X)\times(W_\tau(X))^n \to W_\tau(X)$ is defined by the following steps:

(i) If $t = x_i; 1 \leq i \leq n$, then $S^n(x_i, t_1, ..., t_n) := t_i$.

(ii) If $t = x_i$; $n < i \in N^+$, then $S^n(x_i, t_1, ..., t_n) := x_i$.

(iii) If $t = f_i(s_1, ..., s_{ni})$, then $S^n(f_i(s_1, ..., s_{ni}), t_1, ..., t_n) :=$ $f_i(S^n(s_1, t_1, ..., tn), ..., Sn(sni, t1, ..., tn))$,

supposed that $S^n(s_k, t_1, ..., t_n)$ are already defined for $1 \leq k \leq ni$.

Next, we recall the concept of the operation R^n which extends this generalized superposition to relational terms [1] as follows.

Definition 1.1 [1] Let $n \in N^+$. The operation

$$R^n: W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) \times (W_\tau(X))^n \rightarrow W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

are defined by the following inductive steps:

Let $t_1, ..., t_n \in W\tau(X)$ and S^n be the generalized superposition of terms.

(i) If $t \in W_\tau(X)$, then we defined $R^n(t, t_1, ..., t_n) := S^n(t, t_1, ..., t_n)$.

(ii) If $j \in J$ and $s_1, ..., s_{nj} \in W_\tau(X)$, then $R^n(\gamma_j(s_1, ..., s_{nj}), t_1, ..., t_n) :=$ $\gamma_j(S^n(s_1, t_1, ..., t^n), ..., S^n(s^{nj}, t_1, ..., t_n))$.

In [6], the properties of Rn are proved that satisfied Theorem 1.2 (FC1) and (FC2) in the following theorem.

Theorem 1.2 [6] Let $\beta \in W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$. The operation R^n satisfies:

(FC1): $R^n(R^p(\beta, t_1, ..., t_p), s_1, ..., s_n) = R^p(\beta, R^n(t_1, s_1, ..., s_n), ..., R^n(t_p, s_1, ..., s_n))$, whenever $t_1, ..., t_p$ and $s_1, ..., s_n \in W_\tau(X)$.

(FC3): $R^n(\beta, x_1, ..., x_n) = \beta$.

On the other hand, the concept of a generalized hyper substitution of a given type τ for universal algebra was the first introduced [5]. The set $Hyp_G(\tau)$ of all generalized hypersubstitutions of type τ together with a binary operation '$\circ_G$' is a monoid. (it's mean that $(Hyp_G(\tau); \circ_G)$ is a monoid.) In [7] defined a new binary operation '$+_G$' on the set $Hyp_G(\tau)$ by $(\sigma_1 +_G \sigma_2)(f_i) := S^{ni}(\sigma_2(f_i), \underbrace{\sigma_1(f_i), ..., \sigma_1(f_i)}_{n_i-terms}) \in W_\tau(X)$.

Further, $(Hyp_G(\tau); +_G)$ forms a semigroup. In the present paper, we will define a new concept that generalized the idea in [7] of a hyper substitution (for universal algebras) of type τ to algebraic systems of type (τ,τ').

II. Methodology

Now, we will define a generalized hypersubstitution for algebraic systems of type $(m;n)$.

Definition 2.1 Any mapping

$$\sigma_{t,F}: \{f\} \cup \{\gamma\} \rightarrow W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

which maps m-ary operation symbols to terms does not necessarily preserve the arity and n-ary relational symbols to relational terms does not necessarily preserve the arity is called a *generalized hypersubstitution for algebraic systems of type* $(m;n)$.

Any generalized hyper substitution for algebraic systems of type $(m;n)$ consists of an operational part $\sigma_{t,F}(f) := \sigma_t:\{f\} \rightarrow W_\tau(X)$ which is a usual hypersubstitution and a relational part $\sigma_{t,F}(\gamma) := \sigma_F: \{\gamma\} \rightarrow \gamma\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$. We denote the set of all generalized hyper substitutions for algebraic systems by $HypG(m;n)$. We define the extension of $\sigma_{t,F} \in Hyp_G(m;n)$ as follows:

$$\hat{\sigma}_{t,F}: W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) \rightarrow W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

inductively defined as follows:

(i) $\hat{\sigma}_{t,F}[x] := x$ for any variable $x \in X$,

(ii) $\hat{\sigma}_{t,F}[f_i(s_1, ..., s_{ni})] :=$ $S^{ni}(\sigma_t(fi), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_{ni}])$ for $i \in I$ and $s1, ..., sni \in W\tau(X)$,

(iii) $\hat{\sigma}_{t,F}[\gamma_j(s_1, ..., s_{nj})] :=$ $R^{nj}(\sigma_F(\gamma_j), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_{nj}])$ for $j \in J$ and $s_1, ..., s_{nj} \in W_\tau(X)$.

Then $\hat{\sigma}_{t,F}$ is called the extension of a generalized hypersubstitution for algebraic system $\sigma_{t,F}$. And defined a binary operation '$\circ_G$' on $Hyp_G(\tau,\tau')$ by $\sigma_{t,F} \circ_G \sigma'_{t',F'} := \hat{\sigma}_{t,F} \circ \sigma'_{t',F'}$ where '$\circ$' denotes the usual composition of mapping and $\sigma_{t,F}, \sigma'_{t',F'} \in Hyp_G(\tau,\tau')$. In [6], an important property for the extension is proved. Let σid be the generalized hyper substitution for algebraic systems mapping the operation symbols f_i to the terms $f_i(x_1, ..., x_{ni})$ for all $i \in I$, and the relational symbols γ_j to the relational term $\gamma_j(x_1, ..., x_{nj})$ for all $j \in J$. We have the proposition as follows:

Proposition 2.2 [6] Let $\sigma_{t,F} \in Hyp_G(\tau,\tau')$ and $n \in N^+$ and $t_1, ..., t_n \in W_\tau(X)$. Then

$\hat{\sigma}_{t,F}[R^n(\beta, t_1, ..., t_n)] =$
$R^n(\hat{\sigma}_{t,F}[\beta], \hat{\sigma}_t[t_1], ..., \hat{\sigma}_t[t_n])$ for any
$\beta \in W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$.

Lemma 2.3 [6] For any $\sigma_{t,F}, \sigma'_{t',F'} \in Hyp_G(\tau,\tau')$, we have $(\sigma_{t,F} \circ_G \sigma'_{t',F'})\hat{} = \hat{\sigma}_{t,F} \circ_G \hat{\sigma}'_{t',F'}$.

Lemma 2.4 [6] For any term $t \in W_\tau(X)$ and formula $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$, we have, $\hat{\sigma}_{id}[t] = t$ and $\hat{\sigma}_{id}[F] = F$.

Theorem 2.5 [6] The set $Hyp_G(\tau,\tau')$ forms the monoid $\mathcal{H}yp_G(\tau,\tau') := (Hyp_G(\tau,\tau'); \circ_G, \sigma_{id})$.

In this section, we define a binary operation '$+_G$' on the set of all generalized hypersubstitutions for algebraic systems of type $(m;n)$ and show that $(Hyp_G(m;n); +_G)$ is a semigroup.

Definition 2.6 Let $m,n \geq 1$ and $\sigma_{t,F}, \sigma_{t',F'} \in Hyp_G(m;n)$. We define the binary operation
$+_G : (Hyp_G(m;n))^2 \to Hyp_G(m;n)$
by (i) $(\sigma_{t,F} +_G \sigma_{t',F'})(f) = S^m(t', \underbrace{t, ..., t}_{m-terms})$,

(ii) $(\sigma_{t,F} +_G \sigma_{t',F'})(\gamma) = R^n(F', \underbrace{t, ..., t}_{n-terms})$.

Example 2.7 Let $(m;n) = (3;2)$,
$t = f(x_1,x_5,x_4)$, $t' = x_2$, $F = \gamma(x_1, x_4)$
and $F' = \gamma(x_1, x_2)$. Then
$(\sigma_{t,F} +G\, \sigma_{t',F'})(f)$
$= S^3(x_2, f(x_1, x_5, x_4), f(x_1, x_5, x_4), f(x_1, x_5, x_4))$
$= f(x_1, x_5, x_4)$.
$(\sigma_{t,F} +_G \sigma_{t',F'})(\gamma)$
$= R^2(\gamma(x_1, x_2), f(x_1, x_5, x_4), f(x_1, x_5, x_4), f(x_1, x_5, x_4))$
$= \gamma(f(x_1, x_5, x_4), (f(x_1, x_5, x_4))$.
Therefore, $\sigma_f(x_1,x_5,x_4), \gamma(x_1,x_4) +_G \sigma_{x_2,\gamma}(x_1,x_2)$
$= \sigma_f(x_1,x_5,x_4), \gamma(f(x_1,x_5,x_4), (f(x_1,x_5,x_4))$.

Now, we will show that $(Hyp_G(m;n); +_G)$ is a semigroup.

Proposition 2.8 For arbitrary $\sigma_{t_1,F_1}, \sigma_{t_2,F_2}, \sigma_{t_3,F_3} \in Hyp_G(m;n)$, we have $(\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2}) +_G \sigma_{t_3,F_3} = \sigma_{t_1,F_1} +_G (\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3})$.

Proof. Let $\sigma_{t_1,F_1}, \sigma_{t_2,F_2}$ and $\sigma_{t_3,F_3} \in Hyp_G(m;n)$.
Then $((\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2}) +_G \sigma_{t_3,F_3})(f) = S^m(t_3, \underbrace{((\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2})(f), ..., (\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2})(f)))}_{m-terms}$
$= S^m(t_3, S^m(t_2, \underbrace{t_1, ..., t_1}_{m-terms}), ..., S^m(t_2, \underbrace{t_1, ..., t_1}_{m-terms}))$
$= S^m(S^m(t_3, \underbrace{t_2, ..., t_2}_{m-terms}), \underbrace{t_1, ..., t_1}_{m-terms})$
$= S^m((\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3})(f), \underbrace{t_1, ..., t_1}_{m-terms})$
$= (\sigma_{t_1,F_1} +_G (\sigma_{t_2,F_2} +G\, \sigma_{t_3,F_3}))(f)$.

And $((\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2}) +_G \sigma_{t_3,F_3})(\gamma) = R^n(F_3, \underbrace{((\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2})(f), ..., (\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2})(f)))}_{n-terms}$
$= R^n(F_3, S^m(t_2, \underbrace{t_1, ..., t_1}_{m-terms}), ..., S^m(t_2, \underbrace{t_1, ..., t_1}_{m-terms}))$
$= R^m(R^n(F_3, \underbrace{t_2, ..., t_2}_{n-terms}), \underbrace{t_1, ..., t_1}_{m-terms})$
$= R^m((\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3})(\gamma), \underbrace{t_1, ..., t_1}_{m-terms})$ (by(FC1))
$= (\sigma_{t_1,F_1} +_G (\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3}))(\gamma)$.

III. Results

In this section, we describe idempotent and regular elements in $\mathcal{H}yp_G(m;n)$.

1) *Definition 3.1*
Let $(Hyp_G(m;n); +_G)$ be a semigroup. An element $\sigma_{t,F} \in Hyp_G(m;n)$ is called an idempotent if $\sigma_{t,F} +_G \sigma_{t,F} = \sigma_{t,F}$. We denote $E(Hyp_G(m;n))$ be the set of all idempotent elements of $Hyp_G(m;n)$.

2) *Definition 3.2*
Let $(Hyp_G(m;n); +_G)$ be a semigroup. An element $\sigma_{t,F} \in Hyp_G(m;n)$ is called a regular if there exist $\sigma'_{t',F'} \in Hyp_G(m;n)$ such that $\sigma_{t,F} +_G \sigma'_{t',F'} +_G \sigma_{t,F} = \sigma_{t,F}$. We denote $R(Hyp_G(m;n))$ be the set of all regular elements of $Hyp_G(m;n)$.
Let $Length(t)$ be the number of operation symbols occuring in the term t.

3) *Proposition 3.3*
The element $\sigma_{t,F} \in Hyp_G(m;n)$ is an idempotent if and only if there is $i \in N^+$ *such that* $t = x_i \in X_n$ *for* $i \in \{1, ..., n\}$ and $F = \gamma(s_1, ..., s_n)$ *with* $s_i \in W_i(\{X_i\})$.
Proof. Assume that $\sigma_{t,F} \in Hyp_G(m;n)$ is an idempotent. Suppose that $t \notin X$, then $S(t, t, ..., t) \neq t$ is clearly. Hence $t \in X$.
We consider $(\sigma_{t,F} +_G \sigma_{t,F})(\gamma)$
$= R^n(\gamma(s_1, ..., s_n), t, ..., t) = \gamma(r_1, ..., r_n)$ with $r_i \in W_i(\{t\})$ for $i \in \{1, ..., n\}$.
Conversely, it's easy to see that if $t = x_i \in X_n$ for $i \in 1, ..., n$ and $F = \gamma(s_1, ..., s_n)$ with $s_i \in W_i(X_i)$, we get $(\sigma_{x_i,\gamma}(s_1,...,s_n) +_G \sigma_{x_i,\gamma(s_1,...,s_n)})(f)$
$= S^m(x_i, x_i, ..., x_i) = x_i$ and
$(\sigma_{x_i,\gamma}(s_1, ..., s_n) +_G \sigma_{x_i,\gamma(s_1, ..., s_n)})(\gamma)$
$= R^n(\gamma(s_1, ..., s_n), x_i, ..., x_i) = \gamma(s_1, ..., s_n)$.
Therefore $\sigma_{t,F}$ is an idempotent element.
Now, we will describe regular elements of $(Hyp_G(m;n); +_G)$.

4) *Proposition 3.4*

$E(Hyp_G(m;n)) = R(Hyp_G(m;n))$.

Proof. Since $E(Hyp_G(m;n)) \subseteq R(Hyp_G(m;n))$, we will show that $R(Hyp_G(m;n)) \subseteq E(Hyp_G(m;n))$. Let $\sigma_{t,F} \in R\ (Hyp_G(m;n))$. Then, there exists $\sigma_{t',F'} \in Hyp_G(m;n)$ such that $\sigma_{t,F} +_G \sigma_{t',F'} +_G \sigma_{t,F} = \sigma_{t,F}$. By Definition 2.6, we get $t = S^m(S^m(t, t', ..., t')$ and $F = R^n(F, S^m(t', t, ..., t), ..., S^m(t', t, ..., t))$, where $S^m(t', t, ..., t) \in W(\{t\})$. Assume that, t is not a variable (i, e. $t \notin X$), then $Length(t) = Length\ (S^m(S^m(t, t', ..., t'))$.

But $Length(t) > Length\ (S^m(t, t, ..., t)) > Length(t)$. We get a contradiction. Hence, t is a variable. From, $F = R^n(F, S^m(t', t, ..., t), ..., S^m(t', t, ..., t))$, where $S^m(t', t, ..., t) \in W(\{t\})$ and $t \in X$, we get $F = \gamma(s_1, ..., s_n)$, where $s_i \in W(\{t\})$.

By Proposition 3.3, $\sigma_{t,F}$ is an idempotent element. Therefore, $E\ (Hyp_G(m;n)) = R(Hyp_G(m;n))$.

Next, we investigate an algebraic structure property of the set of all generalized hypersubstitutions for algebraic systems of type $(m;n)$. We first recall the definition of a left (right) seminear-ring [8].

5) *Definition 3.5*

[7] A nonempty set S together with two binary operations, denoted by '$+$' and '$\cdot$' respectively, is said to be a *left(right) seminear-ring* if $(S;+)$ and $(S;\cdot)$ are semigroups and satisfy the left (right) distributive law, i.e., for all $a, b, c \in S$,

$a\cdot(b + c) = a \cdot b + a \cdot c$

$((a + b) \cdot c = a \cdot c + b \cdot c)$.

6) *Proposition 3.6*

For arbitrary σ_{t_1,F_1}, σ_{t_2,F_2} and σ_{t_3,F_3} in $Hyp_G(m;n)$, we have $\sigma_{t_1,F_1} \circ_G(\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3}) = (\sigma_{t_1,F_1} \circ_G \sigma_{t_2,F_2}) +_G (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})$.

Proof. Let $\sigma_{t_1,F_1}, \sigma_{t_2,F_2}, \sigma_{t_3,F_3} \in Hyp_G(m;n)$. We consider,

$(\sigma_{t_1,F_1} \circ_G(\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3}))(f)$

$= \hat{\sigma}_{t_1,F_1}[(\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3})(f)]$

$= \hat{\sigma}_{t_1,F_1}[S^m(t_3, t_2, ..., t_2)]$

$= S^m(\hat{\sigma}_{t_1,F_1}[t_3], \hat{\sigma}_{t_1,F_1}[t_2], ..., \hat{\sigma}_{t_1,F_1}[t_2])$

(by Proposition 2.2)

$= S^m(\hat{\sigma}_{t_1,F_1}1[\sigma_{t_3,F_3}(f)], \hat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(f)], ..., \hat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(f)])$

$= S^m((\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})(f), (\sigma_{t_1,F_1} \circ_G \sigma_{t_2,F_2})(f), ..., (\sigma_{t_1,F_1} \circ_G \sigma_{t_2,F_2})(f))$

$= ((\sigma_{t_1,F_1} \circ_G \sigma_{t_2,F_2}) +_G (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3}))(f)$.

And $(\sigma_{t_1,F_1} \circ_G (\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3}))(\gamma)$

$= \hat{\sigma}_{t_1,F_1}[(\sigma_{t_2,F_2} +_G \sigma_{t_3,F_3})(\gamma)]$

$= \hat{\sigma}_{t_1,F_1}[R^n(F_3, t_2, ..., t_2)]$

$= R^n(\hat{\sigma}_{t_1,F_1}[F_3], \hat{\sigma}_{t_1,F_1}[t_2], ..., \hat{\sigma}_{t_1,F_1}[t_2])$

(by Proposition 2.2)

$= R^n(\hat{\sigma}_{t_1,F_1}[\sigma_{t_3,F_3}(\gamma)], \hat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(f)], ..., \hat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(f)])$

$= R^n((\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})(\gamma), (\sigma_{t_1,F_1} \circ_G \sigma_{t_2,F_2})(f), ..., (\sigma_{t_1,F_1} \circ_G \sigma_{t_2,F_2})(f))$

$= ((\sigma_{t_1,F_1} \circ_G \sigma_{t_2,F_2}) +_G (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3}))(\gamma)$.

By Proposition 3.6, it shows that $Hyp_G(m;n); \circ_G, +_G)$ is a left seminear-ring. But the equation $(\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2}) \circ_G \sigma_{t_3,F_3} = (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3}) +_G (\sigma_{t_2,F_2} \circ_G \sigma_{t_3,F_3})$ is not true for arbitrary generalized hypersubstitutions $\sigma_{t_1,F_1}, \sigma_{t_2,F_2}$ and σ_{t_3,F_3} in $Hyp_G(m;n)$. For example, we consider the type $(2;3)$, i.e., there are one binary operation symbol f and one binary relational symbol γ. Let $\sigma_{t_1,F_1}(f) := t_1 = x_4$, $\sigma_{t_2,F_2}(f) := t_2 = f(x_1, x_2)$ and $\sigma_{t_3,F_3}(f) := t_3 = f(f(x_2, x_2), x_3)$. And let $\sigma_{t_1,F_1}(\gamma) := F_1 = \gamma(x_1, x_2, x_3)$,$\sigma_{t_2,F_2}(\gamma) := F_2 = \gamma(x_2, x_3, x_4)$ and $\sigma_{t_3,F_3}(\gamma) := F_3 = \gamma(x_4, x_5, x_6)$.

Consider, $((\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2}) \circ_G \sigma_{t_3,F_3})(f)$

$= (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[\sigma_{t_3,F_3}(f)]$

$= (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[f(f(x_2, x_2), x_3)]$

$= S^2((\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2})(f), (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[f(x_2, x_2)], (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}\ [x_3])$

$= S^2((\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2})(f), S^2((\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2})(f), (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[x_2], (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}\ [x]), [x_3])$

$= S^2(S^2(f(x_1, x_2), x_4, x_4), S^2\ (S^2(f(x_1, x_2), x_4, x_4), x_2, x_2), x_3)$

$= S^2(f(x_4, x_4), f(x_4, x_4), x_3)$

$= f(x_4, x_4)$.

And, $(\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3}) +_G (\sigma_{t_2,F_2} \circ_G \sigma_{t_3,F_3})(f)$

$= S^2((\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})(f), (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})(f), (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})(f))$

$= S^2(\hat{\sigma}_{t_2,F_2}[\sigma_{t_3,F_3}(f)], \hat{\sigma}_{t_1,F_1}[\sigma_{t_3,F_3}(f)], \hat{\sigma}_{t_1,F_1}[\sigma_{t_3,F_3}(f)])$

$= S^2(\hat{\sigma}_{t_2,F_2}[f(f(x_2, x_2), x_3)], \hat{\sigma}_{t_1,F_1}[f(f(x_2,x_2),x3)], \hat{\sigma}_{t1,F1}[f(f(x_2, x_2), x_3)])$

$= S^2(S^2(\sigma_{t_2,F_2}(f), \hat{\sigma}_{t_2,F_2}[f(x_2, x_2)], \hat{\sigma}_{t_2,F_2}[x_3]), S^2(\sigma_{t_1,F_1}(f), \hat{\sigma}_{t_1,F_1}[f(x_2, x_2)], \hat{\sigma}_{t_1,F_1}[x_3]), S^2(\sigma_{t_1,F_1}(f), \hat{\sigma}_{t_1,F_1}[f(x_2, x_2)], \hat{\sigma}_{t_1,F_1}[x_3])$

$= S^2(S^2(f(x_1, x_2)\ , \hat{\sigma}_{t_2,F_2}[f(x_2, x_2)], \hat{\sigma}_{t_2,F_2}[x_3]), S^2(x_4, \hat{\sigma}_{t_1,F_1}[f(x_2, x_2)], \hat{\sigma}_{t_1,F_1}[x_3]), S^2(x_4, \hat{\sigma}_{t_1,F_1}[f(x_2, x_2)], \hat{\sigma}_{t_1,F_1}[x_3])$

$= S^2(S^2(f(x_1, x_2), S^2(f(x_1, x_2), x_2, x_2), x_3),$
$S^2(x_4, S_2(x_4, x_2, x_2), x_3),$
$S^2(x_4, S_2(x_4, x_2, x_2), x_3))$
$= S^2(S^2(f(x_1, x_2), f(x_2, x_2), x_3), S^2(x_4, x_4, x_3),$
$S^2(x_4, x_4, x_3))$
$= S^2(f(f(x_2, x_2), x_3), x_4, x_4)$
$= f(f(x_4, x_4), x_3).$

The proceed in the similarly way, we get that $((\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2}) \circ_G \sigma_{t_3,F_3})(\gamma)$
$= (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[\sigma_{t_3,F_3}(\gamma)]$
$= (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[\gamma(x_4, x_5, x_6)]$
$= R^3((\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2})(\gamma), (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[x_4],$
$(\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[x_5], (\sigma_{t_1,F_1} +_G \sigma t_{t_2,F_2}\hat{)}[x_6])$
$= R^3(R^3(\gamma(x_2, x_3, x_4), x_4, x_4, x_4), x_4, x_5, x_6)$
$= R^3(\gamma(x_4, x_4, x_4), x_4, x_5, x_6)$
$= \gamma(x_4, x_4, x_4).$

And, $(\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3}) +_G (\sigma_{t_2,F_2} \circ_G \sigma_{t_3,F_3})(\gamma)$
$= R^3((\sigma_{t_2,F_2} \circ_G \sigma_{t_3,F_3})(\gamma), (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})(f),$
$(\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3})(f))$
$= R^3(\hat{\sigma}_{t_2,F_2}[\sigma_{t_3,F_3}(\gamma)], \hat{\sigma}_{t_1,F_1}[\sigma_{t_3,F_3}(f)],$
$\hat{\sigma}_{t_3,F_3}[\sigma_{t_3,F_3}(f)], \hat{\sigma}_{t_3,F_3}[\sigma_{t_3,F_3}(f)])$
$= R^3(\hat{\sigma}_{t_2,F_2}[\gamma(x_4, x_5, x_6)], \hat{\sigma}_{t_1,F_1}[f(f(x_2, x_2), x_3)],$
$\hat{\sigma}_{t_1,F_1}[f(f(x_2, x_2), x_3)], \hat{\sigma}_{t_1,F_1}[f(f(x_2, x_2), x_3)])$
$= R^3(R^3(\sigma_{t_2,F_2}(\gamma), x_4, x_5, x_6), S^2(\sigma_{t_1,F_1}(f),$
$\hat{\sigma}_{t_1,F_1}[f(x_2, x_2)], x_3), S^2(\sigma_{t_1,F_1}(f), \hat{\sigma}_{t_1,F_1}[f(x_2, x_2)],$
$x_3), S^2(\sigma_{t_1,F_1}(f), \hat{\sigma}_{t_1,F_1}[f(x_2, x_2)], x_3)$
$= R^3(R^3(\gamma(x_2, x_3, x_4), x_4, x_5, x_6),$
$S^2(x_4, S^2(x_4, x_2, x_2), x_3), S^2(x_4, S^2(x_4, x_2, x_2), x_3),$
$S^2(x_4, S^2(x_4, x_2, x_2), x_3)))$
$= R^3(\gamma(x_5, x_6, x_4), S^2(x_4, x_4, x_3),$
$S^2(x_4, x_4, x_4), S^2(x_4, x_4, x_3))$
$= R^3(\gamma(x_5, x_6, x_4), x_4, x_4, x_4)$
$= \gamma(x_5, x_6, x_4).$

Therefore, $((\sigma_{t_1,F_1} +_G \sigma_{t_2,F_2}) \circ_G \sigma_{t_3,F_3}) \neq (\sigma_{t_1,F_1} \circ_G \sigma_{t_3,F_3}) +_G (\sigma_{t_2,F_2} \circ_G \sigma_{t_3,F_3})$.

It means that the structure $(HypG(2;3); \circ_G, +_G)$ is not a right seminear-ring.

IV. Conclusion

The main result of the paper is the characterization idempotent and regular elements on a semigroup of the pairs of terms and relational terms of type $(m;n)$. This paper shown that any regular elements are idempotent elements. The proof of these results is based on the semigroup theory. Our results differ from universal algebras [7].

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