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Sums of Reciprocal Triangular Numbers

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Abstract

We consider the infinite sums of the reciprocals of the triangular numbers T_{n^2} and $T_n{}^2$. Then, by applying the floor function to the reciprocals of these sums, we obtain the new identities involving the triangular numbers. Further, we give a formula for an alternating sum of the reciprocals of triangular numbers.

Keywords: triangular number, reciprocal sums

1. Introduction

Let $\{a_k\}_{k\geq 1}$ be a strictly increasing positive sequence such that $\sum_{k=1}^{\infty}(a_k)^{-1}$ is convergent. Many authors studied the partial infinite sums of reciprocal a_k,

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{a_k{}^s}\right)^{-1}\right\rfloor,$$

where $s > 1$ and $\lfloor x\rfloor$ is the greatest integer not exceeding x.

Ohtsuka and Nakamura (5) derived the formulas for the integer part of sums of reciprocal Fibonacci numbers, as follows:

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{F_k}\right)^{-1}\right\rfloor = \begin{cases} F_{n-2}, & n \text{ even} \\ F_{n-2}-1, & n \text{ odd,} \end{cases} \qquad (1.1)$$

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{F_k{}^2}\right)^{-1}\right\rfloor = \begin{cases} F_{n-1}F_n - 1, & n \text{ even} \\ F_{n-1}F_n, & n \text{ odd.} \end{cases}$$

Holliday and Komatsu (2) generalized a formula (1.1) for the generalized Fibonacci numbers. Similar properties were investigated in several different ways; see (3, 6). In (3), the author gave a similar formula (1.1) for alternating sums of reciprocal Fibonacci numbers as

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{F_k}\right)^{-1}\right\rfloor = (-1)^n F_{n+1} - 1, \qquad (1.2)$$

and the generalized Fibonacci numbers are shown in (4). Anantakitpaisal and Kuhapatanakul (1) generalized the formulas (1.1) and (1.2) to the tribonacci numbers.

Xin (7) studies the reciprocal sums related to the Riemann zeta function and showed that

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{-1}\right\rfloor = n-1,$$

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)^{-1}\right\rfloor = 2n(n-1).$$

Recently, Xu (8) gave the formulas for the sums of reciprocal k^4 and k^5.

Naturally, a following question arises: is there a similar formula for the triangular numbers? The triangular number T_n is a number obtained by adding all positive integers less than or equal to a given positive integer n,

$$T_n = \frac{n(n+1)}{2}$$

We know that the series $\sum \frac{1}{T_n{}^p}$ converses if $p > \frac{1}{2}$.

It is easy to verify that

$$\sum_{k=n}^{\infty}\frac{1}{T_k} = \sum_{k=n}^{\infty}\frac{2}{k(k+1)} = 2\sum_{k=n}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \frac{2}{n},$$

we get that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_k} \right)^{-1} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$$

The main purpose of this paper is to study the reciprocal sums of the triangular numbers T_{2n}, T_{2n-1}, T_{n^2} and the alternating sums of reciprocal triangular numbers.

2. Materials and Experiment

2.1 Applying the methods of proof of Xin (7) and Xu (8) to derive the formulas for integer part of inverse of sums and alternating sums of reciprocal triangular numbers.

3. Results and Discussion (Heading 1 Style)

3.1 We begin with the formulas for reciprocal sums of the even-indexed and odd-indexed triangular numbers.

Theorem 2.1 For any positive integer n, we have that

$$(i) \quad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_{2k}} \right)^{-1} \right\rfloor = 2n - 1$$

$$(ii) \quad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_{2k-1}} \right)^{-1} \right\rfloor = 2n - 2.$$

Proof. Consider

$$\begin{aligned} \frac{1}{2n} &= \sum_{k=n}^{\infty} \left(\frac{1}{2k} - \frac{1}{2k+2} \right) \\ &= \sum_{k=n}^{\infty} \frac{2}{(2k)(2k+2)} \\ &< \sum_{k=n}^{\infty} \frac{2}{(2k)(2k+1)} = \sum_{k=n}^{\infty} \frac{1}{T_{2k}} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2n-1} &= \sum_{k=n}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \\ &= \sum_{k=n}^{\infty} \frac{2}{(2k-1)(2k+1)} \\ &> \sum_{k=n}^{\infty} \frac{2}{(2k)(2k+1)} = \sum_{k=n}^{\infty} \frac{1}{T_{2k}}, \end{aligned}$$

we get that

$$\frac{1}{2n} < \sum_{k=n}^{\infty} \frac{1}{T_{2k}} < \frac{1}{2n-1}.$$

Hence

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_{2k}} \right)^{-1} \right\rfloor = 2n - 1.$$

On the other hand,

$$\begin{aligned} \frac{1}{2n-1} &= \sum_{k=n}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \\ &= \sum_{k=n}^{\infty} \frac{2}{(2k-1)(2k+1)} \\ &< \sum_{k=n}^{\infty} \frac{2}{(2k-1)(2k)} = \sum_{k=n}^{\infty} \frac{1}{T_{2k-1}} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2n-2} &= \sum_{k=n}^{\infty} \left(\frac{1}{2k-2} - \frac{1}{2k} \right) \\ &= \sum_{k=n}^{\infty} \frac{2}{(2k-2)(2k)} \\ &> \sum_{k=n}^{\infty} \frac{2}{(2k-1)(2k)} = \sum_{k=n}^{\infty} \frac{1}{T_{2k-1}}, \end{aligned}$$

we get that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_{2k-1}} \right)^{-1} \right\rfloor = 2n - 2,$$

so the part (ii) is always valid.

Next, we give a formula for an alternating sum of reciprocal triangular numbers.

Theorem 2.2 For any positive integer n we have

$$\left\lfloor \left(\left| \sum_{k=n}^{\infty} \frac{(-1)^k}{T_k} \right| \right)^{-1} \right\rfloor = n^2.$$

Proof. Observe that

$$\frac{1}{2\mathrm{k}+1} < \frac{2\mathrm{k}+1}{4\mathrm{k}(\mathrm{k}+1)},$$

then we have

$$\frac{1}{T_{2k}} - \frac{1}{T_{2k+1}} = \frac{1}{k(k+1)(2k+1)} < \frac{4(2k+1)}{(2k)^2(2k+2)^2} = \frac{1}{(2k)^2} - \frac{1}{(2k+2)^2}.$$

Likewise, it is easy to verify that

$$\frac{1}{k(k+1)(2k+1)} > \frac{4(2k+1)}{((2k)^2+1)((2k+2)^2+1)}.$$

We obtain that

$$\frac{1}{(2k)^2+1} - \frac{1}{(2k+2)^2+1} < \frac{1}{T_{2k}} - \frac{1}{T_{2k-1}} < \frac{1}{(2k)^2} - \frac{1}{(2k+2)^2},$$

and so

$$\frac{1}{4m^2+1} < \sum_{k=m}^{\infty}\left(\frac{1}{T_{2k}} - \frac{1}{T_{2k-1}}\right) < \frac{1}{4m^2}.$$

Therefore,

$$\left\lfloor\left(\left|\sum_{k=n}^{\infty}\frac{(-1)^k}{T_k}\right|\right)^{-1}\right\rfloor = (2m)^2.$$

Similarly, we can verify that

$$\frac{1}{(2m-1)^2+1} < \sum_{k=m}^{\infty}\left(\frac{1}{T_{2k-1}} - \frac{1}{T_{2k}}\right) < \frac{1}{(2m-1)^2}.$$

Therefore,

$$\left\lfloor\left(\left|\sum_{k=n}^{\infty}\frac{(-1)^k}{T_k}\right|\right)^{-1}\right\rfloor = (2m-1)^2.$$

This completes the proof.

Finally, we give the formula for the reciprocal sum of the squares of triangular numbers.

Theorem 2.3 For any positive integer n, we have

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{{T_k}^2}\right)^{-1}\right\rfloor$$

$$= \begin{cases} 6m^3 + \left\lfloor\frac{3}{10}m\right\rfloor, & if\ n = 2m, \quad m \neq 0(mod 10) \\ 6m^3 + \frac{3}{10}m - 1, & if\ n = 2m, \quad m \equiv 0(mod 10) \\ 6m^3 - 9m^2 + 5m - 1 - \left\lfloor\frac{m+4}{5}\right\rfloor, & if\ n = 2m-1. \end{cases}$$

Proof. For the case $n = 2m - 1$, we will show that

$$\frac{1}{6m^3 - 9m^2 + \frac{24}{5}m - \frac{9}{8}} < \sum_{k=m}^{\infty}\left(\frac{1}{{T_{2k-1}}^2} + \frac{1}{{T_{2k}}^2}\right) < \frac{1}{6m^3 - 9m^2 + \frac{24}{5}m - \frac{9}{5}}. \quad (2.1)$$

Set

$$f(k) = 6k^3 - 9k^2 + \frac{24}{5}k - \frac{9}{8},$$
$$g(k) = 6k^3 - 9k^2 + \frac{24}{5}k - \frac{9}{5}.$$

Observe that

$$\frac{1}{{T_{2k-1}}^2} + \frac{1}{{T_{2k}}^2} = \frac{8k^2+2}{k^2(2k-1)^2(2k+1)^2},$$
$$\frac{1}{f(k)} - \frac{1}{f(k+1)} = \frac{18k^2 + \frac{9}{5}}{f(k)f(k+1)}$$

and

$$\frac{1}{g(k)} - \frac{1}{g(k+1)} = \frac{18k^2 + \frac{9}{5}}{g(k)g(k+1)}.$$

We can easily check that

$$\frac{1}{f(k)} - \frac{1}{f(k+1)} < \frac{1}{{T_{2k-1}}^2} + \frac{1}{{T_{2k}}^2} < \frac{1}{g(k)} - \frac{1}{g(k+1)}.$$

Summing inequalities from $k = m$ to ∞, we get the inequality (2.1).
For the case $n = 2m$ and $m \equiv 0(mod 10)$, we will show that

$$\frac{1}{6m^3 - \frac{3}{10}m} < \sum_{k=m}^{\infty}\left(\frac{1}{{T_{2k}}^2} + \frac{1}{{T_{2k+1}}^2}\right) < \frac{1}{6m^3 - \frac{3}{10}m - 1}. \quad (2.2)$$

Set

$$f(k) = 6k^3 - \frac{3}{10}k,$$
$$g(k) = 6k^3 - \frac{3}{10}k - 1.$$

Observe that

$$\frac{1}{{T_{2k-1}}^2}+\frac{1}{{T_{2k}}^2}=\frac{2k^2+2k+1}{k^2(k+1)^2(2k+1)^2},$$
$$\frac{1}{f(k)}-\frac{1}{f(k+1)}=\frac{18k^2+18k+\frac{63}{10}}{f(k)f(k+1)}$$

and

$$\frac{1}{g(k)}-\frac{1}{g(k+1)}=\frac{18k^2+18k+\frac{63}{10}}{g(k)g(k+1)}.$$

We can easily check that

$$\frac{1}{f(k)}-\frac{1}{f(k+1)}<\frac{1}{{T_{2k}}^2}+\frac{1}{{T_{2k+1}}^2}<\frac{1}{g(k)}-\frac{1}{g(k+1)}.$$

Summing inequalities from $k=m$ to ∞, we get the inequality (2.2).

For the case $n=2m$ and $m\neq 0(mod10)$, $\left\lfloor\frac{3}{10}(m-1)\right\rfloor=\left\lfloor\frac{3}{10}m\right\rfloor$.

We will show that

$$\frac{1}{6m^3-\frac{3}{10}m}<\sum_{k=m}^{\infty}\left(\frac{1}{{T_{2k}}^2}+\frac{1}{{T_{2k+1}}^2}\right)<\frac{1}{6m^3-\frac{3}{10}m-\frac{3}{10}}. \tag{2.3}$$

Set

$$f(k)=6k^3-\frac{3}{10}k\,,$$
$$g(k)=6k^3-\frac{3}{10}k-\frac{3}{10}.$$

Observe that

$$\frac{1}{{T_{2k}}^2}+\frac{1}{{T_{2k+1}}^2}=\frac{2k^2+2k+1}{k^2(k+1)^2(2k+1)^2},$$
$$\frac{1}{f(k)}-\frac{1}{f(k+1)}=\frac{18k^2+18k+\frac{63}{10}}{f(k)f(k+1)}$$

and

$$\frac{1}{g(k)}-\frac{1}{g(k+1)}=\frac{18k^2+18k+\frac{63}{10}}{g(k)g(k+1)}.$$

We can easily check that

$$\frac{1}{f(k)}-\frac{1}{f(k+1)}<\frac{1}{{T_{2k}}^2}+\frac{1}{{T_{2k+1}}^2}<\frac{1}{g(k)}-\frac{1}{g(k+1)}.$$

Summing inequalities from $k=m$ to ∞, we get the inequality (2.3).

4. Conclusions

We prove new formulas for integer part of inverse of sums and alternating sums of reciprocal triangular numbers. The results are showed in the paper.

Declaration of conflicting interests

The authors declared that they have no conflicts of interest in the research, authorship, and this article's publication.

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