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Mathematical Model of MMR Inversion for a DISC Embedded in Overburden

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Abstract

In this paper, inverse problem with the use of optimization technique is proposed. Mathematical model of steady state magnetic field response is formulated. It is accomplished by using analytical method to solve boundary value problems in the wave number domain and then transforming back to the special domain. One dimensional geometric model of a two layered earth is considered. Probe sources of direct current are located perpendicularly in overburden. There is an ore body like a disc of radius c embedded in overburden. Magnetic field response is computed numerically to see their behavior against source-receiver spacing. The results show that, there are some relations between magnetic field responses and conductivity parameters or overburden thickness significantly as mentioned in some related works. Moreover, the magnetic field responses also depend on the size of disc as well. In our inversion process, conjugate gradient can be used to investigate radius of a disc embedded in overburden accurately.

Keywords: Magnetic, Hankel Transforms, Conjugate gradient

1. Introduction

Magnetometric Resistivity Method (MMR) was developed for more than four decades. Following the most popular article "On the theory of Magnetometric Resistivity Methods(MMR)" conducted by Edwards(4), Edwards and et al.(5), Magnetometric Resistivity Method (MMR) is based on the measurement of low-level, low-frequency magnetic fields associated with non-inductive current flow underground surface. There are a few quantitative interpretational schemes for deriving resistivity from MMR data. The magnetic fields produced by the current in wire between the two electrodes effect to the ground surface and effect to any conductivity boundaries in the ground. Historically, for a surface MMR survey, the wire connecting the two current electrodes is typically plugged into ground surface and data measurements are made somewhere in between the electrode spread. Geometric model of ground is designed as a layered earth. Information concerning the conductivity distribution beneath the surface and the layered thickness are then extracted with the aid of optimization techniques. In our study, two layered earth, one dimensional conductivity ground profile are proposed. Here, we assume that there is an ore body embedded in binomially overburden at $z = h_d$, $0 < h_d \le h$. The shape of an ore body is assumed to be a disc (7). This assumption has been proposed in electromagnetic method but not in MMR history. With the use of conjugate gradient, the radius of a disc can be investigated.

2. Governing Equations of Magnetic Field Due to a Semi-infinite Source of Two Layered Earth Model

A semi-infinite vertical wire DC source carries an exciting current I is located on the ground. The electrode AB is placed deliberately at the interface $z = h$ of overburden and host. There is an ore body embedded in overburden region. The shape of an ore body is assumed to be a disc of radius c as mentioned by Siew and Yooyuanyong (7). The location of disc is at $z = h_d$, $0 < h_d \le h$ as shown in Figure 1. The Maxwell's equations can be used to determine the magnetic field intensity $\vec{H}$ as (2, 7)

$$\nabla \times \vec{E} = \vec{0} \tag{2.1}$$

and

$$\nabla \times \vec{H} = \sigma \vec{E} + \vec{J}_d, \tag{2.2}$$

where $\vec{E}$ is electric field intensity, $\vec{H}$ is magnetic field intensity, σ is conductivity of medium, ∇ is the Gradient operator and $\vec{J}_d$ is current density on disc. Using Eq. (2.1) and (2.2), for the case of no disc, yield

$$\nabla \times \frac{1}{\sigma} \nabla \times \vec{H} = \vec{0}. \tag{2.3}$$

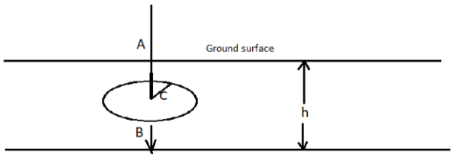

Figure 1 Geometric model of two-layered earth structure with a disc.

In cylindrical coordinates (r, φ, z), Eq. (2.3) can be expressed in terms of three unit vectors $\vec{e}_r$, $\vec{e}_\varphi$ and $\vec{e}_z$ as (8-11)

$$\left\{ \frac{1}{r} \frac{\partial}{\partial \varphi} \frac{1}{\sigma} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r H_\varphi \right) - \frac{1}{r} \frac{\partial H_r}{\partial \varphi} \right] - \frac{\partial}{\partial z} \frac{1}{\sigma} \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right\} \vec{e}_r + \left\{ \frac{\partial}{\partial z} \frac{1}{\sigma} \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right) - \frac{\partial}{\partial r} \frac{1}{\sigma} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r H_\varphi \right) - \frac{1}{r} \frac{\partial H_r}{\partial \varphi} \right] \right\} \vec{e}_\varphi + \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{1}{\sigma} \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \right] - \frac{1}{r} \frac{\partial}{\partial \varphi} \frac{1}{\sigma} \left(\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} \right) \right\} \vec{e}_z = \vec{0},$$

where H_r, H_φ and H_z are magnetic field components in $\vec{e}_r$, $\vec{e}_\varphi$ and $\vec{e}_z$, respectively. Since the problem is axi-symmetric, and $\vec{H}$ has only an azimuthal component in cylindrical coordinate, for simply, we use H to represent H_φ thus, we obtain

$$\frac{\partial}{\partial z} \left(\frac{1}{\sigma} \frac{\partial H}{\partial z} \right) + \frac{\partial}{\partial r} \left(\frac{1}{\sigma r} \frac{\partial}{\partial r} (rH) \right) = 0. \tag{2.4}$$

For simply, we denote σ as a function of depth z only, thus, Eq. (2.4) becomes

$$\frac{\partial^2 H}{\partial z^2} + \sigma \left(\frac{\partial}{\partial z} \left(\frac{1}{\sigma} \right) \right) \left(\frac{\partial H}{\partial z} \right) + \frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} - \frac{1}{r^2} H = 0. \tag{2.5}$$

We introduce the Hankel transforms pair (1), as

$$\tilde{H}(\lambda, z) = \int_0^\infty r H(r, z) J_1(\lambda r) dr \tag{2.6}$$

and

$$H(r, z) = \int_0^\infty \lambda \tilde{H}(\lambda, z) J_1(\lambda r) d\lambda, \tag{2.7}$$

where J_1 is the Bessel function of the first kind of order one. λ is a Hankel variable. Applying Eq. (2.6) to Eq. (2.5), we obtain

$$\frac{d^2 \tilde{H}}{dz^2} + \sigma \frac{d}{dz} \left(\frac{1}{\sigma} \right) \frac{d\tilde{H}}{dz} - \lambda^2 \tilde{H} = 0. \tag{2.8}$$

3. Derivation of Magnetic Field Response from Transitional Ground Profile

In the nature of the Earth structure, ground has various structures. A commonly structure that usually can be found is an overburden located on host rock. The conductivity of an overburden can be denoted by

$$\sigma(z) = a(1+z)^{\frac{m}{n}}, \ 0 \le z \le h, \ 0 < a \in R, \ m, n \in Z, n \ne 0,$$

where R is the set of real number and Z is the set of integer. Host rock has high resistivity and the small positive value of constant conductivity can be approximately used. Substituting $\sigma(z) = a(1+z)^{\frac{m}{n}}$ into Eq. (2.8), we obtain ordinary differential equation as (9)

$$\frac{d^2 \tilde{H}}{dz^2} - \frac{m}{n(1+z)} \frac{d\tilde{H}}{dz} - \lambda^2 \tilde{H} = 0. \tag{3.1}$$

The solution to Eq. (3.1) is given by (9)

$$\tilde{H}(\lambda, z) = (1+z)^{\frac{m+n}{2n}} \begin{bmatrix} A_1 I_{\frac{m+n}{2n}} \left(\lambda(1+z) \right) \\ + B_1 K_{\frac{m+n}{2n}} \left(\lambda(1+z) \right) \end{bmatrix}, \tag{3.2}$$

where A_1 and B_1 are arbitrary constants. $I_{\frac{m+n}{2n}}$ and $K_{\frac{m+n}{2n}}$ are the modified Bessel functions of the first and second kind of order $\frac{m+n}{2n}$, respectively.

4. Derivation of Magnetic Field Response from Homogeneous Ground Profile

In host region, the conductivity is constant and denoted by $\sigma(z) = b, z > h$, b and h are non-negative real number. The Eq. (2.8) can be simplified to be

$$\frac{d^2\tilde{H}}{dz^2} - \lambda^2 \tilde{H} = 0 \,,$$

and the solution is denoted by (9)

$$\tilde{H}(\lambda, z) = A_2 e^{\lambda(z-h)} + B_2 e^{-\lambda(z-h)}, \qquad (4.1)$$

where A_2 and B_2 are arbitrary constants that can be determined by using boundary conditions. In our study, we consider for two-layered earth model as shown in Figure 1. We design the conductivity of ground for overburden and host rock, respectively, as

$$\sigma_{over}(z) = a(1+z)^{\frac{m}{n}}, \, 0 \le z \le h,$$
$$\sigma_{host}(z) = b, \, z > h$$

where a and b are non-negative constants. For the first layer, magnetic field consists of three parts caused by ground, probe source and disc. The first part of magnetic field is responded from overburden and given by (9)

$$\tilde{H}(\lambda, z) = (1+z)^{\frac{m+n}{2n}} \begin{bmatrix} A_1 I_{\frac{m+n}{2n}}\left(\lambda(1+z)\right) \\ +B_1 K_{\frac{m+n}{2n}}\left(\lambda(1+z)\right) \end{bmatrix}$$

The second part, magnetic field dues to the probe source (8), which is defined by Ampere's law (9)

$$H(r,z) = \frac{I}{2\pi r}.$$

By using the Hankel Transforms (1) as described in Eq. (2.6), we obtain

$$\tilde{H}(\lambda, z) = \frac{I}{2\pi\lambda}.$$

The third part, magnetic field dues to the disc embedded in overburden can be derived from the Maxwell's equation as in Eq. (2.1) and Eq. (2.2). With the use of some algebraic operation, we obtain

$$\nabla \times \vec{H} = \vec{J}_d \,.$$

In cylindrical coordinates (r, φ, z), we rewrite above equation in $\vec{e}_r$, $\vec{e}_\varphi$ and $\vec{e}_z$ components as

$$\left(\vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\varphi}{r}\frac{\partial}{\partial \varphi} + \vec{e}_z \frac{\partial}{\partial z}\right) \times \left(H_r \vec{e}_r + H_\varphi \vec{e}_\varphi + H_z \vec{e}_z\right)$$
$$= J^r \vec{e}_r + J^\varphi \vec{e}_\varphi + J^z \vec{e}_z.$$

Since the magnetic field has only $\vec{e}_\varphi$ component, thus, the above equation becomes

$$\frac{\partial H_\varphi}{\partial z} = J^r = \left(\sigma_d - \sigma_{ove}\right) E_d^r$$
$$= \frac{I(\omega) c \delta(r-a) \delta(z-h)}{r} \,,$$

where σ_d is conductivity on disc, E_d^r is electric field on disc in radial direction, $I(\omega)$ is electric current on disc, c is radius of disc and δ is Dirac Delta function. Therefore, the magnetic field in overburden can be written as

$$\tilde{H}_{over}(\lambda, z) = \frac{I}{2\pi\lambda} + (1+z)^{\frac{m+n}{2n}} \times$$
$$\left[A_1 I_{\frac{m+n}{2n}}\left(\lambda(1+z)\right) + B_1 K_{\frac{m+n}{2n}}\left(\lambda(1+z)\right)\right] + \frac{I(\omega) c}{\lambda}.$$

For the second layer, host region, the magnetic field solution can be written by

$$\tilde{H}_{host}(\lambda, z) = A_2 e^{\lambda(z-h)} + B_2 e^{-\lambda(z-h)} \,.$$

5. Boundary Conditions

The arbitrary constants in magnetic field solutions obtained from equation (3.1) can be found by using the following boundary conditions (10, 11):

(1) The magnetic field is continuous at the interface of each layer

$$H_{over}(r,z)\,|_{z=h^-} = H_{host}(r,z)\,|_{z=h^+}$$

(2) The radial component of electric field is continuous at the interface of each layer

$$\lim_{z \to h^-} E^r_{over}(r,z) = \lim_{z \to h^+} E^r_{host}(r,z) \,,$$

where E^r_{over} and E^r_{host} are radial electric fields in overburden and host rock, respectively.

(3) As the depth z tends to infinity, the magnetic field tends to zero.

(4) Since no current across the Air-Earth interface, then $\sigma_{over}(z) E^z_{over}(r,z)\,|_{z=0} = 0$, where E^z_{over} is an electric field in vertical direction in overburden.

Applying the above boundary conditions and taking inverse Hankel Transforms as in Eq. (2.7), with $m=1, n=2, a=1, b \to 0^+, h=3$ we obtain the magnetic field solutions on the ground surface as

$$H_{over}(r,0)=\int_0^\infty \lambda\left[\frac{I}{2\pi\lambda}+A_1 I_{\frac{3}{4}}(\lambda)+B_1 K_{\frac{3}{4}}(\lambda)+\frac{I(\omega)c}{\lambda}\right]J_1(\lambda r)\,d\lambda, \quad (5.1)$$

where

$$A_1=\frac{\frac{I}{8}\left[(2\pi-1)\sqrt{2}K_{\frac{3}{4}}(\lambda)-(2\pi-1)4K_{\frac{3}{4}}(4\lambda)\right]}{\lambda\pi\left[I_{\frac{3}{4}}(4\lambda)K_{\frac{3}{4}}(\lambda)-K_{\frac{3}{4}}(4\lambda)I_{\frac{3}{4}}(\lambda)\right]},$$

and

$$B_1=\frac{\frac{\sqrt{2}I}{8}\left[(2\pi-1)2\sqrt{2}I_{\frac{3}{4}}(4\lambda)-(2\pi-1)I_{\frac{3}{4}}(\lambda)\right]}{\lambda\pi\left[I_{\frac{3}{4}}(4\lambda)K_{\frac{3}{4}}(\lambda)-K_{\frac{3}{4}}(4\lambda)I_{\frac{3}{4}}(\lambda)\right]}.$$

6. Numerical Experiments

The magnetic field as described in Eq. (5.1) can be computed by using Chave's Algorithm (3). In general, the behavior of magnetic field response from high conductive ground will be very strong. We hope to see some signal from graph to indicate the information underground surface such the thickness of overburden and conductivity parameters. In our numerical experiments, we perform the size of the conductive disc effect to magnitude of magnetic field to support our mathematical model. For our initial case, we start with overburden thickness $h=3\,meters$, the conductivity parameters $a=1\,S/m, b=0\,S/m, \ m=1, n=2.$ $I(\omega)$ is electric current on disc which is approximately to be the maximum value of the electric current flow from probe source. The radius of disc is varied such $c=0,1,2,3m$. Numerical results for magnetic fields on ground surface are performed as in Figure 2.

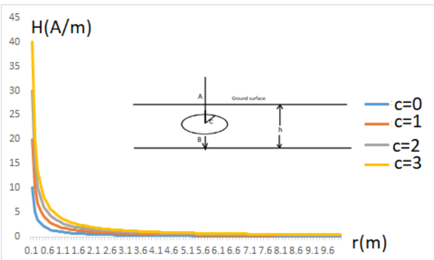

Figure 2 Graph of magnetic fields against source-receiver spacing, using $I=1$ *Ampere*.

As shown in Figure 2, the curve of magnetic fields against source-receiver spacing (r) are plotted at various size of disc, $c=0,1,2,3\,meters$ with the used of electric current (I) equal to 1 *Ampere*. We can see that the magnetic fields drop very fast to zero as the source-receiver spacing is increased. This corresponds to the work done by Khonkhem and Yooyuanyong (6). At larger size of disc, the magnetic field is stronger response from ground and can be shown in Figure 2. For our second case, the electric current is varied, $I=1,2,3\,Amperes$. Radius of disc is given by $c=3\,meters$. Overburden thickness is $h=3\,meters$ and the conductivity parameters are denoted by $a=1\,S/m, b=0\,S/m, \ m=1, n=2.$ $I(\omega)$ is electric current on disc which is approximately to be the maximum value of the electric current flow from probe source. Numerical results for magnetic fields on ground surface are performed as in Figure 3.

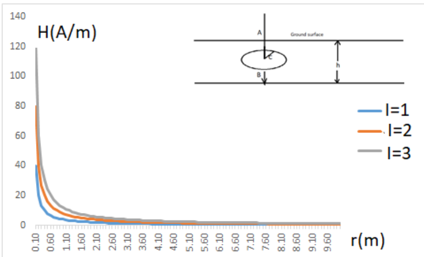

Figure 3. Graph of magnetic fields against source-receiver spacing at various electric currents.

7. Inversion Process

The most important problem in mining is how much of an ore buried under ground. In our context here, the calculation and measurement of magnetic fields are compared to find the size of ore body. Conjugate gradient method is an algorithm for our numerical solution of particular systems of linear equations. The conjugate gradient method is implemented as an iterative algorithm. The relative error of magnetic fields are used to terminate the iterative process. In our inversion example, synthesis data is formulated by using Eq. (5.1). The parameters used in our inversion example are the electric current $I=3\,Amperes$, radius of disc $c=3\,meters$, overburden thickness $h=3\,meters$ and the conductivity parameters denoted by $a=1\,S/m, b\approx 0\,S/m, m=1, n=2.$ $I(\omega)$ is electric current on disc which is approximately to be the maximum value of the electric current flow from probe source. Two percent of Gauss error is added to perturb our data as noise signal. Inversion process is started by using initial guess $c=1m.$ As shown in

Table 1, conjugate gradient is used only 6 iterations to get solution accurately. To confirm our mathematical model, the second example is proposed by using initial guess $c = 5\,m.$ The results are shown in Table 1 accurately with only 6 iterations as well on Desktop Intel®Core™I5-8250U CPU@1.8 GHz. The speed of convergence is similar to the work done by Yooyuanyong (9).

Table 1 Inversion results of conjugate gradient using initial guess c=1 m. and c=5 m.

Calculation Results	**1st(*initial guess*)**	**2nditeration**	**3rditeration**	**4thiteration**	**5thiteration**	**6thiteration**
Radius of disc	c=1.000	c=4.372	c=3.460	c=2.790	c=2.899	c=3.001
Relative error	5.65E-03	9.62E-04	1.27E-04	4.49E-05	6.54E-07	2.26E-08
Radius of disc	c=5.000	c=4.175	c=1.966	c=2.644	c=2.949	c=3.001
Relative error	9.01E-04	6.75E-04	5.57E-04	4.86E-05	2.26E-07	2.25E-08

8. Conclusions

In this paper, we propose a new approach to investigate an ore body buried under the ground by using Magnetometric Resistivity Method (MMR). Following Siew and Yooyuanyong (7) in electromagnetic method, the body of an ore is assumed to be a disc of radius c at $z = h_d$ beneath ground surface in overburden. It is accomplished by solving a boundary value problem in the wave number domain and then transforming back to the spatial domain. We consider two layered Earth model in our study. The magnetic field can be computed to see the behavior by using Chave's algorithm (3). Numerical results due to Direct Current source on the ground surface are shown in Figure 2 and Figure 3. In our forward modeling, with the use of varying radius of the disc $c = 0, 1, 2, 3\ meters$, the curves of magnetic field drop rapidly as we increase the source-receiver spacing (r). With the use of three values of electric current $I = 1, 2\ and\ 3\ Amperes$, the curves of magnetic field drop in a similar manner. Unfortunately, the curves of magnetic field do not give any fluctuation related to the conductivity profile of the ground. There are very few relations between magnitude of magnetic fields and conductivity parameters which imply the conductive of ground as mention in above section. In our inversion for radius of the disc, conjugate gradient is used. Two examples are performed to show very good convergence of the solutions at 6 iterations only.

Declaration of conflicting interests

The authors declared that they have no conflicts of interest in the research, authorship, and this article's publication.

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