SciTech Research Journal, 8(3) : September-December 2025
(ปีที่ 8 ฉบับที่ 3 กันยายน-ธันวาคม 2568)

Generalized Algebras Type Sigma

Sawinee Srikwa[1] & Kittisak Saengsura[2]
[1,2]Department Of Mathematics, Faculty of Science Mahasarakham University.
*Corresponding author: sy.sawinee@gmail.com

Received	Reviewed	Revised	Accepted
27/11/2025	27/12/2025	30/12/2025	30/12/2025

Abstract

In this research, the concepts of generalized algebra type sigma is introduced. We explore some basic properties of this algebra. The concepts of subalgebra, congruence relation, homomorphism and isomorphism are discussed on a generalized algebra type sigma. Some properties concerning to such concepts are obtained.

Keywords: generalized sigma algebra; generalized sigma subalgebra; congruence relation; homomorphism; isomorphism.

Introduction

Classical universal algebra, fundamentally shaped by Birkhoff's celebrated HSP theorem [4, 5], traditionally investigates algebras defined by finitary operations that is, operations whose arities are natural numbers. Building upon a rich history of extending these foundational concepts, exemplified by the work of Arworn and Denecke on hypersubstitutions [11, 12, 18] and Leeratanavalee on generalized groupoid structures, this research [12], introduces a significant generalization: the generalized sigma algebra. This new structure extends the classical notion by permitting operations to be of arbitrary arity, indexed by any non-empty set rather than being restricted to the finite cardinals, thus allowing for operations of potentially uncountable arity. The primary objective of this investigation is therefore to systematically develop the foundational theory for these generalized sigma algebras. We extend the classical concepts of subalgebras, congruences, and homomorphisms to this broader setting, thereby enabling an inquiry into which fundamental theorems of universal algebra are preserved and what new structural properties emerge in this more general framework.

2. Preliminaries

In the foundational framework of universal algebra, an algebra $\mathcal{A}$ is formally defined as a pair $\mathcal{A} := (A; (f_i^A)_{i\in\mathcal{I}})$, where A is a non-empty set, termed the universe, and $(f_i^A)_{i\in\mathcal{I}}$. is an indexed family of finitary operations on A, [11]. Each operation

$f: A^n \to A$ is characterized by its arity n_i a natural number, and the sequence of these $\tau := (n_i)_{i\in\mathcal{J}}$, constitutes the **type** of the algebra. This structure provides a unified language to describe a vast array of mathematical systems.

Central to the study of these algebras are the concepts of substructures and mappings that preserve their operational integrity. A **subalgebra** $\mathcal{A}$ of an algebra $\mathcal{B}$ of the same type is an algebra whose universe A, is a subset of B, and whose operations are the restrictions of the operations of $\mathcal{B}$ to A. [6, p.33].

The Subalgebra Criterion provides a practical test for this relationship, stating that a non-empty subset $A \subseteq B$. forms the universe of a subalgebra if and only if it is closed under all fundamental operations of $\mathcal{B}$. The collection of all subalgebras of a given algebra forms a complete lattice under set inclusion, and the intersection of any family of subalgebras is itself a subalgebra. The relationships between algebras are formally captured by **homomorphisms**, which are mappings between two algebras of the same type that commute with their respective operations. Specifically, a function $h: A \to B$ is a homomorphism $\mathcal{A}$ into $\mathcal{B}$ if for every operation if for all $i \in \mathcal{J}$ we have

$$h\left(f_i^A(a_1, a_2, \dots a_{n_i})\right) = f_i^B\left((h(a_1), h(a_2), \dots, h(a_{n_i})\right),$$

for all $(a_1, a_2, \dots a_{n_i}) \in A$. [6., p.35] A bijective homomorphism is termed an **isomorphism**, signifying structural equivalence between two algebras. The image of a subalgebra under a homomorphism is always a subalgebra, as is the preimage of a subalgebra. Associated with every homomorphism is a **congruence relation**, defined as the kernel of the map $h: kerh := \{(a,b) \in A^2 \mid h(a) = h(b)\}$. We may alternately express this as $kerh = h^{-1} \circ h$, where h^{-1} is the inverse relation of h. The kernel of any homomorphism $h: \mathcal{A} \to \mathcal{B}$., is a congruence relation on $\mathcal{A}$. More generally, Let $\mathcal{A} = (A; (f_i^A)_{i\in\mathcal{J}})$ be an algebra of type τ. An equivalence relation θ on A is called a *congruence relation* on $\mathcal{A}$ if all the fundamental operations f_j^A are compatible with θ. We denote by $Con\mathcal{A}$ the set of all congruence relations of the algebra $\mathcal{A}$. For every algebra $\mathcal{A} = (A; (f_i^A)_{i\in\mathcal{J}})$ the trivial equivalence relations

$$\Delta := \{(a,a) \mid a \in A\} \text{ and } \nabla_A = A \times A.$$

are congruence relations. An algebra which has no congruence relations except Δ_A and ∇_A is called simple [2].

The interplay between homomorphisms, congruences, and quotient algebras is elegantly described by the Isomorphism Theorems. **[14].** The **First Isomorphism**

Theorem establishes a fundamental connection, stating that the homomorphic image of an algebra $\mathcal{A}$. under a homomorphism h is isomorphic to the quotient algebra of $\mathcal{A}$. by the kernel of h, i.e., $\mathrm{A}/\ker h \cong h(\mathrm{A})$.

. The **Second Isomorphism Theorem** further elaborates on the structure of quotient algebras, asserting that congruences on an algebra $\mathcal{A}$, with $\theta_1 \subseteq \theta_2$. Then the relation θ_1/θ_2 is a congruence relation on $\mathcal{A}/\theta_1$, and the function

$$\varphi: (\mathcal{A}/\theta_1)/(\theta_2/\theta_1) \to \mathcal{A}/\theta_2, \text{defined by } [[a]_{\theta_1}]_{\theta_2/\theta_1} \mapsto [a]_{\theta_2},$$

is an isomorphism. [16].

3. Result

3.1 Generalized sigma algebra

In this section, we will introduce the concepts of a generalized sigma algebra, and give an example to show that a generalized sigma algebra is distinct from a classical algebra and every classical algebra can be obtained as a generalized sigma algebra. We will investigate the properties of generalized sigma algebra.

Before providing the definition of a generalized sigma algebra, we will first review and summarize the construction process of a classical algebra.

Definition 3.1.1 Let A be a nonempty set, and let Ω be a any nonempty index set and let $\Sigma = \{\Sigma_j : j \in \Omega\}$ nonempty index set. We defined $f^A_{\Sigma_j}: A^{\Sigma_j} \to A$, where

$$\begin{aligned} A^{\Sigma_j} &= \prod_{\beta \in \Sigma_j} A_\beta \ ; A_\beta = A \text{ for all } \beta \in \Sigma_j \\ &= \{\sigma_\beta | \sigma_\beta : \Sigma_j \to A, \sigma_\beta \in \Sigma_j\}. \end{aligned}$$

We called $f^A_{\Sigma_j}: A^{\Sigma_j} \to A$ for all $j \in \Omega$ the $|\Sigma_j|$- *ary operation* defined on A, and is said to have *arity* $|\Sigma_j|$. We let $\left(f^A_{\Sigma_j}\right)_{j\in\Omega}$ be a function which assigns to every element of Ω and $|\Sigma_j|$*-ary operation* $f^A_{\Sigma_j}$ defined on A^{Σ_j} Then the pair $\mathcal{A}_\Sigma := \left(A; \left(f^A_{\Sigma_j}\right)_{j\in\Omega}\right)$ is called (indexed) *Generalized algebra* (indexed by the set Σ_j) for all $\Sigma_j \in \Sigma$; for all $j \in \Omega$. The set A is called the *base* or *carrier set* or *universe* of A and $\left(f^A_{\Sigma_j}\right)_{j\in\Omega}$ is called *the sequence of fundamental operations* of A, for each $j \in \Omega$ the index Σ_j is called arity of

$\left(f^A_{\Sigma_j}\right)_{j\in\Omega}$. The sequence $\tau := \left(|\Sigma_j|_{j\in\Omega}\right)$ of all arities is called *type* of the generalized algebra. We use the name $Alg(\Sigma)$ for the calss of all algebras of a given type

$\tau := \left(|\Sigma_j|_{j\in\Omega}\right)$.

Now we will give some examples of generalized sigma algebra with uncountable arity and that is not classical algebras.

Example 3.1.2. (An Algebra with Uncountable Arity)

This example demonstrates a generalized sigma algebra that cannot be represented as a classical algebra due to its uncountable arity.

Let the base set be $A = \{a, b, c\}$. and the index set for the operations be $\Omega = \{1,2\}$. We defined the arity sets as real intervals: $\Sigma_1 := [0,1]$, $\Sigma_2 := [2,3]$. Observe that, the arity of our operations is determined by sets of real numbers, not natural numbers. If we given that for each $j \in \Omega$, the carrier of the operation

$A^{\Sigma_j} = \left\{\varphi | \varphi: \Sigma_j \to \{a, b, c\}\right\}$. So we have $A^{\Sigma_1} = \{\varphi | \varphi: \Sigma_1 \to \{a, b, c\}\}$ and

$A^{\Sigma_2} = \{\varphi | \varphi: \Sigma_2 \to \{a, b, c\}\}$. Hence A^{Σ_1} and A^{Σ_2} are the sets of all functions from the arity set Σ_j to the base set A, where $j = 1,2$. For the arity set $\Sigma_1 = [0,1]$, we define $f^A_{\Sigma_1}$ based on the value of the input function φ at the specific point $\beta = 1$:

$$f^A_{\Sigma_1}(\varphi) = \begin{cases} a & \text{if } \varphi(1) = a \\ b & \text{otherwise.} \end{cases}$$

For the arity set $\Sigma_2 = [2,3]$, we defined $f^A_{\Sigma_2}$ as follows:

$$f^A_{\Sigma_2}(\varphi) = \begin{cases} b & \text{if } \varphi(2) = b \\ c & \text{otherwise.} \end{cases}$$

Therefore, $\mathcal{A}_\Sigma = (\{a, b, c\}; (f^A_{[0,1]}, f^A_{[2,3]}))$ is a well-defined generalized sigma algebra of type $\tau = (|\mathbb{R}|, |\mathbb{R}|) = |\mathbb{R}|$ Since the arity of the operation $f^A_{[0,1]}$ is the set $[0,1]$. The cardinality of this set is $|[0,1]| = |\mathbb{R}|$, which is an uncountable infinity. Since classical algebras require all arities to be natural numbers ($n \in \mathbb{N}$), there is no way to represent this structure in the classical framework. This example clearly demonstrates that the class of Generalized Sigma Algebras is a proper extension of the class of classical algebras.

3.2 Generalized sigma algebra

Definition 3.2.1 Let $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B\right)_{j\in\Omega}\right)$ be a generalized sigma algebra of type $\tau := \left(\left|\Sigma_j\right|_{j\in\Omega}\right)$. Then an generalized sigma algebra $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A\right)_{j\in\Omega}\right)$. is called a generalized sigma subalgebra of $\mathcal{B}_\Sigma$, written as $\mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma$., if the following conditions are satisfied:

i. $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A\right)_{j\in\Omega}\right)$. is an algebra of type $\tau := \left(\left|\Sigma_j\right|_{j\in\Omega}\right)$.

ii. $A \subseteq B$.

iii. For all $j \in \Omega$, $f_{\Sigma_j}^B|_{A^{\Sigma_j}} = f_{\Sigma_j}^A$.

Then $\mathcal{A}_\Sigma \preccurlyeq \mathcal{B}_\Sigma$.

Whether one generalized sigma algebra is a generalized sigma subalgebra of another generalized sigma algebra can be checked by the following criterion:

Lemma 3.2.1. (Generalized Sigma Subalgebra Criterion)
Let $\mathcal{B}_\Sigma := \left(B; \left(f_{\Sigma_j}^B\right)_{j\in\Omega}\right)$ be a generalized sigma algebra of type $\tau := \left(\left|\Sigma_j\right|_{j\in\Omega}\right)$ and let $A \subseteq B$, then $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A\right)_{j\in\Omega}\right)$ is a generalized sigma subalgebra of $\mathcal{B}_\Sigma$ iff A is closed with respect to all the operations $f_{\Sigma_j}^B$ for all $j \in \Omega$; that is, if $f_{\Sigma_j}^B|_{A^{\Sigma_j}}(\sigma) \in A$ for all $\sigma \in A^{\Sigma_J}$ and for all $j \in \Omega$

Proof. Straightforward from Definition 3.2.1 ☐

4. Homomorphisms, Congruences, and Isomorphisms

4.1 Generalized Homomorphisms

A fundamental pursuit in any algebraic theory is not merely to study individual structures in isolation, but to understand the rich web of relationships that connect them. The primary tools for this exploration are structure-preserving maps, known as generalized homomorphisms. These maps allow us to compare different algebras, identify shared properties, and understand how one structure can be mapped into another.

Definition 4.1.1 (Generalized Homomorphisms.)
Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j\in\Omega})$ and $\mathcal{B}_\Sigma = (B; (f_{\Sigma_j}^B)_{j\in\Omega})$ be generalized sigma algebras of the same type. A mapping $h: A \to B$ is called a *generalized homomorphism* from $\mathcal{A}_\Sigma$ to $\mathcal{B}_\Sigma$ if for every $j \in \Omega$ and for every function $\psi \in A^{\Sigma_j}$, the following condition holds:

$$h\left(f_{\Sigma_j}^A(\psi)\right) = f_{\Sigma_j}^B(h \circ \psi)$$

where $h \circ \psi$ denotes the function composition. This composition results in a new function $h \circ \psi: \Sigma_j \to B$, which is a valid input for the operation $f_{\Sigma_j}^B$, and is defined by $(h \circ \psi)(\sigma) = h(\psi(\sigma))$ for all $\sigma \in \Sigma_j$. If the function h is bijective, that is both one-to-one (injective) and "onto" (surjective), then the generalized homomorphism $h: \mathcal{A}_\Sigma \to \mathcal{B}_\Sigma$ is called a *generalized isomorphism* from $\mathcal{A}_\Sigma$ onto $\mathcal{B}_\Sigma$. An *injective generalized homomorphism* from $\mathcal{A}_\Sigma$ into $\mathcal{B}_\Sigma$ is also called a *generalized embedding* of $\mathcal{A}_\Sigma$ into $\mathcal{B}_\Sigma$. A generalized homomorphism $h: \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$ of a generalized sigma algebra $\mathcal{A}_\Sigma$ into itself is called a *generalized endomorphism* of $\mathcal{A}_\Sigma$, and a *generalized isomorphism* $h: \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$ from $\mathcal{A}_\Sigma$ onto $\mathcal{A}_\Sigma$ is called a *generalized automorphism* of $\mathcal{A}_\Sigma$.

Therefore, the map k is a generalized homomorphism. Since k is injective (one-to-one), it is a generalized embedding of $\mathcal{C}_\Sigma$ into $\mathcal{A}_\Sigma$. It is not a generalized isomorphism because it is not surjective (the element $c \in A$ is not in the image of k).

Theorem 4.1.1 Let $\mathcal{A}_\Sigma, \mathcal{B}_\Sigma$, and $\mathcal{C}_\Sigma$ be generalized sigma algebras of the same type.

i. The identity map on A, denoted $id_A: A \to A$, is a generalized homomorphism from $\mathcal{A}_\Sigma$ to $\mathcal{A}_\Sigma$ (i.e., a generalized endomorphism).

ii. If $h: \mathcal{A}_\Sigma \to \mathcal{B}_\Sigma$, and $k: \mathcal{B}_\Sigma \to \mathcal{C}_\Sigma$, are generalized homomorphisms, then their composition $k \circ h: \mathcal{A}_\Sigma \to \mathcal{C}_\Sigma$, is also a generalized homomorphism.

Proof. Let $j \in \Omega$ be an arbitrary index for an operation. We will show that satisfying:

Identity Map: Let $id_A: A \to A$ be the identity map. To show it is a generalized homomorphism, we must verify that for any function $\psi \in A^{\Sigma_j}$, the following condition holds: $id_A(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^A(id_A \circ \psi)$. We analyze both sides of the equation:

LHS : By the definition of the identity map, $id_A(x) = x$ for any element $x \in A$. Since $f_{\Sigma_j}^A(\psi)$ is an element of A, we have: $id_A(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^A(\psi)$.

RHS : Consider the input function for the operation, which is the composition $id_A \circ \psi$. For any $\sigma \in \Sigma_j$, we have $(id_A \circ \psi)(\sigma) = id_A(\psi(\sigma)) = \psi(\sigma)$.

This means that the function $id_A \circ \psi$ is identical to the function ψ. Therefore: $f_{\Sigma_j}^A(id_A \circ \psi) = f_{\Sigma_j}^A(\psi)$. Since LHS = RHS, the identity map is a generalized homomorphism. This is the property we demonstrated in Case 2 of Example 4.2.

Hence if we given that $h: \mathcal{A}_\Sigma \to \mathcal{B}_\Sigma$ and $k: \mathcal{B}_\Sigma \to \mathcal{C}_\Sigma$, be generalized homomorphisms. We must show that for any function $\psi \in A^{\Sigma_j}$, the map $k \circ h$ satisfies the generalized homomorphism condition: $(k \circ h)(f_{\Sigma_j}^A(\psi)) = f_{\Sigma_j}^C((k \circ h) \circ \psi)$.

We start from the left-hand side and transform it step-by-step using the properties of composition and generalized homomorphisms. We have

$$\begin{aligned}(k \circ h)(f_{\Sigma_j}^A(\psi)) &= k\left(h\left(f_{\Sigma_j}^A(\psi)\right)\right) \\ &= k\left(f_{\Sigma_j}^B(h \circ \psi)\right) \\ &= f_{\Sigma_j}^C(k \circ (h \circ \psi)) \\ &= f_{\Sigma_j}^C\big((k \circ h) \circ \psi\big).\end{aligned}$$

The final expression is the right-hand side of the condition we needed to prove. Thus, $k \circ h$ is a generalized homomorphism. ☐

The binary relation that defines this partition is called the kernel of the homomorphism. To properly build our framework, we first establish a key property of generalized isomorphisms, which will be essential for developing the Generalized Homomorphism Theorems. With the properties of generalized homomorphisms established, we can now formally define the kernel.

Definition 4.1.2. (Kernel of a Generalized Homomorphism.)
Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j \in \Omega})$ and $\mathcal{B}_\Sigma = (B; (f_{\Sigma_j}^B)_{j \in \Omega})$ be generalized sigma algebras of the same type, and let $h: A \to B$ be a generalized homomorphism. The **kernel of a generalized homomorphism** h, denoted by $\mathrm{ker}h$, is the binary relation on the base set A defined as follows: $\mathrm{ker}h := \{(a_1, a_2) \in A \times A \mid h(a_1) = h(a_2)\}$.

Theorem 4.1.2. (First Generalized Isomorphism Theorem for Generalized Sigma Algebras.)

Let $h: \mathcal{A}_\Sigma \to \mathcal{B}_\Sigma$ be a generalized homomorphism. Then the quotient algebra $\mathcal{A}_\Sigma/\mathrm{ker}h$ is isomorphic to the image algebra $h(\mathcal{A}_\Sigma)$. Specifically, there exists a unique isomorphism $f: \mathcal{A}_\Sigma/\mathrm{ker}h \to h(\mathcal{A}_\Sigma)$, such that the following diagram commutes, i.e., $h|_{\mathrm{im}(h)} = f \circ nat(\mathrm{ker}h)$, where $nat(\mathrm{ker}h)$ is the natural generalized homomorphism and $h|_{\mathrm{im}(h)}$ is the generalized homomorphism h with its codomain restricted to its image.

Proof. Let $\theta = \text{ker}h$. To prove the theorem, we must define the map f and show that it is satisfying (1) well-defined, (2) a generalized homomorphism, (3) injective, and (4) surjective.

Let an element of the quotient algebra $\mathcal{A}_\Sigma/\theta$ be denoted by an equivalence class $[a]_\theta$ for some $a \in A$. We define the map $f: \mathcal{A}_\Sigma/\theta \to h(A).$, as follows: $f([a]_\theta) := h(a)$. We assume $[a]_\theta = [b]_\theta$ for $a, b \in A$. Hence

$$\begin{aligned} [a]_\theta = [b]_\theta &\Leftrightarrow (a,b) \in \theta \\ &\Leftrightarrow (a,b) \in \text{ker}h \\ &\Leftrightarrow h(a) = h(b) \\ &\Leftrightarrow f([a]_\theta) = f([b]_\theta). \end{aligned}$$

Thus, f is well-defined. We see that, f is a generalized homomorphism: Because if we let $j \in \Omega$ and let Ψ be a function from Σ_j into the quotient algebra $\mathcal{A}_\Sigma/\theta$. The operation in the quotient algebra is defined as $f_{\Sigma_j}^{\mathcal{A}/\theta}(\Psi) = [f_{\Sigma_j}^{A}(\psi)]_\theta$, where ψ is any function such that $nat(\theta) \circ \psi = \Psi$. We need to show $f(f_{\Sigma_j}^{\mathcal{A}/\theta}(\Psi)) = f_{\Sigma_j}^{h(A)}(f \circ \Psi)$.

$$\begin{aligned} f(f_{\Sigma_j}^{\mathcal{A}/\theta}(\Psi)) &= f([f_{\Sigma_j}^{A}(\psi)]_\theta) \\ &= h(f_{\Sigma_j}^{A}(\psi)) \\ &= f_{\Sigma_j}^{B}(h \circ \psi). \end{aligned}$$

Note that $(f \circ \Psi)(\sigma) = f(\Psi(\sigma)) = f([\psi(\sigma)]_\theta) = h(\psi(\sigma))$. Thus, the function $f \circ \Psi$ is identical to $h \circ \psi$. The operation $f_{\Sigma_j}^{h(A)}$ is just the restriction of $f_{\Sigma_j}^{B}$ to the subalgebra $h(\mathcal{A}_\Sigma)$. So, $f_{\Sigma_j}^{B}(h \circ \psi) = f_{\Sigma_j}^{h(A)}(f \circ \Psi)$. The condition holds.

Assume $f([a]_\theta) = f([b]_\theta)$. From the chain of equivalences in step 2, the reverse implication shows:

$$f([a]_\theta) = f([b]_\theta) \Longrightarrow h(a) = h(b) \Longrightarrow (a,b) \in \text{ker}h \Longrightarrow [a]_\theta = [b]_\theta.$$

Thus, f is injective. Let y be an arbitrary element in the codomain $h(A)$. By definition of the image set, there must exist some $a \in A$ such that $h(a) = y$. The element $[a]_\theta$ is in the domain $\mathcal{A}_\Sigma/\theta$, and by our definition of f, we have $f([a]_\theta) = h(a) = y$. Thus, for any element y in the codomain, we have found a preimage. So, f is surjective. This completes the proof. □

4.2 Congruence Relations

Definition 4.2.1. (Compatibility in Generalized Sigma Algebras) Let A be a set, let θ be an equivalence relation on A, and let $f_{\Sigma_j}^A: A^{\Sigma_j} \to A$ be a generalized operation. We say that $f_{\Sigma_j}^A$ is **compatible** with θ if for all functions $\alpha, \beta \in A^{\Sigma_j}$, the following implication holds:

If $(\alpha(\sigma), \beta(\sigma)) \in \theta$ for all $\sigma \in \Sigma_j$, then $(f_{\Sigma_j}^A(\alpha), f_{\Sigma_j}^A(\beta)) \in \theta$.

Definition 4.2.2. (Generalized Congruence Relations)
Let $\mathcal{A}_\Sigma := \left(A; \left(f_{\Sigma_j}^A\right)_{j\in\Omega}\right)$. be a generalized sigma algebra of type $\tau := \left(|\Sigma_j|_{j\in\Omega}\right)$. An equivalence relation θ on A is called a *generalized congruence relations* on $\mathcal{A}_\Sigma$ if all the fundamental operations $f_{\Sigma_j}^A$ are compatible with θ. We denote by $Con\mathcal{A}_\Sigma$ the set of all generalized congruence relations of the generalized sigma algebra $\mathcal{A}_\Sigma$.

For every generalized sigma algebra $\mathcal{A}_\Sigma; = \left(A; \left(f_{\Sigma_j}^A\right)_{j\in\Omega}\right)$., the trivial equivalence relations: $\Delta_A := \{(a, a) \mid a \in A\}$ and $\nabla_A = A \times A$. are congruence relations.

An algebra which has no congruence relations except Δ_A and ∇_A is called **simple.**

Theorem 4.2.1. (Arbitrary Intersection of Generalized Congruences) Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j\in\Omega})$ be a generalized sigma algebra. Let $\{\theta_i\}_{i\in I}$ be any non-empty family of generalized congruence relations on $\mathcal{A}_\Sigma$, indexed by the set I. Then their intersection, $\theta := \bigcap_{i\in I}$, is also a generalized congruence relation on $\mathcal{A}_\Sigma$.

Proof. Clearly that θ is an equivalence relation on A, and that it is compatible with all fundamental operations of $\mathcal{A}_\Sigma$. ☐

Theorem 4.2.2. (The Natural Homomorphism) Let $\mathcal{A}_\Sigma = (A; (f_{\Sigma_j}^A)_{j\in\Omega})$. be a generalized sigma algebra and let θ be a generalized congruence relation on it. The mapping $nat_\theta: A \to A/\theta$, defined by $nat_\theta(a) := [a]_\theta$. for all $a \in A$, is a surjective generalized homomorphism from $\mathcal{A}_\Sigma$ onto the quotient algebra $\mathcal{A}_\Sigma/\theta$. This map is called the **natural generalized homomorphism** (or canonical generalized homomorphism) associated with θ.

Proof. By its definition, the map nat_θ is surjective, as every equivalence class $[a]_\theta$ in the codomain A/θ has at least one preimage, namely the element a itself.

It is easy to show that nat_θ is a generalized homomorphism, by show that for any operation index $j \in \Omega$ and for any function $\psi: \Sigma_j \to A$, the following condition holds: $nat_\theta(f_{\Sigma_j}^{A}(\psi)) = f_{\Sigma_j}^{A/\theta}(nat_\theta \circ \psi)$. □

Theorem 4.2.3. (Generalized Homomorphism Theorem)

Let $h: \mathcal{A}_\Sigma \to \mathcal{B}_\Sigma$. and $g: \mathcal{A}_\Sigma \to \mathcal{C}_\Sigma$. be generalized homomorphisms between generalized sigma algebras of the same type, and assume that g is surjective.

A generalized homomorphism $f: \mathcal{C}_\Sigma \to \mathcal{B}_\Sigma$. satisfying the condition $f \circ g = h$ exists if and only if $\ker g \subseteq \ker h$.

Furthermore, if such a generalized homomorphism f exists, it satisfies the following:

(i) f is unique.

(ii) f is injective if and only if $\ker g = \ker h$.

(iii) f is surjective if and only if h is surjective.

Proof. ($\Rightarrow$) Existence of f implies $\ker g \subseteq \ker h$. Assume a homomorphism $f: \mathcal{C}_\Sigma \to \mathcal{B}_\Sigma$ exists such that $f \circ g = h$. Let (a_1, a_2) be an arbitrary pair in $\ker g$.

$$\begin{aligned}(a_1, a_2) \in \ker g &\Rightarrow g(a_1) = g(a_2)\\ &\Rightarrow f(g(a_1)) = f(g(a_2))\\ &\Rightarrow (f \circ g)(a_1) = (f \circ g)(a_2)\\ &\Rightarrow h(a_1) = h(a_2)\\ &\Rightarrow (a_1, a_2) \in \ker h.\end{aligned}$$

Since any pair in $\ker g$ is also in $\ker h$, we conclude that $\ker g \subseteq \ker h$.

($\Leftarrow$) $\ker g \subseteq \ker h$ implies Existence of f. Assume $\ker g \subseteq \ker h$. We must define the map f and show it is a well-defined homomorphism.

(1). Define the map f. Let c be an arbitrary element of the set C. Since g is surjective, there exists at least one element $a \in A$ such that $g(a) = c$. We define the map $f: C \to B$ as follows: $f(c) := h(a)$ where a is any element such that $g(a) = c$.

(2). f is well-defined. We must show that the definition of $f(c)$ is independent of the choice of the preimage a. Suppose $a_1, a_2 \in A$ are two elements such that $g(a_1) = c$ and $g(a_2) = c$.

$$g(a_1) = g(a_2) \Longrightarrow (a_1, a_2) \in \ker g$$
$$\Longrightarrow (a_1, a_2) \in \ker h, \ker g \subseteq \ker h$$
$$\Longrightarrow h(a_1) = h(a_2).$$

This shows that the value of $h(a)$ is the same for all possible choices of a. Thus, f is well-defined.

(3). f is a homomorphism. Let $j \in \Omega$ and let $\Phi: \Sigma_j \to C$ be a function. We need to show $f(f^C_{\Sigma_j}(\Phi)) = f^B_{\Sigma_j}(f \circ \Phi)$. Since g is surjective, for the function Φ, there exists a representative function $\psi: \Sigma_j \to A$ such that $g \circ \psi = \Phi$. Now, let's evaluate the left-hand side (LHS): $f(f^C_{\Sigma_j}(\Phi)) = f(f^C_{\Sigma_j}(g \circ \psi)) = f(g(f^A_{\Sigma_j}(\psi)))$. By the definition of f, this becomes $h(f^A_{\Sigma_j}(\psi))$. For the right-hand side (RHS): We first analyze the function $f \circ \Phi$. For any $\sigma \in \Sigma_j$, $(f \circ \Phi)(\sigma) = f(\Phi(\sigma)) = f(g(\psi(\sigma)))$. By definition of f, this is $(\psi(\sigma))$. So, the function $f \circ \Phi$ is identical to $h \circ \psi$. The RHS is therefore $f^B_{\Sigma_j}(f \circ \Phi) = f^B_{\Sigma_j}(h \circ \psi)$. Since h is a generalized homomorphism, we know that $h(f^A_{\Sigma_j}(\psi)) = f^B_{\Sigma_j}(h \circ \psi)$. Since LHS = RHS, we conclude that f is a generalized homomorphism.

The properties (i)-(iii). Is easy to prove. □

4.3 The Generalized Isomorphism

This section explores the three classical Isomorphism Theorems, which are cornerstones of the subject, adapted to the framework of generalized sigma algebras. These theorems reveal deep structural relationships between subalgebras, quotient algebras, and generalized homomorphic images.

Theorem 4.3.2. (Second Isomorphism Theorem)
Let $\mathcal{A}_\Sigma$ be a generalized sigma algebra, let $\mathcal{S}$ be a subalgebra of $\mathcal{A}_\Sigma$ with carrier set S, and let θ be a congruence on $\mathcal{A}_\Sigma$. Then:

(i) The set $S \vee \theta := \{a \in A \mid \exists s \in S, (a, s) \in \theta\}$. is the carrier set of a subalgebra of $\mathcal{A}_\Sigma$, denoted $\mathcal{S} \vee \theta$.
(ii) The relation $\theta_S := \theta \cap (S \times S)$ is a congruence on the subalgebra $\mathcal{S}$.
(iii) The corresponding quotient algebras are isomorphic: $(\mathcal{S} \vee \theta)/\theta \cong \mathcal{S}/\theta_S$.

Proof. The proofs for parts (i) and (ii) involve straightforward checks of the definitions of subalgebra and congruence, and are omitted here for brevity. We will focus on proving the isomorphism in part (iii).

The strategy is to cleverly define a surjective generalized homomorphism from $\mathcal{S}$ onto $(\mathcal{S} \vee \theta)/\theta$ and then apply the First Generalized Isomorphism Theorem.

To define the map: Let's define a map $h: S \to (S \vee \theta)/\theta$ by the rule:

$$h(s) := [s]_\theta \quad \text{for all } s \in S.$$

This map takes an element from the subalgebra $\mathcal{S}$ and maps it to its equivalence class in the larger quotient algebra $(\mathcal{S} \vee \theta)/\theta$.

To show h is a generalized homomorphism: We must show that h is a generalized homomorphism from the algebra $\mathcal{S}$ to $(\mathcal{S} \vee \theta)/\theta$. Let $j \in \Omega$ and let $\psi: \Sigma_j \to S$ be a function.

$$h(f_{\Sigma_j}^{\mathcal{S}}(\psi)) = [f_{\Sigma_j}^{\mathcal{S}}(\psi)]_\theta = [f_{\Sigma_j}^{A}(\psi)]_\theta$$

Now, consider the right-hand side, $f_{\Sigma_j}^{(\mathcal{S}\vee\theta)/\theta}(h \circ \psi)$. The function : $h \circ \psi: \Sigma_j \to (S \vee \theta)/\theta.$

maps $\sigma \mapsto [\psi(\sigma)]_\theta$. By the definition of a quotient operation, $f_{\Sigma_j}^{(\mathcal{S}\vee\theta)/\theta}(h \circ \psi) = [f_{\Sigma_j}^{A}(\psi)]_\theta$. Since LHS = RHS, h is a generalized homomorphism. To show h is surjective: Let $[x]_\theta$ be an arbitrary element in the codomain $(S \vee \theta)/\theta$. By definition of the set $S \vee \theta$, this means there exists some element $s \in S$ such that $(x, s) \in \theta$. This implies that $[x]_\theta = [s]_\theta$. The element S is in the domain S of our map h. Then $h(s) = [s]_\theta = [x]_\theta$. We have found a preimage $s \in S$ for any element $[x]_\theta$ in the codomain. Thus, h is surjective. To determine the kernel of h : We now find the kernel of this generalized homomorphism h. Let $s_1, s_2 \in S$.

$$\begin{aligned}(s_1, s_2) \in \ker h &\Leftrightarrow h(s_1) = h(s_2) \text{(by definition of kernel)}\\ &\Leftrightarrow [s_1]_\theta = [s_2]_\theta \text{ (by definition of } h\text{)}\\ &\Leftrightarrow (s_1, s_2) \in \theta. \text{ (by property of equivalence classes)}\end{aligned}$$

Since s_1 and s_2 must both be in S, the condition is that $(s_1, s_2) \in \theta$ and

$(s_1, s_2) \in S \times S$. This means $\ker h = \theta \cap (S \times S)$., which is precisely the definition of θ_S.

Apply the First Generalized Isomorphism Theorem: We have now established that $h: \mathcal{S} \to (\mathcal{S} \vee \theta)/\theta.$ is a surjective generalized homomorphism with $\ker h = \theta_S$. By the First Generalized Isomorphism Theorem (Theorem 4.21), we have:

$\mathcal{S}/\ker h \cong h(\mathcal{S})$. Substituting what we found: $\mathcal{S}/\theta_S \cong (\mathcal{S} \vee \theta)/\theta$.

(Since h is surjective, its image $h(\mathcal{S})$ is the entire codomain $(\mathcal{S} \vee \theta)/\theta$). This completes the proof of a generalized isomorphism. □

5. Conclusion

This research has successfully achieved its primary goal: to introduce and develop the foundational theory of generalized sigma algebras, a novel extension of classical algebraic structures. By generalizing the notion of arity from a natural number to an arbitrary index set, this work provides a framework for studying operations of potentially uncountable arities, thereby broadening the scope of universal algebra.

The core contributions of this research are the rigorous definitions and systematic investigation of fundamental algebraic concepts within this new framework. We have defined generalized sigma subalgebras, generalized congruence relations, generalized homomorphisms, and generalized isomorphisms and have proven that many of the cornerstones of classical algebra such as the Isomorphism Theorems are preserved in this more abstract setting.

References

Arworn. (2000).Groupoids of Hypersubstitutions and G-solid Varieties. Shaker Verlag, Aachen.

Bergman. (2015). An Invitation to General Algebra and Universal Constructions, (2nd ed). Springer, Cham.

Birkhoff and J. D. Lipson.(1970). “Heterogeneous algebras,” Journal of Combinatorial Theory, vol. 8, pp. 115–133.

Birkhoff and O. Frink.(1948). “Representations of lattices by sets,” Transactions of the American Mathematical Society, vol. 64, pp. 299–316.

Bodirsky. (2023. Graph Homomorphisms and Universal Algebra Course Notes. TU Dresden, Germany.

Burris and H. P. Sankappanavar. (1981). A Course in Universal Algebra. Springer-Verlag, New York

Cohn. (1965). Universal Algebra. Harper & Row, New York.

Csákány. (2005). “Minimal clones—a minicourse,” Algebra Universalis, vol. 54, pp. 73–89.

Denecke and Leeratanavalee, S. (2003). “The Semantical Kernel Of a Generalized Hypersubstitution,” Southeast Asian Bulletin of Mathematics, vol. 27, no. 2, pp. 223–233.

Denecke and S. L. Wismath. (2002).Universal Algebra and Applications in Theoretical Computer Science. Chapman & Hall/CRC, Boca Raton.

Grätzer and W. A. Lampe . (1967). “On subalgebra lattices of universal algebras,” Journal of Algebra, vol. 7, pp. 263–270.

Grätzer. (1968). Universal Algebra. D. Van Nostrand Co., Inc., London

Higgins. (1963). “Algebras with a scheme of operators,” Mathematische Nachrichten, vol. 27, pp. 115–132.

McKenzie. (1987).G. F. McNulty, and W. F. Taylor, Algebras, Lattices, Varieties. Vol. I. Wadsworth & Brooks/Cole, Monterey, CA.

Whitehead, A. (1897.Treatise on Universal Algebra with Applications. Cambridge University Press, Cambridge.