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**On the value distribution of two Dirichlet  $L$ -functions  
over sums of two zeros of the Riemann zeta-function**

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**Abstract**

Let  $\rho = \beta + i\gamma$  and  $\rho' = \beta' + i\gamma'$  denote the non-trivial zeros of the Riemann zeta-function  $\zeta(s)$ , with  $\gamma, \gamma' > 0$  and  $\sigma \in (1/2, 1)$ . Under the assumption of the Riemann Hypothesis, we show that for a positive proportion of  $1/2 + i(\gamma + \gamma')$ , the values of the Dirichlet  $L$ -function  $L(\sigma + i(\gamma + \gamma'), \chi_1)$  and  $L(\sigma + i(\gamma + \gamma'), \chi_2)$  associated with the primitive characters  $\chi_1$  and  $\chi_2$  to different prime moduli are linearly independent over  $\mathbb{R}$ .

**Keywords:** Dirichlet  $L$ -function, Riemann zeta-function, non-trivial zero

**1. Introduction**

Let  $s = \sigma + it$  be a complex variable.

The Riemann zeta-function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad \sigma > 1.$$

In most application of zeta-function theory, information about  $\zeta(s)$  for  $\sigma \leq 1$  is of interest.

Riemann [5] proved that  $\zeta(s)$  is regular for all

values of  $s$  except a simple pole at  $s = 1$  with residue 1 and for all complex  $s$ ,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

This shows that  $\zeta(-2n) = 0$  for all  $n \in \mathbb{N}$ . The even negative integers are called trivial zeros of  $\zeta(s)$ . The non-trivial zeros of  $\zeta(s)$  lie inside the strip  $0 < \sigma < 1$  and denoted by  $\rho = \beta + i\gamma$ .

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Thus, the investigation the location of non-trivial zeros of  $\zeta(s)$  is importance for application of zeta-function theory. The most famous conjectures of mathematics, the Riemann hypothesis, states that all non-trivial zeros of  $\zeta(s)$  have a real part  $\sigma = 1/2$ . Lindelöf hypothesis, another famous conjecture which is a consequence of the Riemann hypothesis, asserts that  $\zeta(1/2 + it) \ll t^\varepsilon$  for any  $\varepsilon > 0$  and  $|t| \rightarrow \infty$ . Titchmarsh [ 7] gave equivalent formulations of the Lindelöf hypothesis in the sense of the value distribution of  $\zeta(s)$ , namely

$$\frac{1}{T} \int_1^T |\zeta(1/2 + it)|^{2k} dt = O(T^\varepsilon), \quad k \in \mathbb{N}.$$

Thus, the value distribution of  $\zeta(s)$  attracted many authors [ 6] . In this paper, we focus on the distribution of their zeros ( non- trivial) on the product of Dirichlet  $L$ -function. The Dirichlet  $L$ -function defined for  $\sigma > 1$  by the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s},$$

where  $\chi(n)$  denote the Dirichlet character. Let  $\rho_\chi = \beta_\chi + i\gamma_\chi$  denote the non-trivial zeros of  $L(s, \chi)$ . A well-known conjecture, the Grand Simplicity Hypothesis (GSH), asserts that the set

$$\{\gamma_\chi \geq 0 \mid L(1/2 + i\gamma_\chi, \chi) = 0\},$$

where  $\chi$  runs through all primitive characters are linearly independent over  $\mathbb{Q}$  . This implies that all zeros of Dirichlet  $L$ - functions to primitive characters are simple and two Dirichlet  $L$ -functions with distinct primitive characters cannot share any

non- trivial zeros. A zero of the product of two functions is called distinct if it is a zero of only one of the two or it is a zero of both but with different multiplicities for each function. In 1976, Fujii [2] showed that a positive proportion of zeros of  $L(s, \psi)L(s, \chi)$  are distinct, where  $\psi$  and  $\chi$  are distinct primitive character to the same moduli. Two non-zero complex numbers  $z$  and  $w$  are called linearly independent over  $\mathbb{R}$  if and only if the quotient  $z/w$  are non- real or  $|z\bar{w} - \bar{z}w| > 0$  . Recently, N. Laaksonen and Y.N. Petridis [3] studied the linear independence over  $\mathbb{R}$  of the values of  $L(\rho + \sigma, \chi_1)$  and  $L(\rho + \sigma, \chi_2)$  , where  $\sigma \in (0, 1/2)$  is fixed and  $\chi_1, \chi_2$  are two fixed primitive characters to distinct prime moduli. They proved that:

**Theorem 1.1.** Assume the Riemann Hypothesis. Let  $\chi_1, \chi_2$  be two primitive characters modulo distinct primes  $q$  and  $\ell$ , respectively. Let  $\sigma \in (1/2, 1)$  , then, for a positive proportion of the non- trivial zeros of  $\zeta(s)$  with  $\gamma > 0$  , the values of the Dirichlet  $L$ - functions  $L(\sigma + i\gamma, \chi_1)$  and  $L(\sigma + i\gamma, \chi_2)$  are linearly independent over  $\mathbb{R}$  .

In this paper, we shall extend following the results of N. Laaksonen and Y.N. Petridis in [3] to Dirichlet  $L$ - function  $L(\sigma + i(\gamma + \gamma'), \chi_1)$  and  $L(\sigma + i(\gamma + \gamma'), \chi_2)$  associated with the primitive characters  $\chi_1$  and  $\chi_2$  to different prime moduli and  $\sigma \in (1/2, 1)$  .

### 2. Preliminaries

In this section, basic definitions, terminologies, and assumptions throughout this work are gathered.

**Lemma 2.1** [1]. For  $T > T_0$ , we have

$$\begin{aligned}
 N_2(T) &:= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} 1 \\
 &= \frac{1}{8\pi^2} T^2 \log^2 T - \frac{1}{8\pi^2} (3 + 2 \log 2\pi) T^2 \log T \\
 &\quad + \frac{1}{16\pi^2} (7 + 6 \log 2\pi + 2 \log^2 2\pi - 2\zeta(2)) T^2 \\
 &\quad + O\left(T \frac{\log^2 T}{\log \log T}\right).
 \end{aligned}$$

**Lemma 2.2** [1]. Assume the Riemann Hypothesis.

For any  $\varepsilon > 0$ ,  $x > 1$  and  $T > T_0$ , we have

$$\begin{aligned}
 \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} x^{1+i(\gamma+\gamma')} &= \frac{1}{8\pi^2} \Lambda^2(x) T^2 \\
 &\quad + O(x^{3/2+\varepsilon} T \log^2 T) \\
 &\quad + O\left(x^{1/2+\varepsilon} T \log T \min\left(T, \frac{x}{\langle x \rangle}\right)\right),
 \end{aligned}$$

where  $\langle x \rangle$  denotes the distance from  $x$  to the nearest prime power other than  $x$  itself and  $\Lambda(x) = \log p$  if  $x = p^k$  for some prime  $p$  and integer  $k \geq 1$ , and  $\Lambda(x) = 0$  for all other real values of  $x$ .

Let  $q$  be a positive integer and  $\chi$  be a Dirichlet character mod  $q$ . We denote the Gauss sum by

$$G(k, \chi) = \sum_{a=1}^q \chi(a) e^{2\pi i a k / q}$$

and also write  $G(1, \chi) = G(\chi)$ .

**Theorem 2.3** [4]. Let  $\chi$  be a primitive character mod  $q$ . For  $s = \sigma + it$  with  $0 < \sigma < 1$ ,  $t > 0$ , and  $x = \Delta \sqrt{\frac{qt}{2\pi}}$ ,  $y = \Delta^{-1} \sqrt{\frac{qt}{2\pi}}$  and  $\Delta \geq 1$ ,  $\Delta \in \mathbb{N}$ ,

we have

$$\begin{aligned}
 L(s, \chi) &= \sum_{n \leq x} \frac{\chi(n)}{n^s} \\
 &\quad + \varepsilon(\chi) \left(\frac{q}{\pi}\right)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1-s+\alpha}{2}\right)}{\Gamma\left(\frac{s+\alpha}{2}\right)} \sum_{n \leq y} \frac{\bar{\chi}(n)}{n^{1-s}} \\
 &\quad + R_{xy},
 \end{aligned}$$

with

$$R_{xy} \ll \sqrt{q} y^{-\sigma} + x^{\sigma-1} (qt)^{1/2-\sigma} \log 2t,$$

and in particular, for  $x = y$ ,

$$R \ll x^{-\sigma} \sqrt{q} \log 2t.$$

Here  $\varepsilon(\chi) = q^{-1/2} i^\alpha G(\chi)$ , and

$$\alpha = \frac{1 - \chi(-1)}{2}.$$

In fact, this formula is valid for  $\Delta \in \mathbb{R}_{\geq 0}$  [3].

Throughout this paper, we consider  $\chi_1$  and  $\chi_2$  as primitive characters modulo with distinct primes  $q$  and  $\ell$ , respectively. For Proposition 3.5, 3.6 and 3.7, we denote the set of primes smaller than prime  $P$  in an increasing order by  $\mathcal{P} = \{p_1, p_2, \dots, P\}$  such that  $P > \max(q, \ell)$ . We need the weight product

$$B(s, P) = \prod_{p \leq P} (1 - \chi_1(p) p^{-s})(1 - \chi_2(p) p^{-s})$$

in order to make sums balanced. Suppose that this product have the expansion

$$B(s, P)L(s, \chi_1) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

and

$$B(s, P)L(s, \chi_2) = \sum_{n=1}^{\infty} \frac{e(n)}{n^s}.$$

By [3], we have

$$d(n) = \begin{cases} \chi_1(n) & \text{if } p \nmid n \text{ for all } p \in \mathcal{P}, \\ (-1)^k \chi_1(m)\chi_2(k), & \text{if } p_{i_j} \parallel n \text{ for all } j, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$e(n) = \begin{cases} \chi_2(n) & \text{if } p \nmid n \text{ for all } p \in \mathcal{P}, \\ (-1)^k \chi_2(m)\chi_1(k), & \text{if } p_{i_j} \parallel n \text{ for all } j, \\ 0, & \text{otherwise,} \end{cases}$$

where  $m = n / (p_{i_1} \cdots p_{i_k})$  and  $k = p_{i_1} \cdots p_{i_k}$ .

Then,  $d(n) \ll 1$  and  $e(n) \ll 1$ . Suppose that for a fixed  $t$ ,

$$B(s, P) \sum_{n \leq \sqrt{\frac{qt}{2\pi}}} \chi_1(n)n^{-s} = \sum_{n \leq R\sqrt{\frac{qt}{2\pi}}} d'(n)n^{-s},$$

for some  $R$  depending on  $P$ . By [3], we have

$$d'(n) \ll 1.$$

### 3. Main Theorem

The proof will follow the steps of [3], so the following lemmas and propositions are needed.

**Lemma 3.1.** Assume the Riemann Hypothesis; let  $\sigma \in (1/2, 1)$ , let  $c$  be a positive constant, let  $a(n)$  and  $b(n)$  be arithmetic functions such that  $a(n) \ll 1$  and  $b(n) \ll 1$  for all  $n \in \mathbb{N}$ . Then,

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{n, m \leq c\sqrt{\gamma + \gamma'} \\ n \neq m}} \frac{a(m)b(n)}{(nm)^{1-\sigma}} \left(\frac{n}{m}\right)^{i(\gamma + \gamma')} \tag{3.1}$$

$$\ll T^{\sigma-1/2} N_2(T).$$

*Proof.* Denote this sum by  $S$ . By the same method of [3], we have

$$S = \sum_{n \leq c\sqrt{T}} \sum_{m < n} \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \binom{n}{c} \leq \gamma + \gamma' \leq T}} \left( \frac{a(m)b(n)}{n^{2-\sigma} m^{-\sigma}} \left(\frac{n}{m}\right)^{\rho + \rho'} + \frac{a(n)b(m)}{n^{2-\sigma} m^{-\sigma}} \left(\frac{n}{m}\right)^{\rho + \rho'} \right)$$

$$= \sum_{n \leq c\sqrt{T}} \sum_{m < n} \left( \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < \gamma + \gamma' \leq T}} - \sum_{\substack{0 < \gamma, \gamma' \leq \left(\frac{n}{c}\right)^2 \\ 0 < \gamma + \gamma' \leq \left(\frac{n}{c}\right)^2}} \right)$$

$$\left( \frac{a(m)b(n)}{n^{2-\sigma} m^{-\sigma}} \left(\frac{n}{m}\right)^{\rho + \rho'} + \frac{a(n)b(m)}{n^{2-\sigma} m^{-\sigma}} \left(\frac{n}{m}\right)^{\rho + \rho'} \right).$$

By Lemma 2.2, we have

$$S = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 = \frac{1}{8\pi^2} T^2 \sum_{n \leq c\sqrt{T}} \sum_{m < n} \frac{a(m)b(n) + a(n)b(m)}{n^{2-\sigma} m^{-\sigma}} \Lambda^2 \times \left(\frac{n}{m}\right),$$

$$S_2 \ll \sum_{n \leq c\sqrt{T}} \sum_{m < n} n^{\sigma+2} m^{\sigma} \Lambda^2 \left(\frac{n}{m}\right),$$

$$S_3 \ll T \log^2 T \sum_{n \leq c\sqrt{T}} \sum_{m < n} \frac{1}{n^{1/2-\sigma-\varepsilon} m^{3/2-\sigma+\varepsilon}},$$

and

$$S_4 \ll T \log T \times \sum_{n \leq c\sqrt{T}} \sum_{m < n} \frac{1}{n^{3/2-\sigma-\varepsilon} m^{1/2-\sigma+\varepsilon}} \min\left(T, \frac{n/m}{\langle n/m \rangle}\right).$$

First, we consider  $S_1$ . The term  $\Lambda(n/m)$  maybe non-vanish when  $m \mid n$ . Thus, by writing  $n = km$ , we have

$$\begin{aligned}
 S_1 &\ll T^2 \sum_{k \leq c\sqrt{T}} \sum_{m \leq \frac{c}{k}\sqrt{T}} \frac{\Lambda^2(k)}{k^{2-\sigma} m^{2(1-\sigma)}} \\
 &\ll T^2 \sum_{k \leq c\sqrt{T}} k^{\sigma-2+2\varepsilon} \sum_{m \leq \frac{c}{k}\sqrt{T}} m^{2(\sigma-1)} \\
 &\ll T^2 \sum_{k \leq c\sqrt{T}} k^{\sigma-2+2\varepsilon} \left( \left( \frac{1}{k} \right)^{2\sigma-1} + 1 \right) \\
 &\ll T^{\sigma+3/2} \sum_{k \leq c\sqrt{T}} k^{2\varepsilon-\sigma-1} \ll T^{\sigma+3/2},
 \end{aligned}$$

since  $\Lambda(k) \ll k^\varepsilon$  for any  $\varepsilon > 0$  and the last sums are convergent. Similarly, for  $S_2$ , we have

$$\begin{aligned}
 S_2 &\ll \sum_{k \leq c\sqrt{T}} \sum_{m < \frac{c}{k}\sqrt{T}} k^{\sigma+2} \Lambda^2(k) m^{2(\sigma+1)} \\
 &\ll \sum_{k \leq c\sqrt{T}} k^{\sigma+2+2\varepsilon} \sum_{m < \frac{c}{k}\sqrt{T}} m^{2(\sigma+1)} \\
 &\ll T^{\sigma+3/2} \sum_{k \leq c\sqrt{T}} k^{2\varepsilon-\sigma-1} \ll T^{\sigma+3/2}.
 \end{aligned}$$

Next, we see that

$$S_3 \ll T \log^2 T \sum_{n \leq c\sqrt{T}} n^{2\sigma-1} \ll T^{1+\sigma} \log^2 T.$$

For  $S_4$ , we write  $n = um + r$ , where

$-m/2 < r \leq m/2$ , and obtain

$$\begin{aligned}
 &\langle u + r/m \rangle \\
 &= \begin{cases} \frac{|r|}{m}, & \text{if } u \text{ is a prime power} \\ & \text{and } r \neq 0, \\ \geq \frac{1}{2}, & \text{otherwise} \end{cases} \quad (3.2)
 \end{aligned}$$

(cf. [3]). Thus,

$$\begin{aligned}
 S_4 &\ll T \log T \\
 &\times \sum_{n \leq c\sqrt{T}} \sum_{m < n} \frac{1}{n^{3/2-\sigma-\varepsilon} m^{1/2-\sigma+\varepsilon}} \frac{n \langle n/m \rangle^{-1}}{m} \\
 &\ll T \log T \\
 &\times \sum_{m \leq c\sqrt{T}} \sum_{u \leq \lfloor c\sqrt{T}/m \rfloor + 1} \sum_{-\frac{m}{2} < r \leq \frac{m}{2}} \frac{\langle u + r/m \rangle^{-1}}{m^{3/2-\sigma+\varepsilon} (um+r)^{1/2-\sigma-\varepsilon}}.
 \end{aligned}$$

By determining whether the term  $u$  is a prime power or not, we have

$$\begin{aligned}
 S_4 &\ll T \log T \\
 &\times \sum_{m \leq c\sqrt{T}} \sum_{u \leq \lfloor c\sqrt{T}/m \rfloor + 1} \left( \frac{\Lambda(u) \frac{m \log m}{m^{3/2-\sigma+\varepsilon} (um)^{1/2-\sigma-\varepsilon}}}{\frac{m}{m^{3/2-\sigma+\varepsilon} (um)^{1/2-\sigma-\varepsilon}}} \right) \\
 &\ll T \log T \sum_{m \leq c\sqrt{T}} \frac{\log m}{m^{1-2\sigma}} \sum_{u \leq c\sqrt{T}/m} \frac{u^\varepsilon}{u^{1/2-\sigma-\varepsilon}} \\
 &\ll T^{1+\sigma+\varepsilon}.
 \end{aligned}$$

Finally, we combine these results to complete the proof. □

**Lemma 3.2.** Assume the Riemann Hypothesis; let  $\sigma \in (1/2, 1)$ , let  $c$  be a positive constant, let  $a(n)$  and  $b(n)$  be arithmetic functions such that  $a(n), b(n) \ll 1$  and  $\bar{a}(n), \bar{b}(n) \ll 1$  for all  $n \in \mathbb{N}$ . Then,

$$\begin{aligned}
 &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{m, n, \mu, \nu \leq c\sqrt{\gamma + \gamma'} \\ mn \neq \mu\nu}} \left( \frac{a(m)b(n)\bar{a}(\mu)\bar{b}(\nu) \left( \frac{\mu\nu}{mn} \right)^{i(\gamma + \gamma')}}{(mn\mu\nu)^{1-\sigma}} \right) \\
 &\ll T^{2\sigma-1+\varepsilon} N_2(T).
 \end{aligned} \quad (3.3)$$

*Proof.* Denote this sum by  $S$ . According to the proof of Lemma 3.1, we set  $mn = s, \mu\nu = r$ , and split into two cases;  $s < r$  and  $s > r$ . For the case  $s < r$ , we have

$$S = \sum_{r \leq c^2 T} \sum_{s < r} \sum_{m|s, \mu|r} \frac{a(m)b(s/m)\bar{a}(\mu)\bar{b}(r/\mu)}{(rs)^{1-\sigma}} \times \sum_{\substack{0 < \gamma, \gamma' \leq T \\ K \leq \gamma + \gamma' \leq T}} \left(\frac{r}{s}\right)^{i(\gamma + \gamma')}$$

$$S_2 \ll T \log^2 T \sum_{r \leq T} \frac{r^{2\epsilon}}{r^{1/2-\sigma}} \sum_{s < r} \frac{1}{s^{3/2-\sigma}} \ll T \log^2 T \sum_{r \leq T} r^{2\sigma-1+2\epsilon} \ll T^{2\sigma+1+\epsilon} \log^2 T.$$

where

$$K = \min(T, (1/c^2)) \max(m^2, s^2/m^2, \mu^2, r^2/\mu^2).$$

By Lemma 2.2, we have

$$S = \sum_{r \leq c^2 T} \sum_{s < r} \sum_{m|s, \mu|r} \frac{a(m)b(s/m)\bar{a}(\mu)\bar{b}(r/\mu)}{r^{2-\sigma} s^{-\sigma}} \times \left( \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left(\frac{r}{s}\right)^{\rho+\rho'} - \sum_{\substack{0 < \gamma, \gamma' \leq K \\ \gamma + \gamma' \leq K}} \left(\frac{r}{s}\right)^{\rho+\rho'} \right) = S_1 + S_2 + S_3,$$

where

$$S_1 = \sum_{r \leq T} \sum_{s < r} \sum_{m|s, \mu|r} \frac{a(m)b(s/m)\bar{a}(\mu)\bar{b}(r/\mu)}{r^{2-\sigma} s^{-\sigma}} \times \left( \frac{T^2 - K^2}{8\pi^2} \Lambda^2 \left(\frac{r}{s}\right) \right), S_2 \ll T \log^2 T \sum_{r \leq T} \sum_{s < r} \sum_{m|s, \mu|r} \frac{1}{r^{1/2-\sigma-\epsilon} s^{3/2-\sigma+\epsilon}},$$

and

$$S_3 \ll T \log T \times \sum_{r \leq c^2 T} \sum_{s < r} \sum_{m|s, \mu|r} \frac{a(m)b(s/m)\bar{a}(\mu)\bar{b}(r/\mu)}{r^{3/2-\sigma-\epsilon} s^{1/2-\sigma+\epsilon}} \times \min \left( T, \frac{r/s}{\langle r/s \rangle} \right).$$

First, we estimate  $S_1$ . Writing  $r = sk$  and using

$d(x) \ll x^\epsilon$  and  $K \leq T$ , we have

$$S_1 \ll T^2 \sum_{k \leq T} \sum_{s \leq T/k} \frac{\Lambda^2(k) k^\epsilon s^\epsilon}{k^{2-\sigma} s^{2(1-\sigma)}} \ll T^{2\sigma+1+\epsilon}.$$

For  $S_2$ , we find that

Next, we rewrite  $S_3$  as

$$T \log T \times \sum_{r \leq c^2 T} \frac{(a * b)(r)}{r^{3/2-\sigma-\epsilon}} \sum_{s < r} \frac{(a * b)(s)}{s^{1/2-\sigma+\epsilon}} \min \left( T, \frac{r/s}{\langle r/s \rangle} \right),$$

where  $*$  is Dirichlet convolution. We write  $r = us + t$ , where  $-s/2 < t \leq s/2$ , and consider the case that  $u$  is a prime power or not separately. Denote the sums  $S_3$  whose the term  $u$  is not a prime power and  $u$  is a prime power by  $S_{3,1}$  and  $S_{3,2}$ , respectively.

By (3.2), we have

$$S_{3,1} \ll T \log T \times \sum_{s \leq c^2 T} \sum_{u \leq T/s} \sum_{|t| < s/2} \frac{s^\epsilon (us + t)^\epsilon}{s^{1/2-\sigma+\epsilon} (us + t)^{3/2-\sigma-\epsilon}} \ll T \log T \times \sum_{s \leq c^2 T} \frac{s^{2\epsilon}}{s^{2-2\sigma}} \sum_{u \leq T/s} \sum_{|t| < s/2} \left( u + \frac{t}{s} \right)^{\sigma-3/2+2\epsilon}.$$

Since  $(u + t/s)^{\sigma-3/2+2\epsilon} \leq (u-1)^{\sigma-3/2+2\epsilon}$ , we have

$$S_{3,1} \ll T \log T \sum_{s \leq c^2 T} s^{2\sigma-1+2\epsilon} \left( \frac{T}{s} \right)^{\sigma-1/2+\epsilon} \ll T^{2\sigma+1+\epsilon}.$$

In order to estimate  $S_{3,2}$ , we rewrite  $S_{3,2}$  in the following way,

$$T \log T \times \sum_{s \leq c^2 T} \frac{(a * b)(s)}{s^{2-2\sigma}} \times \sum_{\substack{u \leq \frac{cT}{s} + 1 \\ 0 \neq |t| < s/2}} \frac{\overline{(a * b)(us + t)}}{(u + t/s)^{3/2-\sigma-\epsilon}} \min \left( T, \frac{us + t}{|t|} \right),$$

where  $\sum'$  means that the sum runs over prime

powers. Working similarly  $S_{3,1}$ , we get

$$\begin{aligned} S_{3,2} &\ll T \log T \sum_{s \ll T} s^{2\sigma-1+\varepsilon} \sum_{\substack{u \ll \frac{T}{s} \\ u \text{ prime}}} u^{\sigma-1/2+\varepsilon} \\ &\ll T^{\sigma+1/2+\varepsilon} \log^2 T \sum_{s \ll T} s^{\sigma-1/2} \\ &\ll T^{2\sigma+1+\varepsilon}. \end{aligned}$$

Therefore,  $S_3 = O(T^{2\sigma-1+\varepsilon} N_2(T))$ .

For the case  $s > r$ , we conjugate (3.3) and obtain the same estimates. Finally, we combine these results to complete the proof.  $\square$

**Lemma 3.3.** Assume the Riemann Hypothesis; let  $\sigma \in (1/2, 1)$ , let  $c$  be a positive constant, let  $a(n)$  and  $b(n)$  be arithmetic functions such that  $a(n) \ll 1$  and  $b(n) \ll 1$  for all  $n \in \mathbb{N}$ . Then,

$$\begin{aligned} &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{n, m \leq c\sqrt{\gamma + \gamma'} \\ n \neq m}} \frac{a(m)b(n)}{(nm)^\sigma} \left(\frac{n}{m}\right)^{i(\gamma + \gamma')} \\ &= o(N_2(T)). \end{aligned} \tag{3.4}$$

*Proof.* The proof has the same method as the proof of Lemma 3.1.  $\square$

**Lemma 3.4.** Assume the Riemann Hypothesis; let  $\sigma \in (1/2, 1)$ , let  $c$  be a positive constant, let  $a(n)$  and  $b(n)$  be arithmetic functions such that  $a(n), b(n) \ll 1$  and  $\bar{a}(n), \bar{b}(n) \ll 1$  for all  $n \in \mathbb{N}$ .

Then,

$$\begin{aligned} &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{m, n, \mu, \nu \leq c\sqrt{\gamma + \gamma'} \\ mn \neq \mu\nu}} \left( \frac{a(m)b(n)\bar{a}(\mu)\bar{b}(\nu)}{(mn\mu\nu)^\sigma} \left(\frac{\mu\nu}{mn}\right)^{i(\gamma + \gamma')} \right) \\ &\ll N_2(T). \end{aligned} \tag{3.5}$$

*Proof.* The proof has the same method as the proof of Lemma 3.2.  $\square$

**Proposition 3.5.** Suppose  $s = \sigma + i(\gamma + \gamma')$ ,  $\sigma \in (1/2, 1)$ ,  $\gamma, \gamma' > 0$ . Then, under the Riemann Hypothesis,

$$\begin{aligned} &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} B(s, P) L(s, \chi_1) \overline{L(s, \chi_2)} \\ &\sim N_2(T) \sum_{n=1}^{\infty} \frac{d(n) \bar{\chi}_2(n)}{n^{2\sigma}}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} B(s, P) \overline{L(s, \chi_1)} L(s, \chi_2) \\ &\sim N_2(T) \sum_{n=1}^{\infty} \frac{e(n) \bar{\chi}_1(n)}{n^{2\sigma}}, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} B(s, P) L(s, \chi_1) &= \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \\ B(s, P) L(s, \chi_2) &= \sum_{n=1}^{\infty} \frac{e(n)}{n^s}. \end{aligned} \tag{3.8}$$

*Proof.* By Theorem 2.3 with  $\Delta = \sqrt{\ell}$ , we have

$$\begin{aligned} L(s, \chi_1) &= W_1 + X(s, \chi_1) W_2 \\ &\quad + O((\gamma + \gamma')^{-1/4} \log(\gamma + \gamma')), \end{aligned} \tag{3.9}$$

where

$$X(s, \chi) = \varepsilon(\chi) \left(\frac{q}{\pi}\right)^{1/2-s} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}.$$

And also, for  $\chi_2$  with  $\Delta = \sqrt{qR}$ ,

$$\begin{aligned} L(s, \chi_2) &= Y_1 + \overline{X(s, \chi_2)} Y_2 \\ &\quad + O((\gamma + \gamma')^{-1/4} \log(\gamma + \gamma')). \end{aligned} \tag{3.10}$$

Then,

$$\begin{aligned} & \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} B(s, P) L(s, \chi_1) \overline{L(s, \chi_2)} \\ &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} B(s, P) \\ & \times \left[ W_1 + X(s, \chi_1) W_2 \right. \\ & \quad \left. + O((\gamma + \gamma')^{-1/4} \log(\gamma + \gamma')) \right] \\ & \times \left[ Y_1 + \overline{X(s, \chi_2)} Y_2 \right. \\ & \quad \left. + O((\gamma + \gamma')^{-1/4} \log(\gamma + \gamma')) \right]. \end{aligned} \tag{3.11}$$

We begin compute a product of each the first term in (3.11) and denote by  $M(T)$ . Then,

$$\begin{aligned} M(T) &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} B(s, P) \sum_{\substack{n \leq \sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}}} \chi_1(n) n^{-s} \\ & \times \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}}} \overline{\chi_2(n)} n^{-\bar{s}} \\ &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}}} d'(n) n^{-s} \\ & \times \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}}} \overline{\chi_2(n)} n^{-\bar{s}} \\ &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}}} d'(n) n^{-\sigma - i(\gamma + \gamma')} \\ & \times \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}}} \overline{\chi_2(n)} n^{-\sigma + i(\gamma + \gamma')} \\ &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left( \sum_{\substack{m, n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}} } \frac{d'(m) \overline{\chi_2(n)}}{(nm)^\sigma} \left( \frac{n}{m} \right)^{i(\gamma + \gamma')} \right). \end{aligned}$$

We split the inner sum into two terms. Then

$$\begin{aligned} M(T) &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left( \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}} } \frac{d'(n) \overline{\chi_2(n)}}{n^{2\sigma}} \right. \\ & \quad \left. + \sum_{\substack{n, m \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}} \\ n \neq m}} \frac{d'(m) \overline{\chi_2(n)}}{(nm)^\sigma} \left( \frac{n}{m} \right)^{i(\gamma + \gamma')} \right) \\ &= Z_1 + Z_2. \end{aligned}$$

For  $Z_1$ , we have

$$\begin{aligned} Z_1 &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left( \sum_{n=1}^{\infty} \frac{d(n) \overline{\chi_2(n)}}{n^{2\sigma}} \right. \\ & \quad - \sum_{\substack{n > R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}} } \frac{d(n) \overline{\chi_2(n)}}{n^{2\sigma}} \\ & \quad \left. + \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}} } \frac{(d'(n) - d(n)) \overline{\chi_2(n)}}{n^{2\sigma}} \right) \\ &= N_2(T) \sum_{n=1}^{\infty} \frac{d(n) \overline{\chi_2(n)}}{n^{2\sigma}} + C_1 + C_2. \end{aligned}$$

For  $C_1$ , we have

$$\begin{aligned} & \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{n > \sqrt{\gamma + \gamma'}} n^{-2\sigma} \\ & \ll \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} (\gamma + \gamma')^{1/2 - \sigma} = o(N_2(T)). \end{aligned}$$

For  $C_2$ , we have

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{n > \sqrt{\gamma + \gamma'}} n^{-2\sigma} = o(N_2(T)).$$

Thus,

$$M(T) = N_2(T) \sum_{n=1}^{\infty} \frac{d(n) \overline{\chi_2(n)}}{n^{2\sigma}} + o(N_2(T)).$$

In order to estimate the remaining terms in (3.11), we need to estimate the following summations.

Working similarly to  $M(T)$ , we have

$$\begin{aligned} & \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left| \sum_{\substack{n \leq R\sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}}} } d'(n) n^{-\sigma - i(\gamma + \gamma')} \right|^2 \\ & \ll N_2(T), \end{aligned} \tag{3.12}$$

and

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left| \sum_{n \leq R \sqrt{\frac{q(\gamma + \gamma')}{2\pi}}} \bar{\chi}_2(n) n^{-\sigma + i(\gamma + \gamma')} \right|^2 \ll N_2(T). \tag{3.13}$$

With the change of the above argument, we have

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left| \sum_{n \leq \sqrt{\frac{q(\gamma + \gamma')}{2\pi\ell}}} \chi_1(n) n^{\sigma - 1 + i(\gamma + \gamma')} \right|^2 \ll T^{\sigma - 1/2} N_2(T), \tag{3.14}$$

since

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{n \leq \sqrt{\frac{q(\gamma + \gamma')}{2\pi}}} \frac{1}{n^{2(1-\sigma)}} \ll \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} (\gamma + \gamma')^{\sigma - 1/2} \ll T^{\sigma - 1/2} N_2(T)$$

and Lemma 3.1. Similarly to above, we get

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \left| \sum_{n \leq \frac{1}{R} \sqrt{\frac{\ell(\gamma + \gamma')}{2\pi q}}} \bar{\chi}_2(n) n^{\sigma - 1 - i(\gamma + \gamma')} \right|^2 \ll T^{\sigma - 1/2} N_2(T). \tag{3.15}$$

We use

$$|X(s, \chi)|^2 \ll t^{1-2\sigma},$$

and

$$\frac{d}{dt} |X(s, \chi)|^2 \ll t^{-2\sigma}$$

for  $s = \sigma + it$ ,  $|t| \rightarrow \infty$  (cf. [3]),  $N_2(t) = O(t^2 \log^2 t)$ , (3.14), (3.15) and summation by parts to find that

$$\begin{aligned} & \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |X(\sigma + i(\gamma + \gamma'), \chi_1)|^2 \\ & \times \left| \sum_{n \leq \sqrt{\frac{q(\gamma + \gamma')}{2\pi\ell}}} \chi_1(n) n^{\sigma - 1 + i(\gamma + \gamma')} \right|^2 \tag{3.16} \\ & = o(N_2(T)), \end{aligned}$$

and

$$\begin{aligned} & \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |X(\sigma + i(\gamma + \gamma'), \bar{\chi}_2)|^2 \\ & \times \left| \sum_{n \leq \frac{1}{R} \sqrt{\frac{\ell(\gamma + \gamma')}{2\pi q}}} \bar{\chi}_2(n) n^{\sigma - 1 - i(\gamma + \gamma')} \right|^2 \tag{3.17} \\ & = o(N_2(T)). \end{aligned}$$

To complete the proof, we use Cauchy-Schwarz inequality, (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17) to estimate the remaining terms in (3.11) as  $o(N_2(T))$ .  $\square$

**Proposition 3.6.** Suppose  $s = \sigma + i(\gamma + \gamma')$ ,  $\sigma \in (1/2, 1)$ ,  $\gamma, \gamma' > 0$  and let  $A(\gamma + \gamma') = B(s, P) \times L(s, \chi_1) \overline{L(s, \chi_2)} - \overline{L(s, \chi_1)} L(s, \chi_2)$ .

Then, under the Riemann Hypothesis,

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |A(\gamma + \gamma')|^2 \ll N_2(T).$$

*Proof.* Since

$$|A(\gamma + \gamma')|^2 \ll |L(s, \chi_1)|^2 |L(s, \chi_2)|^2,$$

it is sufficient to show that

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |L(s, \chi_1)|^2 |L(s, \chi_2)|^2 \ll N_2(T). \tag{3.18}$$

By (3.4) and (3.5), we have

$$\begin{aligned} L(s, \chi_1) &= W_1 + X(s, \chi_1) W_2 \\ &+ O((\gamma + \gamma')^{-1/4} \log(\gamma + \gamma')), \end{aligned}$$

and

$$\begin{aligned} L(s, \chi_2) &= Y_1 + \overline{X(s, \chi_2)} Y_2 \\ &+ O((\gamma + \gamma')^{-1/4} \log(\gamma + \gamma')). \end{aligned}$$

Let  $M(T) = \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} Y_1 \bar{Y}_1 W_1 \bar{W}_2$ . Then

$$\begin{aligned}
 M(T) &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{m, n, \mu, \nu \leq \sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}} \\ mn = \mu\nu}} \left( \frac{\chi_1(m)\chi_2(n)\bar{\chi}_1(\mu)\bar{\chi}_2(\nu)}{(mn\mu\nu)^\sigma} \left(\frac{\mu\nu}{mn}\right)^{i(\gamma + \gamma')} \right) \\
 &= \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{m, n, \mu, \nu \leq \sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}} \\ mn = \mu\nu}} \left( \frac{\chi_1(m)\chi_2(n)\bar{\chi}_1(\mu)\bar{\chi}_2(\nu)}{(mn)^{2\sigma}} \right) \\
 &\quad + \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{m, n, \mu, \nu \leq \sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}} \\ mn \neq \mu\nu}} \left( \frac{\chi_1(m)\chi_2(n)\bar{\chi}_1(\mu)\bar{\chi}_2(\nu)}{(mn\mu\nu)^\sigma} \left(\frac{\mu\nu}{mn}\right)^{i(\gamma + \gamma')} \right) \\
 &= Z_1 + Z_2.
 \end{aligned}$$

First, we consider  $Z_1$ . Since the number of solutions  $mn = \mu\nu = r$  is less than or equal to  $d^2(r)$ ,

$$Z_1 \ll \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{r=1}^{\infty} \frac{d^2(r)}{r^{2\sigma}} \ll N_2(T).$$

By Lemma 3.4, we have

$$M(T) \ll N_2(T).$$

Working similarly to  $M(T)$ , we obtain

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |W_1|^4 \ll N_2(T), \tag{3.19}$$

and

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |Y_1|^4 \ll N_2(T). \tag{3.20}$$

With the change of the above argument, we get

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |W_2|^4 \ll T^{2\sigma - 1 + \varepsilon} N_2(T), \tag{3.21}$$

since

$$\begin{aligned}
 &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{\substack{m, n, \mu, \nu \leq \sqrt{\frac{q\ell(\gamma + \gamma')}{2\pi}} \\ mn = \mu\nu}} \frac{\chi_1(m)\chi_2(n)\bar{\chi}_1(\mu)\bar{\chi}_2(\nu)}{(mn)^{2(1-\sigma)}} \\
 &\ll \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} \sum_{r \leq \frac{q\ell(\gamma + \gamma')}{2\pi}} \frac{d^2(r)}{r^{2-2\sigma}} \\
 &\ll \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} (\gamma + \gamma')^{2\sigma - 1 + \varepsilon} \\
 &\ll T^{2\sigma - 1 + \varepsilon} N_2(T)
 \end{aligned}$$

and because of Lemma 3.2. Similarly to the above method, we obtain

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |Y_2|^4 \ll T^{2\sigma - 1 + \varepsilon} N_2(T). \tag{3.22}$$

We use

$$\frac{d}{dt} |X(s, \chi)|^4 \ll t^{1-4\sigma}$$

for  $s = \sigma + it, |t| \rightarrow \infty$  (cf. [3]),  $N_2(t) = O(t^2 \log^2 t)$ , (3.21), (3.22) and summation by parts to find that

$$\begin{aligned}
 &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |X(s, \chi)|^4 |W_2|^4 \\
 &= O(T^{1-2\sigma + \varepsilon} N_2(T)),
 \end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
 &\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |X(s, \chi)|^4 |Y_2|^4 \\
 &= O(T^{1-2\sigma + \varepsilon} N_2(T)).
 \end{aligned} \tag{3.24}$$

To finish the proof, we use the Cauchy-Schwarz inequality, (3.19), (3.20), (3.21), (3.22), (3.23), and (3.24) to estimate the remaining terms in (3.18) as  $O(N_2(T))$ .  $\square$

**Proposition 3.7.** Suppose  $s = \sigma + i(\gamma + \gamma')$ ,  $\sigma \in (1/2, 1)$ ,  $\gamma, \gamma' > 0$  and let

$$A(\gamma + \gamma') = B(s, P) \times \overline{L(s, \chi_1)L(s, \chi_2)} - \overline{L(s, \chi_1)L(s, \chi_2)}.$$

Then, under the Riemann Hypothesis,

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} A(\gamma + \gamma') \sim C \cdot N_2(T)$$

for some non-zero constant  $C$ .

*Proof.* The proof is similar to the proof of Proposition 2.3 in [3].  $\square$

**Theorem 3.8.** Assume the Riemann Hypothesis; let  $\sigma \in (1/2, 1)$ . Then, for a positive proportion of  $1/2 + i(\gamma + \gamma')$ , with  $\gamma, \gamma' > 0$ , the values of the Dirichlet  $L$ -functions  $L(\sigma + i(\gamma + \gamma'), \chi_1)$  and  $L(\sigma + i(\gamma + \gamma'), \chi_2)$  associated with primitive characters  $\chi_1$  and  $\chi_2$  to different prime moduli are linearly independent over  $\mathbb{R}$ .

*Proof.* By the Cauchy-Schwarz inequality, we have

$$\left( \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T \\ A(\gamma + \gamma') \neq 0}} 1 \right) \left( \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |A(\gamma + \gamma')|^2 \right) \geq \left| \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} A(\gamma + \gamma') \right|^2.$$

By Proposition 3.6 and 3.7, we have

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T \\ A(\gamma + \gamma') \neq 0}} 1 \geq \frac{\left| \sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} A(\gamma + \gamma') \right|^2}{\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} |A(\gamma + \gamma')|^2} \gg \frac{|C|^2 N_2(T)^2}{N_2(T)} = |C|^2 N_2(T). \quad \square$$

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