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Generalized Nonexpansive Mappings in CAT(0) spaces

Pim Sanboonsiri[1*] and Paiwan Wongsasinchai[1]

[1]Department of Mathematics, Faculty of Science and Technology,
Rambhai Barni Rajabhat University, Muang, Chanthaburi 22000, Thailand
*E-mail: pim.m@rbru.ac.th

Abstract

In this paper, we introduce the modified algorithm in frame of a CAT(0) space for total asymptotically nonexpansive mapping and prove strong convergence. Moreover, we have numerical example for the proposed algorithm to compare speed of convergence among the existing iterative algorithm.

Keywords: generalized nonexpansive mappings, CAT(0) spaces , strong convergence .

1. Introduction

A CAT(0) space plays a primary role in various mathematic areas [1-3]. Moreover, it is also beneficial to biology and computer science [4-5]. A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison of triangle in the Euclidean plane. The CAT(0) space is the well-known method that provides complete, simply connected Riemannian manifold, showing non-positive sectional curvature. The complex Hilbert ball with a hyperbolic metric is the CAT(0) space [6]. Other examples of the CAT(0) space include preHilbert spaces, R-trees [1] and Euclidean buildings [7].

Kirk proposed fixed point theory in a CAT(0) space [8-9]. He presented every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space, always having a fixed point. Since then the fixedpoint theory in a CAT(0) space has been continueously developed, there have been a lot of reports for the CAT(0) space

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application e.g., [8], [9], [10], [11], [12], [13], [14], [15], [16]. The Noor iteration [17] is defined as $u_1 \in K$ and

$$\begin{cases} z_n &= (1-\eta_n)u_n + \eta_n \mathrm{T} u_n, \\ y_n &= (1-\vartheta_n)u_n + \vartheta_n \mathrm{T} z_n, \\ x_{n+1} &= (1-\xi_n)u_n + \xi_n \mathrm{T} y_n \end{cases} \tag{1.1}$$

for all $n \geq 1$, where $\{\xi_n\}$, $\{\vartheta_n\}$ and $\{\eta_n\}$ are sequences in $[0,1]$. If we take $\vartheta_n = \eta_n = 0$ for all n, (1.1) reducing to the Mann iteration [18], and $\eta_n = 0$ for all n, (1.1) is taken that reduces to the Ishikawa iteration [19].

The new two-step iteration [21] is defined as $u_1 \in K$ and

$$\begin{cases} y_n &= (1-\vartheta_n)u_n + \vartheta_n \mathrm{T} u_n, \\ x_{n+1} &= (1-\xi_n)y_n + \xi_n \mathrm{T} y_n \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\xi_n\}$, $\{\vartheta_n\}$ and $\{\eta_n\}$ are sequences in $[0,1]$.

Phuengrattana and Suantai [22] defined by the SP-iteration as follows:

$$\begin{cases} z_n &= (1-\eta_n)u_n + \eta_n \mathrm{T} u_n, \\ y_n &= (1-\vartheta_n)z_n + \vartheta_n \mathrm{T} z_n, \\ x_{n+1} &= (1-\xi_n)y_n + \xi_n \mathrm{T} y_n \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $u_1 \in K$, $\{\xi_n\}$, $\{\vartheta_n\}$ and $\{\eta_n\}$ are sequences in $[0,1]$. They reported that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges better than those of the others for the continuous and nondecreasing functions class. The new two-step and Mann iterations are clearly special cases of the SP-iteration.

Kitkuan and Padcharoen [26] have modified SP-iteration(1.3) in frame of a CAT(0) space as follows:

$$\begin{cases} z_n &= (1-\eta_n)u_n \oplus \eta_n \mathrm{T} u_n, \\ y_n &= (1-\vartheta_n)z_n \oplus \vartheta_n \mathrm{T} z_n, \\ x_{n+1} &= (1-\xi_n)y_n \oplus \xi_n \mathrm{T} y_n \end{cases} \tag{1.4}$$

for all $n \geq 1$, where K is a nonempty convex subset of a CAT(0) space, $u_1 \in K$, $\{\xi_n\}$, $\{\vartheta_n\}$ and $\{\eta_n\}$ are sequences in $[0,1]$.

In this paper, we introduce the modified algorithm in frame of a CAT(0) space for total asymptotically nonexpansive mapping as follows:

$$\begin{cases} z_n &= (1-\vartheta_n)u_n \oplus \vartheta_n \mathrm{T}^n u_n, \\ y_n &= \mathrm{T}^n z_n, \\ x_{n+1} &= \xi_n \mathrm{T}^n z_n \oplus (1-\xi_n)\mathrm{T}^n y_n \end{cases} \tag{1.5}$$

2. Preliminaries and lemmas

The definitions and known results are recalled in the existing literature on this concept. K is a nonempty subset of a CAT(0) space X and $T: K \to K$ is a mapping. A point $u \in K$ is called a fixed point of T if $Tu = u$.

Let we recall some basics for nonlinear mappings on metric spaces. Let (X, d) be a metric space and K be its non-empty subset. Then $T: K \to K$ is said a:

• Asymptotically nonexpansive if for a sequence $\{a_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} a_n = 0$ such that

$$d(T^n u,\ T^n y) \le (1 + a_n) d(u, y)$$

for all $u, y \in K$ and $n \ge 1$.

• Uniformly L-Lipschitzian if there exist $L > 0$ such that $d(T^n u,\ T^n y) \le L d(u, y)$ for all $u, y \in K$ and $n \ge 1$.

Definition 2.1. [33] Let (X, d) be a CAT (0) space, K be a non-empty closed convex subset and let $T : K \to K$ be a mapping. T is said to be total asymptotically nonexpansive mapping if there exist nonnegative real sequences $\{a_n\}$, $\{b_n\}$ with $a_n \to 0$, $b_n \to 0$ and strictly increasing continuous function $\xi : [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$ such that $d(T^n u,\ T^n y) \le d(u, y) + a_n \xi(d(u, y)) + b_n$ for all $u, y \in K$ and $n \ge 1$.

Let (X, d) be a metric space. A geodesic path joining $u \in X$ to $y \in X$ (or more briefly, a *geodesic* from u to y) is a map t from a closed interval $[0, b] \subset \mathbb{R}$ to X such that $t(0) = u$, $t(b) = y$ and $d(t(a),\ t(a')) = |a - a'|$ for all $a,\ a' \in [0, b]$. A map t is an isometry and $d(u, y) = b$. The t image is called as a geodesic (or metric) segment joining u and y. When it is unique, this geodesic is denoted by $[u, y]$. If every two points of X are joined by a geodesic, the space, (X, d) is a geodesic space. More over, X is a uniquely geodesic space when there is exactly one geodesic joining u to y for each $u, y \in X$.

A geodesic triangle $\triangle(u_1, u_2, u_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(u_1, u_2, u_3)$ in (X, d) is a triangle $\overline{\triangle}(u_1, u_2, u_3) = \triangle(\overline{u}_1, \overline{u}_2, \overline{u}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{u}_i, \overline{u}_j) = d(u_i, u_j)$ for $i, j \in \{1,\ 2,\ 3\}$.

A geodesic metric space is said to be a CAT(0) space [1] if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): $\triangle$ is defined as a geodesic triangle in X and $\overline{\triangle}$ as a comparison triangle for $\triangle$. If for all $u, y \in \triangle$ and all comparison points $\overline{u}, \overline{y} \in \overline{\triangle}$, $\triangle$ satisfies the *CN* inequality: $d(u, y) = d_{\mathbb{R}^2}(\overline{u}, \overline{y})$.

Finally, we observe that if u, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the *CN* inequality implies

$$d(u,\ y_0)^2 \le \tfrac{1}{2} d(u, y_1)^2 + \tfrac{1}{2} d(u, y_2)^2 - \tfrac{1}{4} d(y_1, y_2)^2. \qquad (2.1)$$

The equality holds for the Euclidean metric. In actual fact [1], if and only if it satisfies inequality (2.1) (which is known as the *CN* inequality of Bruhat and Tits, a geodesic metric space is a CAT(0) space [23]).The ensuing lemmas can be found in [12].

Lemma 2.2. [12] Let X be a CAT(0) space. Then $d((1-\varepsilon)x \oplus \varepsilon y, z) \le (1-\varepsilon) d(x, z) + \varepsilon d(y, z)$ for all $\varepsilon \in [0,1]$ and $x, y, z \in X$.

Lemma 2.3. [12] Let X be a CAT(0) space.

Then

$$d((1-\varepsilon)x \oplus \varepsilon y, z)^2 \le (1-\varepsilon)d(x,z)^2 + \varepsilon d(y,z)^2 - \varepsilon(1-\varepsilon)d(x,y)^2$$

for all $\varepsilon \in [0,1]$ and $x, y, z \in X$.

Now, some definitions are recalled. X is a complete CAT(0) space and $\{u_n\}$ is a bounded sequence in X. For $x \in X$, set

$$r(x,\{u_n\}) = \limsup_{n\to\infty} d(x,u_n).$$

The *asymptotic radius* $r(\{u_n\})$ of $\{u_n\}$ is defined as

$$r(\{u_n\}) = \inf\{r(x,\{u_n\}) : x \in X\}.$$

The *asymptotic center* $A(\{u_n\})$ of $\{u_n\}$ is the set

$$A(\{u_n\}) = \{x \in X : r(x,\{u_n\}) = r(\{u_n\})\}.$$

It is known that in a complete CAT(0) space, $A(\{u_n\})$ consists of exactly one point (see [10]). Also, every CAT(0) space has the *Opial* property, i.e., if $\{u_n\}$ is a sequence in K and $\Delta-\lim_{n\to\infty} u_n = x$, then for each $y \ne x \in K$, $\limsup_{n\to\infty} d(u_n,x) < \limsup_{n\to\infty} d(u_n,y)$.

Definition 2.4. [8] A sequence $\{u_n\}$ in a CAT(0) space X is convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{u_n\}$. For this case, $\Delta-\lim_{n\to\infty} u_n = x$ and x is given by the $\Delta-$ limit of $\{u_n\}$.

The concept of $\Delta-$ convergence in a fundamental metric space was reported by Lim [25]. Kirk and Panyanak [16] recently used the notion of $\Delta-$ convergence begin by Lim [25] to prove on the CAT(0) space analogous of some Banach space results, which relate to weak convergence. Furthermore, Dhompongsa and Panyanak [12] acheived $\Delta-$ convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space.

Lemma 2.5. [12]

(i) Every bounded sequence in a complete CAT(0) space always contains a $\Delta-$ convergent subsequence.

(ii) Let K be a nonempty closed convex subset of a complete CAT(0) space and let $\{u_n\}$ be a bounded sequence in K. Then the asymptotic center of $\{u_n\}$ is in K.

(iii) Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $f : K \to X$ be a nonexpansive mapping. Then the conditions, $\{u_n\}$ $\Delta-$ converges to x and $d(u_n,\{u_n\}) \to 0$, imply $x \in K$ and $f(x) = x$.

Lemma 2.6. [24] Define $\{a_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ as the nonnegative numbers sequences such that

$$a_{n+1} \le (1+\lambda_n)a_n + c_n$$

for all $n \ge 1$. If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. Whenever, if there exists a subsequence $\{a_{ni}\} \subseteq \{a_n\}$ such that $\{a_{ni}\} \to 0$ as $i \to \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. [20] Define x as a point in a CAT(0) space (X,d) and $\{t_n\}$ to be a sequence in a closed interval $[b,d]$ for some $b,d \in (0,1)$. Assume that $\{u_n\}$ and $\{y_n\}$ are two sequences in X such that

$$\limsup_{n\to\infty} d(u_n,x) \le c$$

$$\limsup_{n\to\infty} d(y_n,x) \le c$$

$$\lim_{n\to\infty} d((1-t_n)u_n \oplus t_n y_n, x) = c$$

for some $c \ge 0$. Then $\limsup_{n\to\infty} d(u_n,y_n) = 0.$

Lemma 2.8. [28] Assume (X,d) is a complete CAT(0) space. Consider a uniformly continuous, total asymptotically nonexpansive mapping self mapping T on a nonempty, convex, closed and bounded set $K \subset X$. Then T has a fixed point, and the fixed point set F(T) is closed and convex.

Lemma 2.9. [28] Assume (X,d) is a complete CAT(0) space, and C a closed, convex subset of X. Define $T: K \to K$ is a uniformly continuous and total asymptotically nonexpansive mapping. For every bounded sequence $\{u_n\} \subset K$ such that, $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ and $\Delta - \lim_{n\to\infty} u_n = q$, Then $Tq = q$.

Lemma 2.10. [29] For the complete CAT(0) space (X,d), every bounded sequence in X has $\Delta-$ convergent subsequence.

Lemma 2.11. [30] Define $\{u_n\}$ as a bounded sequence in a closed convex subset C of X. So, the asymptotic center of $\{u_n\}$ is in K, provided that (X,d) is a complete CAT(0) space.

Lemma 2.12. [31] Assume that (X,d) is a complete CAT(0) space. Let $\{u_n\}$ be a bounded sequence in X. If $A(\{u_n\}) = \{p\}$, $\{w_n\}$ is a subsequence of $\{u_n\}$ such that $A(\{w_n\}) = \{w\}$ and $d(u_n, w)$ converges, then $p = w$.

3. Results and Discussion

Theorem 3.1. Define C as a bounded closed convex subset of a complete CAT(0) space (X,d) and $T: K \to K$ as a uniformly L-Lipschitzian and $(\{a_n\},\{b_n\},\varphi)$-total asymptotically nonexpansive mapping. Assume that the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$;

(ii) there exist constants s, k with $0 < s < \xi_n < k < 1$ for every $n \in \mathbb{N}$;

(iii) there exist constants g, e with $0 < g < \vartheta_n < e < 1$ for every $n \in \mathbb{N}$;

(iv) there exists a constant τ such that $\psi(t) \le \tau t$ for every $t \ge 0$.

Then the sequence $\{u_n\}$ defined by (1.5) $\Delta-$ converges to a fixed point of T.

Proof. By using Lemma 2.8, we get $F(T) \ne \varnothing$. Next, we part the show that into three steps.

Step 1. $\lim_{n\to\infty} \sup d(u_n, u)$ is proved to exist for all $u \in F(T)$, where $\{u_n\}$ is defined as (1.5). Let $u \in F(T)$ and $n \in \mathbb{N}$. So,

$$\begin{aligned} d(z_n,u) &= d((1-\vartheta_n)u_n \oplus \vartheta_n T^n u_n, u) \\ &\le (1-\vartheta_n) d(u_n,u) + \vartheta_n d(T^n u_n, u) \\ &\le (1-\vartheta_n) d(u_n,u) + \vartheta_n \{(1+a_n\tau) d(u_n,u) + b_n\} \\ &\le (1+a_n\tau) d(u_n,u) + b_n . \end{aligned} \tag{3.1}$$

Also, we have

$$\begin{aligned} d(y_n,u) &= d(T^n z_n, T^n u) \\ &\le (1+a_n\tau) d(z_n,u) + b_n . \end{aligned} \tag{3.2}$$

By (iv) and (3.1) then,

$$\begin{aligned} d(y_n,u) &\le (1+a_n\tau)\{(1+a_n\tau) d(u_n,u) + b_n\} \\ &\quad + b_n . \end{aligned} \tag{3.3}$$

By from (1.5) and lemma 2.2, then we get

$$d(x_{n+1},u) = d(\xi_n T^n z_n \oplus (1-\xi_n)T^n y_n, u)$$
$$\le \xi_n d(T^n z_n, u) + (1-\xi_n) d(T^n y_n, u)$$
$$= \xi_n d(T^n z_n, T^n u) + (1-\xi_n) d(\xi_n T^n y_n, T^n u)$$
$$\le \xi_n \left\{(1+a_n\tau) d(z_n,u) + b_n\right\}$$
$$+ (1-\xi_n)\left\{(1+a_n\tau) d(y_n,u) + b_n\right\}$$
$$= \xi_n (1+a_n\tau) d(z_n,u) + \xi b_n$$
$$+ (1-\xi_n)(1+a_n\tau) d(y_n,u) + (1-\xi_n) b_n$$
$$= \xi_n (1+a_n\tau) d(z_n,u) + (1-\xi_n)$$
$$\times (1+a_n\tau) d(y_n,u) + \xi_n b_n \quad . \qquad (3.4)$$

Substituting (3.2) to (3.4), then we get

$$d(x_{n+1},u) \le \xi_n (1+a_n\tau) d(z_n,u)$$
$$+ (1-\xi_n)(1+a_n\tau)\left\{(1+a_n\tau) d(z_n,u) + b_n\right\}$$
$$+ \xi_n b_n$$
$$= \xi_n (1+a_n\tau) d(z_n,u) + (1-\xi_n)(1+a_n\tau)^2$$
$$\times d(z_n,u) + (1-\xi_n) b_n + \xi_n b_n$$
$$= \xi_n (1+a_n\tau) d(z_n,u) + (1-\xi_n)(1+a_n\tau)^2$$
$$\times d(z_n,u) + b_n$$
$$\le \xi_n (1+a_n\tau)^2 d(z_n,u) + (1-\xi_n)(1+a_n\tau)^2$$
$$\times d(z_n,u) + b_n$$
$$= (1+a_n\tau)^2 d(z_n,u) + b_n \quad . \qquad (3.5)$$

By (iv) and (3.5) we have

$$d(x_{n+1},u) \le (1+a_n\tau)^2 \left\{(1+a_n\tau) d(u_n,u) + b_n\right\} + b_n$$
$$= (1+a_n\tau)^3 d(u_n,u) + \left\{(1+a_n\tau)^2 + 1\right\} b_n$$
$$= (1+\varphi_n) d(u_n,u) + \theta_n, \qquad (3.6)$$

where $\varphi = 3(a_n\tau) + 3(a_n\tau)^2 + 3(a_n\tau)^3$ and $\theta_n = \left\{(1+a_n\tau)^2 + 1\right\} b_n$. By (iv) , we get $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$,it follows that $\sum_{n=1}^{\infty} \varphi_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. Now, from (3. 6) and lemma 2. 8, consequently $\lim_{n\to\infty} \sup d(u_n,u)$ exists.

Step 2. Next, we will show that $\lim_{n\to\infty} d(u_n,u) = 0$.

By step 1 we have $\lim_{n\to\infty} \sup d(u_n,u)$ exists and $\{u_n\}$ is bounded without loss of generality, we may suppose that $c = \lim_{n\to\infty} d(u_n,u) \ge 0. \qquad (3.7)$

From (3.1) we have $\lim_{n\to\infty} d(z_n,u) \le c. \qquad (3.8)$

By (iv) we have

$$d(T^n z_n, u) \le (1+a_n\tau) d(z_n,u) + b_n. \qquad (3.9)$$

From (3.8) and (3.9), we get

$$\limsup_{n\to\infty} d(T^n z_n, u) \le c$$

(3.10)

and

$$d(T^n u_n, u) \le (1+a_n\tau) d(u_n,u) + b_n. \qquad (3.11)$$

Then,

$$d(x_{n+1},u) = d(\xi_n T^n z_n \oplus (1-\xi_n)T^n y_n, u)$$
$$\le \xi_n d(T^n z_n, u) + (1-\xi_n) d(T^n y_n, u)$$
$$= \xi_n d(y_n,u) + (1-\xi_n) d(T^n y_n, T^n u)$$
$$\le \xi_n d(y_n,u) + (1-\xi_n)\left\{(1+a_n\tau) d(y_n,u) + b_n\right\}$$
$$= \left\{\xi_n d(y_n,u) + (1-\xi_n)(1+a_n\tau)\right\} d(y_n,u)$$
$$+ (1-\xi_n) b_n. \qquad (3.12)$$

Thus, $c = \limsup_{n\to\infty} d(y_n,u). \qquad (3.13)$

We have

$$d(y_n,u) \le (1+a_n\tau) d(z_n,u) + b_n. \qquad (3.14)$$

Then,

$$c \le \limsup_{n\to\infty} d(z_n,u). \qquad (3.15)$$

Hence,

$$c = \limsup_{n\to\infty} d(z_n, u)$$

$$= \limsup_{n\to\infty} d((1-\vartheta_n)u_n \oplus \vartheta_n T^n u_n, u). \quad (3.16)$$

By Lemma 2.7, we have

$$\limsup_{n\to\infty} d(u_n, T^n u_n) = 0. \quad (3.17)$$

Similarly, $\limsup_{n\to\infty} d(y_n, T^n y_n) = 0$ and

$$\limsup_{n\to\infty} d(z_n, T^n z_n) = 0. \quad (3.18)$$

Hence, we get

$$d(x_{n+1}, u) = d(\xi_n T^n z_n \oplus (1-\xi_n) T^n y_n, y_n)$$

$$\leq \xi_n d(y_n, y_n) + (1-\xi_n) d(T^n y_n, y_n)$$

$$\to 0 \text{ as n} \to \infty . \quad (3.19)$$

Similarly, we get

$$d(y_n, z_n) = d(T^n z_n, z_n) \ \to 0 \text{ as n} \to \infty \quad (3.20)$$

and

$$d(z_n, u_n) = d((1-\vartheta_n)u_n \oplus \vartheta_n T^n u_n, u)$$

$$\leq (1-\vartheta_n) d(u_n, u_n) + \vartheta_n d(T^n u_n, u)$$

$$\to 0 \text{ as n} \to \infty . \quad (3.21)$$

Since *T* is uniformly *L*-Lipschitzian, we have

$$d(Tu_n, u_n) \leq d(u_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1})$$

$$+ d(T^{n+1} x_{n+1}, T^{n+1} u_n)$$

$$+ d(T^{n+1} u_n, u_n)$$

$$\leq (1+L) d(u_n, x_{n+1})$$

$$+ d(x_{n+1}, T^{n+1} x_{n+1})$$

$$+ L d(T^n u_n, u_n)$$

$$\to 0 \text{ as n} \to \infty \quad (3.22)$$

which implies

$$\limsup_{n\to\infty} d(T^n u_n, u_n) = 0. \quad (3.23)$$

The proof is completed.

Step 3. After that, we conclude that sequence $\{u_n\}$ $\Delta-$ converges to a fixed point of *T*. Indeed, we proof that $z_\Delta := \bigcup_{\{\vartheta_n\}\subseteq\{u_n\}} A(\{u_n\} \subseteq T(T))$ and $z_\Delta(u_n)$ are collected of exactly one point. Let $w \in z_\Delta(u_n)$. By the definition of $z_\Delta(u_n)$, there exists a subsequence $\{w_n\}$ of $\{u_n\}$ such that $A(\{w_n\}) = \{w\}$. From Lemma 2.10, there is a subsequence $\{u_n\}$ of $\{w_n\}$ which $\Delta - \lim_{n\to\infty} u_n = u$ and $u \in C$. By Lemma 2.9, we have $u \in F(T)$. Since $\{d(w_n, u)\}$ converges, by Lemma 2.11, we get $w = u$. Thus $z_\Delta(u_n) \subseteq F(T)$. Finally, we prove $z_\Delta(u_n)$ comprise exactly one point. Let $\{w_n\}$ be a subsequence of $\{u_n\}$ by the uniqueness asymptotic center such that $A(\{w_n\}) = w$ and allow $A(\{w_n\}) = \{x\}$. Since $w = u \in F(T)$ and $\{d(u_n, u)\}$ converges, by using Lemma 2.12, we see that $x = u \in F(T)$, Therefore $z_\Delta(u_n) = \{x\}$. It refers to completes of the proof □

By using the similar technique as in the proof of Theorem 3.2 as the previous report [32], we get strong convergence theorem without the proof immediately.

Theorem 3.2. Let X, T, K, (i), (ii), (iii), (iv), $\{\xi_n\}, \{\vartheta_n\}$ satisfy the hypothesis of Theorem 3.1. Then, the sequence $\{u_n\}$ which is defined as (1.5) converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(u_n, F(T)) = 0$, where

$$d(u_n, F(T)) = 0, \inf\{d(u_n, u) : u \in F(T)\}.$$

The concept of special self mapping is called Condition(I) proposed by Senter and Dotson [27] as follows.

Definition 3.3. [27] Let (X,d) be a CAT(0) space and K a nonempty subset. A self mapping T with $F(T)\neq 0$ is said to satisfy condition (I) if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(l)>0$ for all $l>0$ such that $d(x,Tx)\geq f(d(x,F(T)))$ for all $x\in K$.

By using Condition (*I*) with the similar technique as in the proof of Theorem 3.3 in Thakur et al. ,[32] we obtain the following result.

Theorem 3.4. Let X,T,K, (i), (ii), (iii), (iv), $\{\xi_n\},\{\vartheta_n\}$ satisfy the presumption of Theorem 3.1 and let self mapping of T satisfy Condition (I). Then the sequence $\{u_n\}$ which is defined as (1.5) converges strongly to a fixed point of T.

Definition 3.5. Let (X,d) be a CAT(0) space and K a nonempty subset. Self mapping T is semicompact if K is closed and for all bounded sequence $\{u_n\}\subset K$ with $\lim_{n\to\infty} d(u_n,Tu_n)=0$, there exists a subsequence $\{x_{n_j}\}$ of $\{u_n\}$ such that $\{x_{n_j}\}\to u\in K$.

Using a similar technique as in the proof of Theorem 22 in Karapinar et al. [28], we obtain the following results.

Theorem 3.6. Let X,T,K, (i), (ii), (iii), (iv), $\{\xi_n\},\{\vartheta_n\}$ satisfy the hypothesis of Theorem 3.1 and define T as semicompact. Then, the sequence $\{u_n\}$, defined as (1.5) converges strongly to a fixed point of T.

4. Numerical Example

Let $X=\mathbb{R}$ be a Euclidean metric space and $K=[1,10]$. Let $T:\mathbb{R}\to\mathbb{R}$ be defined as $Tx=\sqrt[3]{x^2+4}$. It is noticeable that T is a continuously uniform L-Lipschitzian and $F(T)=\{2\}$ Next, we present that T is total asymptotically nonexpansive mapping on [1], [10].

Proof. Observe that the function

$f(x)=\sqrt[3]{x^2+4}-x,\ \forall x\in[1,10]$ has the derivative

$$f'(x)=\frac{1}{3}\left(\frac{1}{\sqrt[3]{(x^2+4)^2}}(2x)\right)-1,\ \forall x\in[1,10].$$

Since $x\geq 1$, we have $\frac{1}{3}\left(\frac{1}{\sqrt[3]{(x^2+4)^2}}(2x)\right)\leq 1$

and so, $f'(x)\leq 1$. Let $x,y\in[1,10]$ with $x\leq y$ which implies that

$f(y)\leq f(x)$. S0,

$$\sqrt[3]{y^2+4}-y \leq \sqrt[3]{x^2+4}-x$$
$$\sqrt[3]{y^2+4}-\sqrt[3]{x^2+4} \leq y-x$$
$$\left|\sqrt[3]{y^2+4}-\sqrt[3]{x^2+4}\right| \leq |y-x|$$

or

$$\left|\sqrt[3]{y^2+4}-\sqrt[3]{x^2+4}\right| \leq |x-y|.$$

Hence, we have $|Tx-Ty| \leq |x-y|$.

Therefore, *T* is nonexpansive mapping referring that *T* is total asymptotically nonexpansive mapping. □

Let $u_1=9$. By using MATLAB reckon the iterates of algorithm (1.4) and our algorithm (1.5) with two different control conditions conditions

$\eta_n = \frac{1}{3n+1}$, $\vartheta_n = \frac{1}{n}$, $\xi_n = \frac{9n}{10n+1}$ and $\eta_n = \frac{4n}{9n+1}$, $\vartheta_n = \frac{n}{10n+5}$, $\xi_n = \frac{5n}{10n+1}$. Then we obtain numerical results in Table 1, Figure 1, Table 2 and Figure 2.

Table 1 By using MATLAB reckon the iterates of algorithm (1.4) and our algorithm (1.5) with two different control conditions conditions $\eta_n = \frac{1}{3n+1}$, $\vartheta_n = \frac{1}{n}$, and $\xi_n = \frac{9n}{10n+1}$.

$\eta_n = \frac{1}{3n+1}$, $\vartheta_n = \frac{1}{n}$, $\xi_n = \frac{9n}{10n+1}$		
Iterative scheme		
Iterate	Algorithm (1.4)	Our Algorithm (1.5)
u_1	9.0000	9.0000
x_2	2.9678	2.0000
x_3	2.2612	-
x_4	2.0807	-
x_5	2.0807	-
x_6	2.0091	-
x_7	2.0032	-
x_8	2.0011	-
x_9	2.0004	-
x_{10}	2.0002	-
CPU Time (s)	0.0245	0.0067

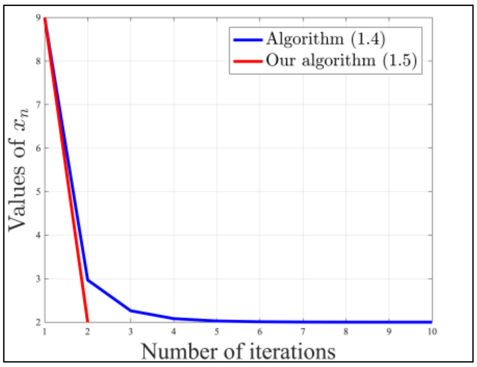

Figure 1 Graphical analysis our algorithm (1.5) faster than algorithm (1.4) in case

$\eta_n = \frac{1}{3n+1}$, $\vartheta_n = \frac{1}{n}$, and $\xi_n = \frac{9n}{10n+1}$.

Table 2 By using MATLAB reckon the iterates of algorithm (1.4) and our algorithm (1.5) with two different control conditions conditions $\eta_n = \frac{4n}{9n+1}$, $\vartheta_n = \frac{n}{10n+5}$ and $\xi_n = \frac{5n}{10n+1}$.

$\eta_n = \frac{4n}{9n+1}$, $\vartheta_n = \frac{n}{10n+5}$, $\xi_n = \frac{5n}{10n+1}$		
Iterative scheme		
Iterate	Algorithm (1.4)	Our Algorithm (1.5)
u_1	9.0000	9.0000
x_2	5.4793	2.0000
x_3	3.6701	-
x_4	2.7845	-
x_5	2.3612	-
x_6	2.1639	-
x_7	2.0737	-
x_8	2.0329	-
x_9	2.0147	-
x_{10}	2.0065	-
x_{11}	2.0029	-
x_{12}	2.0013	-
x_{13}	2.0006	-
x_{14}	2.0003	-
x_{15}	2.0001	-
CPU Time (s)	0.0066	0.0058

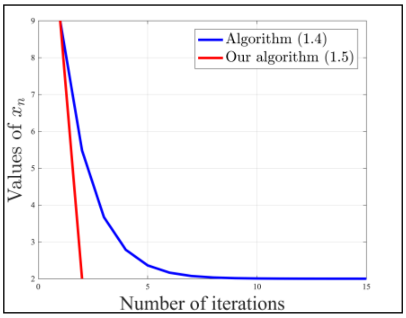

Figure 2 Graphical analysis our algorithm (1.5) faster than algorithm (1.4) in case $\eta_n = \frac{4n}{9n+1}$, $\vartheta_n = \frac{n}{10n+5}$ and $\xi_n = \frac{5n}{10n+1}$.

From Table 1, Figure 1, Table 2 and Figure 2 show that the numerical results of algorithm (1.4), and our algorithm (1.5) which our algorithm (1.5) faster than algorithm (1.4) in case number of iterations and CPU Time (second).

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