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Explicit Formulae for Some Sums of Trinomial Coefficients

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Abstract

For $a, b, c \in \{0,1,2\}, n \in \mathbb{N}, n \geq a+b+c$, and $n \equiv a+b+c \pmod 3$, explicit formulae for $T_{abc}(n) := \sum_{i\equiv a, j\equiv b, k\equiv c \ (\mathrm{mod}\ 3)} (i+j+k)!/i!\,j!\,k!$, where the sum is over all non-negative integers i, j, k such that $n = i+j+k$, are established by Carlitz [1] via properties of the cube root of unity. In this paper, another method to get these formulae is presented. Almost all $T_{abc}(n)$ are also expressed in terms of the cube root of unity.

Mathematics Subject Classification: 05A10, 11B65

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1. Introduction

Some sums of binomial coefficients (see for example [3]) suggest Carlitz [1] to consider simple sums of trinomial coefficients, defined by

$$(i, j, k) = \frac{(i+j+k)!}{i!\,j!\,k!} \quad (i, j, k \in \mathbb{N} \cup \{0\}).$$

Where

$$(x+y+z)^n = \sum_{i+j+k=n} (i, j, k) x^i y^j z^k.$$

For $a, b, c \in \{0,1\}, n \in \mathbb{N}, n \geq a+b+c$, and $n \equiv a+b+c \pmod 2$, define

$$S_{abc}(n) = \sum_{i\equiv a, j\equiv b, k\equiv c \ (\mathrm{mod}\ 2)} (i, j, k),$$

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where the sum is over all non-negative integers i, j, k such that $n = i + j + k$. Then all of the $S_{abc}(n)$ are:

$$S_{000}(n), S_{001}(n) = S_{010}(n) = S_{100}(n),$$
$$S_{011}(n) = S_{101}(n) = S_{110}(n), S_{111}(n).$$

In 1976, Carlitz [1] provided explicit formulae for such S_{abc} as follows:

$$S_{000}(n) = \frac{3^n + 3}{4}, S_{001}(n) = \frac{3^n + 1}{4},$$
$$S_{011}(n) = \frac{3^n - 1}{4}, S_{111}(n) = \frac{3^n - 3}{4}.$$

Later in 1988, a general formula for some sums of multinomial coefficients in terms of values of Krawtchouk polynomials was provided by Fixman [2].

For $a, b, c \in \{0,1,2\}, n \in \mathbb{N}, n \geq a + b + c$, and $n \equiv a + b + c$ (mod 3), define

$$T_{abc}(n) = \sum_{i \equiv a, j \equiv b, k \equiv c \ (\mathrm{mod}\ 3)} (i, j, k),$$

where the sum is over all non-negative integers i, j, k such that $n = i + j + k$. Then all of the $T_{abc}(n)$ are:

$$\begin{aligned}
&T_{000}(n),\\
&T_{001}(n) = T_{010}(n) = T_{100}(n),\\
&T_{002}(n) = T_{020}(n) = T_{200}(n),\\
&T_{011}(n) = T_{101}(n) = T_{110}(n),\\
&T_{012}(n) = T_{021}(n) = T_{102}(n)\\
&\qquad = T_{201}(n) = T_{120}(n)\\
&\qquad = T_{210}(n),\\
&T_{022}(n) = T_{202}(n) = T_{220}(n),\\
&T_{111}(n),\\
&T_{112}(n) = T_{121}(n) = T_{211}(n),\\
&T_{122}(n) = T_{212}(n) = T_{221}(n),\\
&T_{222}(n).
\end{aligned}$$

Carlitz [1] also provided explicit formulae for each $T_{abc}(n)$ by using properties of the cube root of unity as follows:

$$T_{000}(n) = \begin{cases} 3^{n-2} + 2(-1)^{\frac{n}{6}} 3^{\frac{n}{2}-1} & \text{if } n \equiv 0 \ (\mathrm{mod}\ 6), \\ 3^{n-2} & \text{if } n \equiv 3 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{001}(n) = \begin{cases} 3^{n-2} & \text{if } n \equiv 4 \ (\mathrm{mod}\ 6), \\ 3^{n-2} + 2(-1)^{\frac{n-1}{6}} 3^{\frac{n-1}{2}-1} & \text{if } n \equiv 1 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{002}(n) = \begin{cases} 3^{n-2} & \text{if } n \equiv 2 \ (\mathrm{mod}\ 6), \\ 3^{n-2} - 2(-1)^{\frac{n-5}{6}} 3^{\frac{n-5}{2}+1} & \text{if } n \equiv 5 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{011}(n) = \begin{cases} 3^{n-2} + (-1)^{\frac{n-2}{6}} 3^{\frac{n-2}{2}} & \text{if } n \equiv 2 \ (\mathrm{mod}\ 6), \\ 3^{n-2} + (-1)^{\frac{n-5}{6}} 3^{\frac{n-5}{2}+1} & \text{if } n \equiv 5 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{012}(n) = 3^{n-2},$$

$$T_{022}(n) = \begin{cases} 3^{n-2} - (-1)^{\frac{n-4}{6}} 3^{\frac{n-4}{2}+1} & \text{if } n \equiv 4 \ (\mathrm{mod}\ 6), \\ 3^{n-2} - (-1)^{\frac{n-1}{6}} 3^{\frac{n-1}{2}-1} & \text{if } n \equiv 1 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{111}(n) = \begin{cases} 3^{n-2} - (-1)^{\frac{n}{6}} 3^{\frac{n}{2}-1} & \text{if } n \equiv 0 \ (\mathrm{mod}\ 6), \\ 3^{n-2} + (-1)^{\frac{n-3}{6}} 3^{\frac{n-3}{2}+1} & \text{if } n \equiv 3 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{112}(n) = \begin{cases} 3^{n-2} + (-1)^{\frac{n-4}{6}} 3^{\frac{n-4}{2}+1} & \text{if } n \equiv 4 \ (\mathrm{mod}\ 6), \\ 3^{n-2} - (-1)^{\frac{n-1}{6}} 3^{\frac{n-1}{2}-1} & \text{if } n \equiv 1 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{122}(n) = \begin{cases} 3^{n-2} - (-1)^{\frac{n-2}{6}} 3^{\frac{n-2}{2}} & \text{if } n \equiv 2 \ (\mathrm{mod}\ 6), \\ 3^{n-2} + (-1)^{\frac{n-5}{6}} 3^{\frac{n-5}{2}+1} & \text{if } n \equiv 5 \ (\mathrm{mod}\ 6), \end{cases}$$

$$T_{222}(n) = \begin{cases} 3^{n-2} - (-1)^{\frac{n}{6}} 3^{\frac{n}{2}-1} & \text{if } n \equiv 0 \ (\mathrm{mod}\ 6), \\ 3^{n-2} - (-1)^{\frac{n-3}{6}} 3^{\frac{n-3}{2}+1} & \text{if } n \equiv 3 \ (\mathrm{mod}\ 6). \end{cases}$$

Some of $T_{abc}(n)$ are expressed in terms of the cube root of unity. Several examples are also given in [1].

In this work, we present another technique to get these explicit formulae by using the product of some matrices and properties of the cube root of unity. Almost all $T_{abc}(n)$ are also expressed in terms of the cube root of unity.

2. Explicit formulae for $T_{abc}(n)$

Let ω be the cube root of unity, given by

$$\omega = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right).$$

Then

$$1 + \omega + \omega^2 = 0 \text{ and } \omega^3 = 1.$$

Note that

$$(\omega+2)^3=(\omega-1)^3=3(2\omega+1), \tag{2.1}$$

$$\begin{aligned}(2\omega+1)^2&=(\omega+2)^2\omega\\&=(\omega-1)^2(-\omega-1)\\&=(\omega+2)(\omega-1)=-3,\end{aligned} \tag{2.2}$$

$$\begin{aligned}(\omega+2)(\omega+1)&=(1-\omega)\omega\\&=\omega^2(-\omega-2)\\&=2\omega+1,\end{aligned} \tag{2.3}$$

$$(\omega+2)^2=3(\omega+1), \tag{2.4}$$

$$(\omega-1)^2=-3\omega. \tag{2.5}$$

Consequently, if $n \equiv 0 \pmod 6$, then

$$\begin{aligned}(\omega+2)^n+(\omega-1)^n+(2\omega+1)^n\\=(-1)^{\frac{n}{6}}3^{\frac{n}{2}+1}.\end{aligned} \tag{2.6}$$

For $\boldsymbol{n}\in\mathbb{N}$ and $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\{\boldsymbol{1},\boldsymbol{\omega},\boldsymbol{\omega^2}\}$, we have

$$\begin{aligned}(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z})^{\boldsymbol{n}}&=\sum_{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{n}}(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})\boldsymbol{x}^{\boldsymbol{i}}\boldsymbol{y}^{\boldsymbol{j}}\boldsymbol{z}^{\boldsymbol{k}}\\&=\boldsymbol{T}_{\boldsymbol{000}}(\boldsymbol{n})+(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z})\boldsymbol{T}_{\boldsymbol{001}}(\boldsymbol{n})\\&\quad+(\boldsymbol{x}^2+\boldsymbol{y}^2+\boldsymbol{z}^2)\boldsymbol{T}_{\boldsymbol{002}}(\boldsymbol{n})\\&\quad+(\boldsymbol{xy}+\boldsymbol{yz}+\boldsymbol{xz})\boldsymbol{T}_{\boldsymbol{011}}(\boldsymbol{n})\\&\quad+(\boldsymbol{xy}^2+\boldsymbol{xz}^2+\boldsymbol{yx}^2+\boldsymbol{yz}^2+\boldsymbol{zx}^2\\&\quad+\boldsymbol{zy}^2)\boldsymbol{T}_{\boldsymbol{012}}(\boldsymbol{n})\\&\quad+(\boldsymbol{x}^2\boldsymbol{y}^2+\boldsymbol{y}^2\boldsymbol{z}^2+\boldsymbol{x}^2\boldsymbol{z}^2)\boldsymbol{T}_{\boldsymbol{022}}(\boldsymbol{n})\\&\quad+(\boldsymbol{xyz})\boldsymbol{T}_{\boldsymbol{111}}(\boldsymbol{n})\\&\quad+(\boldsymbol{xy}^2\boldsymbol{z}^2+\boldsymbol{x}^2\boldsymbol{yz}^2+\boldsymbol{x}^2\boldsymbol{y}^2\boldsymbol{z})\boldsymbol{T}_{\boldsymbol{122}}(\boldsymbol{n})\\&\quad+(\boldsymbol{xyz}^2+\boldsymbol{xy}^2\boldsymbol{z}+\boldsymbol{x}^2\boldsymbol{yz})\boldsymbol{T}_{\boldsymbol{112}}(\boldsymbol{n})\\&\quad+(\boldsymbol{x}^2\boldsymbol{y}^2\boldsymbol{z}^2)\boldsymbol{T}_{\boldsymbol{222}}(\boldsymbol{n}).\end{aligned} \tag{2.7}$$

Substituting $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ in (2.7) by $(\boldsymbol{1},\boldsymbol{1},\boldsymbol{1}),(\boldsymbol{1},\boldsymbol{1},\boldsymbol{\omega}),(\boldsymbol{1},\boldsymbol{1},\boldsymbol{\omega^2}),(\boldsymbol{1},\boldsymbol{\omega},\boldsymbol{\omega}),(\boldsymbol{1},\boldsymbol{\omega},\boldsymbol{\omega^2}),(\boldsymbol{1},\boldsymbol{\omega^2},\boldsymbol{\omega^2}),(\boldsymbol{\omega},\boldsymbol{\omega},\boldsymbol{\omega}),(\boldsymbol{\omega},\boldsymbol{\omega},\boldsymbol{\omega^2}),(\boldsymbol{\omega},\boldsymbol{\omega^2},\boldsymbol{\omega^2})$, **and** $(\boldsymbol{\omega^2},\boldsymbol{\omega^2},\boldsymbol{\omega^2})$, respectively, we obtain the 10 equations in the indeterminate values

$\boldsymbol{T}_{\boldsymbol{000}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{001}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{002}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{011}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{012}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{022}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{111}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{112}}(\boldsymbol{n}),\boldsymbol{T}_{\boldsymbol{122}}(\boldsymbol{n})$, **and** $\boldsymbol{T}_{\boldsymbol{222}}(\boldsymbol{n})$

with coefficients in terms of $\boldsymbol{\omega}$. We can write this 10-equation system in the form of matrix equation as follows:

$$N=WT, \tag{2.8}$$

where

$$\boldsymbol{N}=\begin{bmatrix}\boldsymbol{3^n}\\(\boldsymbol{\omega}+\boldsymbol{2})^{\boldsymbol{n}}\\(\boldsymbol{1}-\boldsymbol{\omega})^{\boldsymbol{n}}\\(\boldsymbol{2\omega}+\boldsymbol{1})^{\boldsymbol{n}}\\\boldsymbol{0}\\(-\boldsymbol{2\omega}-\boldsymbol{1})^{\boldsymbol{n}}\\(\boldsymbol{3\omega})^{\boldsymbol{n}}\\(\boldsymbol{\omega}-\boldsymbol{1})^{\boldsymbol{n}}\\(-\boldsymbol{\omega}-\boldsymbol{2})^{\boldsymbol{n}}\\(\boldsymbol{3\omega^2})^{\boldsymbol{n}}\end{bmatrix},\boldsymbol{T}=\begin{bmatrix}\boldsymbol{T}_{\boldsymbol{000}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{001}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{002}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{011}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{012}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{022}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{111}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{112}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{122}}(\boldsymbol{n})\\\boldsymbol{T}_{\boldsymbol{222}}(\boldsymbol{n})\end{bmatrix},$$

and

$$W=\begin{bmatrix}1&3&3&3&6&3&1&3&3&1\\1&\omega+2&-\omega+1&2\omega+1&0&-2\omega-1&\omega&\omega-1&-\omega-2&-\omega-1\\1&-\omega+1&\omega+2&-2\omega-1&0&2\omega+1&-\omega-1&-\omega-2&\omega-1&\omega\\1&2\omega+1&-2\omega-1&\omega-1&0&-\omega-2&-\omega-1&-\omega+1&\omega+2&\omega\\1&0&0&0&-3&0&1&0&0&1\\1&-2\omega-1&2\omega+1&-\omega-2&0&\omega-1&\omega&\omega+2&-\omega+1&-\omega-1\\1&3\omega&-3\omega-3&-3\omega-3&6&3\omega&1&3\omega&-3\omega-3&1\\1&\omega-1&-\omega-2&-\omega+1&0&\omega+2&\omega&-2\omega-1&2\omega+1&-\omega-1\\1&-\omega-2&\omega-1&\omega+2&0&-\omega+1&-\omega-1&2\omega+1&-2\omega-1&\omega\\1&-3\omega-3&3\omega&3\omega&6&-3\omega-3&1&-3\omega-3&3\omega&1\end{bmatrix}.$$

From (2.8), we have

$$W^{-1}N=T, \tag{2.9}$$

where

$$W^{-1}=\begin{bmatrix}
\frac{1}{27} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{27} & \frac{1}{9} & \frac{1}{9} & \frac{1}{27}\\
\frac{1}{27} & \frac{\omega+1}{\alpha} & \frac{\omega}{\alpha} & \frac{1}{\alpha} & 0 & \frac{-1}{\alpha} & \frac{-\omega+1}{3\alpha} & \frac{-\omega}{\alpha} & \frac{-\omega-1}{\alpha} & \frac{-\omega-2}{3\alpha}\\
\frac{1}{27} & \frac{\omega}{\alpha} & \frac{\omega+1}{\alpha} & \frac{-1}{\alpha} & 0 & \frac{1}{\alpha} & \frac{-\omega-2}{3\alpha} & \frac{-\omega-1}{\alpha} & \frac{-\omega}{\alpha} & \frac{-\omega+1}{3\alpha}\\
\frac{1}{27} & \frac{1}{\alpha} & \frac{-1}{\alpha} & \frac{-\omega}{\alpha} & 0 & \frac{-\omega-1}{\alpha} & \frac{-\omega-2}{3\alpha} & \frac{\omega}{\alpha} & \frac{\omega+1}{\alpha} & \frac{-\omega+1}{3\alpha}\\
\frac{1}{27} & 0 & 0 & 0 & \frac{-1}{9} & 0 & \frac{1}{27} & 0 & 0 & \frac{1}{27}\\
\frac{1}{27} & \frac{-1}{\alpha} & \frac{1}{\alpha} & \frac{-\omega-1}{\alpha} & 0 & \frac{-\omega}{\alpha} & \frac{-\omega+1}{3\alpha} & \frac{\omega+1}{\alpha} & \frac{\omega}{\alpha} & \frac{-\omega-2}{3\alpha}\\
\frac{1}{27} & \frac{-\omega+1}{\alpha} & \frac{-\omega-2}{\alpha} & \frac{-\omega-2}{\alpha} & \frac{2}{9} & \frac{-\omega+1}{\alpha} & \frac{1}{27} & \frac{-\omega+1}{\alpha} & \frac{-\omega-2}{\alpha} & \frac{1}{27}\\
\frac{1}{27} & \frac{-\omega}{\alpha} & \frac{-\omega-1}{\alpha} & \frac{\omega}{\alpha} & 0 & \frac{\omega+1}{\alpha} & \frac{-\omega+1}{3\alpha} & \frac{-1}{\alpha} & \frac{1}{\alpha} & \frac{-\omega-2}{3\alpha}\\
\frac{1}{27} & \frac{-\omega-1}{\alpha} & \frac{-\omega}{\alpha} & \frac{\omega+1}{\alpha} & 0 & \frac{\omega}{\alpha} & \frac{-\omega-2}{3\alpha} & \frac{1}{\alpha} & \frac{-1}{\alpha} & \frac{-\omega+1}{3\alpha}\\
\frac{1}{27} & \frac{-\omega-2}{\alpha} & \frac{-\omega+1}{\alpha} & \frac{-\omega+1}{\alpha} & \frac{2}{9} & \frac{-\omega-2}{\alpha} & \frac{1}{27} & \frac{-\omega-2}{\alpha} & \frac{-\omega+1}{\alpha} & \frac{1}{27}
\end{bmatrix}$$

and $\alpha=9(2\omega+1)$. Next we calculate all of the $T_{abc}(n)$ from (2.9).

Row 1 in (2.9): In this case, $n>0$ and

$n\equiv 0 \pmod 3$.

$T_{000}(n)$

$$\begin{aligned}
&= \frac{3^n}{27}+\frac{(\omega+2)^n}{9}+\frac{(1-\omega)^n}{9}+\frac{(2\omega+1)^n}{9}+\frac{(-2\omega-1)^n}{9}\\
&\quad+\frac{(3\omega)^n}{27}+\frac{(\omega-1)^n}{9}+\frac{(-\omega-2)^n}{9}+\frac{(3\omega^2)^n}{27}\\
&= 3^{n-2}+(1+(-1)^n)\frac{((\omega+2)^n+(\omega-1)^n+(2\omega+1)^n)}{9}\\
&= \begin{cases} 3^{n-2}+2(-1)^{\frac{n}{6}}3^{\frac{n}{2}-1} & \text{if } n\equiv 0 \pmod 6, \quad \text{(by (2.6))}\\ 3^{n-2} & \text{if } n\equiv 3 \pmod 6.\end{cases}
\end{aligned}$$

Row 2 in (2.9): In this case, $n\geq 1$ and

$n\equiv 1 \pmod 3$.

$T_{001}(n)$

$$\begin{aligned}
&= \frac{3^n}{27}+\frac{(\omega+2)^n(\omega+1)}{9(2\omega+1)}+\frac{(1-\omega)^n\omega}{9(2\omega+1)}+\frac{(2\omega+1)^n}{9(2\omega+1)}\\
&\quad-\frac{(-2\omega-1)^n}{9(2\omega+1)}+\frac{(3\omega)^n(-\omega+1)}{27(2\omega+1)}-\frac{(\omega-1)^n\omega}{9(2\omega+1)}\\
&\quad+\frac{(-\omega-2)^n(-\omega-1)}{9(2\omega+1)}+\frac{(3\omega^2)^n(-\omega-2)}{27(2\omega+1)}\\
&= \frac{3^n}{27}+\frac{(\omega+2)^{n-1}(2\omega+1)}{9(2\omega+1)}+\frac{(1-\omega)^{n-1}(2\omega+1)}{9(2\omega+1)}\\
&\quad+\frac{(2\omega+1)^n}{9(2\omega+1)}-\frac{(-1)^n(2\omega+1)^n}{9(2\omega+1)}+\frac{3^n\omega^{n-1}(2\omega+1)}{27(2\omega+1)}\\
&\quad+\frac{(\omega-1)^{n-1}(2\omega+1)}{9(2\omega+1)}+\frac{(-\omega-2)^{n-1}(2\omega+1)}{9(2\omega+1)}\\
&\quad+\frac{3^n\omega^{2(n-1)}(2\omega+1)}{27(2\omega+1)} \quad \text{(by (2.3))}\\
&= 3^{n-2}+(1+(-1)^{n-1})\\
&\quad\frac{((\omega+2)^{n-1}+(\omega-1)^{n-1}+(2\omega+1)^{n-1})}{9}\\
&= \begin{cases} 3^{n-2} & \text{if } n\equiv 4 \pmod 6,\\ 3^{n-2}+2(-1)^{\frac{n-1}{6}}3^{\frac{n-1}{2}-1} & \text{if } n\equiv 1 \pmod 6. \quad \text{(by (2.6))}\end{cases}
\end{aligned}$$

Row 3 in (2.9): In this case, $n \geq 2$ and $n \equiv 2 \pmod 3$.

$$
\begin{aligned}
&T_{002}(n)\\
&= \frac{3^n}{27} + \frac{(\omega+2)^n\omega}{9(2\omega+1)} + \frac{(1-\omega)^n(\omega+1)}{9(2\omega+1)} - \frac{(2\omega+1)^n}{9(2\omega+1)}\\
&\quad + \frac{(-2\omega-1)^n}{9(2\omega+1)} + \frac{(3\omega)^n(-\omega-2)}{27(2\omega+1)} + \frac{(\omega-1)^n(-\omega-1)}{9(2\omega+1)}\\
&\quad - \frac{(-\omega-2)^n\omega}{9(2\omega+1)} + \frac{(3\omega^2)^n(-\omega+1)}{27(2\omega+1)}\\
&= \frac{3^n}{27} + \frac{-9(\omega+2)^{n-5}(2\omega+1)}{9(2\omega+1)} + \frac{-9(1-\omega)^{n-5}(2\omega+1)}{9(2\omega+1)}\\
&\quad - \frac{9(2\omega+1)^{n-5}(2\omega+1)}{9(2\omega+1)} + \frac{-9(-2\omega-1)^{n-5}(2\omega+1)}{9(2\omega+1)}\\
&\quad + \frac{3^n\omega^{n-2}(2\omega+1)}{27(2\omega+1)} + \frac{-9(\omega-1)^{n-5}(2\omega+1)}{9(2\omega+1)}\\
&\quad - \frac{9(-\omega-2)^{n-5}(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^{2(n-2)}(2\omega+1)}{27(2\omega+1)}\\
&\quad \big(\text{by } (2.1), (2.2), \text{and } (2.3)\big)\\
&= 3^{n-2} - (1+(-1)^{n-5})\\
&\quad ((\omega+2)^{n-5} + (\omega-1)^{n-5} + (2\omega+1)^{n-5})\\
&= \begin{cases} 3^{n-2} & \text{if } n \equiv 2 \pmod 6, \\ 3^{n-2} - 2(-1)^{\frac{n-5}{6}} 3^{\frac{n-5}{2}+1} & \text{if } n \equiv 5 \pmod 6. \end{cases} \quad (\text{by } (2.6))
\end{aligned}
$$

Row 4 in (2.9): In this case, $n \geq 2$ and $n \equiv 2 \pmod 3$.

$$
\begin{aligned}
&T_{011}(n)\\
&= \frac{3^n}{27} + \frac{(\omega+2)^n}{9(2\omega+1)} - \frac{(1-\omega)^n}{9(2\omega+1)} - \frac{(2\omega+1)^n\omega}{9(2\omega+1)}\\
&\quad + \frac{(-2\omega-1)^n(-\omega-1)}{9(2\omega+1)} + \frac{(3\omega)^n(-\omega-2)}{27(2\omega+1)}\\
&\quad + \frac{(\omega-1)^n\omega}{9(2\omega+1)} + \frac{(-\omega-2)^n(\omega+1)}{9(2\omega+1)} + \frac{(3\omega^2)^n(-\omega+1)}{27(2\omega+1)}\\
&= \frac{3^n}{27} + \frac{3(\omega+2)^{n-3}(2\omega+1)}{9(2\omega+1)} + \frac{3(1-\omega)^{n-3}(2\omega+1)}{9(2\omega+1)}\\
&\quad + \frac{3(2\omega+1)^{n-3}\omega(2\omega+1)}{9(2\omega+1)}\\
&\quad + \frac{3(-2\omega-1)^{n-3}(-\omega-1)(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^{n-2}(2\omega+1)}{27(2\omega+1)}\\
&\quad + \frac{3(\omega-1)^{n-3}\omega(2\omega+1)}{9(2\omega+1)} - \frac{3(-\omega-2)^{n-3}(\omega+1)(2\omega+1)}{9(2\omega+1)}\\
&\quad + \frac{3^n\omega^{2(n-2)}(2\omega+1)}{27(2\omega+1)} \quad \big(\text{by } (2.1), (2.2) \text{and } (2.3)\big)\\
&= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(1+(\omega+1)(-1)^{n-2})\\
&\quad + \frac{(\omega-1)^{n-3}}{3}(\omega-(-1)^{n-2})\\
&\quad + \frac{(2\omega+1)^{n-3}}{3}(\omega+(\omega+1)(-1)^{n-2}).
\end{aligned}
$$

If $n \equiv 2 \pmod 6$, then

$$
\begin{aligned}
&T_{011}(n)\\
&= 3^{n-2} + \frac{(\omega+2)^{n-2} + (\omega-1)^{n-2} + (2\omega+1)^{n-2}}{3}\\
&= 3^{n-2} + (-1)^{\frac{n-2}{6}} 3^{\frac{n-2}{2}}. \quad \big(\text{by } (2.6)\big)
\end{aligned}
$$

If $n \equiv 5 \pmod 6$, then

$$
\begin{aligned}
&T_{011}(n)\\
&= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(-\omega) + \frac{(\omega-1)^{n-3}}{3}(\omega+1)\\
&\quad + \frac{(2\omega+1)^{n-3}}{3}(-1)\\
&= 3^{n-2} + (\omega+2)^{n-5} + (\omega-1)^{n-5} + (2\omega+1)^{n-5} \quad \big(\text{by } (2.2)\big)\\
&= 3^{n-2} + (-1)^{\frac{n-5}{6}} 3^{\frac{n-5}{2}+1}. \quad \big(\text{by } (2.6)\big)
\end{aligned}
$$

Row 5 in (2.9): In this case, $n \geq 3$ and $n \equiv 0 \pmod 3$.

$$T_{012}(n) = \frac{3^n}{27} + \frac{(3\omega)^n}{27} + \frac{(3\omega^2)^n}{27} = 3^{n-2}.$$

Row 6 in (2.9): In this case, $n \geq 4$ and $n \equiv 1 \pmod 3$.

$$
\begin{aligned}
&T_{022}(n)\\
&= \frac{3^n}{27} - \frac{(\omega+2)^n}{9(2\omega+1)} + \frac{(1-\omega)^n}{9(2\omega+1)} + \frac{(2\omega+1)^n(-\omega-1)}{9(2\omega+1)}\\
&\quad - \frac{(-2\omega-1)^n\omega}{9(2\omega+1)} + \frac{(3\omega)^n(-\omega+1)}{27(2\omega+1)} + \frac{(\omega-1)^n(\omega+1)}{9(2\omega+1)}\\
&\quad + \frac{(-\omega-2)^n\omega}{9(2\omega+1)} + \frac{(3\omega^2)^n(-\omega-2)}{27(2\omega+1)}\\
&= \frac{3^n}{27} - \frac{3(\omega+2)^{n-3}(2\omega+1)}{9(2\omega+1)} - \frac{3(1-\omega)^{n-3}(2\omega+1)}{9(2\omega+1)}\\
&\quad - \frac{3(2\omega+1)^{n-3}(-\omega-1)(2\omega+1)}{9(2\omega+1)}\\
&\quad - \frac{3(-2\omega-1)^{n-3}\omega(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^{n-1}(2\omega+1)}{27(2\omega+1)}\\
&\quad + \frac{3(\omega-1)^{n-3}(\omega+1)(2\omega+1)}{9(2\omega+1)}\\
&\quad - \frac{3(-\omega-2)^{n-3}\omega(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^{2(n-1)}(2\omega+1)}{27(2\omega+1)}\\
&\quad \big(\text{by } (2.1), (2.2), \text{and } (2.3)\big)\\
&= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(-1-(-1)^{n-1}\omega)\\
&\quad + \frac{(\omega-1)^{n-3}}{3}(\omega+1-(-1)^{n-1})\\
&\quad + \frac{(2\omega+1)^{n-3}}{3}(\omega+1-(-1)^{n-1}\omega).
\end{aligned}
$$

If $n \equiv 4 \pmod 6$, then

$$
\begin{aligned}
&T_{022}(n)\\
&= 3^{n-2} + \frac{(\omega+2)^{n-3}(-1+\omega)}{3} + \frac{(\omega-1)^{n-3}(\omega+2)}{3}\\
&\quad + \frac{(2\omega+1)^{n-3}(2\omega+1)}{3}\\
&= 3^{n-2} + \frac{-3(\omega+2)^{n-4} - 3(\omega-1)^{n-4} - 3(2\omega+1)^{n-4}}{3}\\
&\quad \big(\text{by } (2.2)\big)\\
&= 3^{n-2} - (-1)^{\frac{n-4}{6}} 3^{\frac{n-4}{2}+1}. \quad \big(\text{by } (2.6)\big)
\end{aligned}
$$

If $n \equiv 1 \pmod 6$, then

$T_{022}(n)$

$$= 3^{n-2} + \frac{(\omega+2)^{n-3}(-1-\omega)}{3} + \frac{(\omega-1)^{n-3}\omega}{3} + \frac{(2\omega+1)^{n-3}}{3}$$
$$= 3^{n-2} + \frac{-(\omega+2)^{n-4}(2\omega+1)}{3} + \frac{-(\omega-1)^{n-4}(2\omega+1)}{3} + \frac{(2\omega+1)^{n-3}}{3} \quad (\text{by } (2.3))$$
$$= 3^{n-2} + \frac{-3(\omega+2)^{n-7}(2\omega+1)^2}{3} + \frac{-3(\omega-1)^{n-7}(2\omega+1)^2}{3} + \frac{(2\omega+1)^{n-3}}{3} \quad (\text{by } (2.1))$$
$$= 3^{n-2} + \frac{9(\omega+2)^{n-7} + 9(\omega-1)^{n-7} + 9(2\omega+1)^{n-7}}{3} \quad (\text{by } (2.2))$$
$$= 3^{n-2} + (-1)^{\frac{n-7}{6}} 3^{\frac{n-7}{2}+2} \quad (by\ (2.6))$$
$$= 3^{n-2} - (-1)^{\frac{n-1}{6}} 3^{\frac{n-1}{2}-1}.$$

Row 7 in (2.9): In this case, $n \geq 3$ and $n \equiv 0 \pmod 3$.

$T_{111}(n)$
$$= \frac{3^n}{27} + \frac{(\omega+2)^n(-\omega+1)}{9(2\omega+1)} + \frac{(1-\omega)^n(-\omega-2)}{9(2\omega+1)} + \frac{(2\omega+1)^n(-\omega-2)}{9(2\omega+1)} + \frac{(-2\omega-1)^n(-\omega+1)}{9(2\omega+1)} + \frac{(3\omega)^n}{27} + \frac{(\omega-1)^n(-\omega+1)}{9(2\omega+1)} + \frac{(-\omega-2)^n(-\omega-2)}{9(2\omega+1)} + \frac{(3\omega^2)^n}{27}$$
$$= \frac{3^n}{27} + \frac{3(\omega+2)^{n-3}(-\omega+1)(2\omega+1)}{9(2\omega+1)} - \frac{3(1-\omega)^{n-3}(-\omega-2)(2\omega+1)}{9(2\omega+1)} - \frac{3(2\omega+1)^{n-3}(-\omega-2)(2\omega+1)}{9(2\omega+1)} + \frac{3(-2\omega-1)^{n-3}(-\omega+1)(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^n}{27} + \frac{3(\omega-1)^{n-3}(-\omega+1)(2\omega+1)}{9(2\omega+1)} - \frac{3(-\omega-2)^{n-3}(-\omega-2)(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^{2n}}{27}$$
(by (2.1) and (2.2))
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}\left(-\omega+1+(-1)^n(-\omega-2)\right) + \frac{(\omega-1)^{n-3}}{3}\left(-\omega+1+(-1)^n(-\omega-2)\right) + \frac{(2\omega+1)^{n-3}}{3}\left(\omega+2-(-1)^n(-\omega+1)\right).$$

If $n \equiv 0 \pmod 6$, then

$T_{111}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(-2\omega-1) + \frac{(\omega-1)^{n-3}}{3}(-2\omega-1) + \frac{(2\omega+1)^{n-3}}{3}(2\omega+1)$$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}\frac{(\omega+2)^3}{-3} + \frac{(\omega-1)^{n-3}}{3}\frac{(\omega-1)^3}{-3} + \frac{(2\omega+1)^{n-3}}{3}\frac{(2\omega+1)^3}{-3} \quad (\text{by } (2.1) \text{and } (2.2))$$
$$= 3^{n-2} - (-1)^{\frac{n}{6}} 3^{\frac{n}{2}-1}. \quad (\text{by } (2.6))$$

If $n \equiv 3 \pmod 6$, then

$T_{111}(n)$
$$= 3^{n-2} + (\omega+2)^{n-3} + (\omega-1)^{n-3} + (2\omega+1)^{n-3}$$
$$= 3^{n-2} + (-1)^{\frac{n-3}{6}} 3^{\frac{n-3}{2}+1}. \quad (\text{by } (2.6))$$

Row 8 in (2.9): In this case, $n \geq 4$ and $n \equiv 1 \pmod 3$.

$T_{112}(n)$
$$= \frac{3^n}{27} + \frac{(\omega+2)^n(-\omega)}{9(2\omega+1)} + \frac{(1-\omega)^n(-\omega-1)}{9(2\omega+1)} + \frac{(2\omega+1)^n\omega}{9(2\omega+1)} + \frac{(-2\omega-1)^n(\omega+1)}{9(2\omega+1)} + \frac{(3\omega)^n(-\omega+1)}{27(2\omega+1)} - \frac{(\omega-1)^n}{9(2\omega+1)} + \frac{(-\omega-2)^n}{9(2\omega+1)} + \frac{(3\omega^2)^n(-\omega-2)}{27(2\omega+1)}$$
$$= \frac{3^n}{27} + \frac{3(\omega+2)^{n-3}(-\omega)(2\omega+1)}{9(2\omega+1)} - \frac{3(1-\omega)^{n-3}(-\omega-1)(2\omega+1)}{9(2\omega+1)} - \frac{3(2\omega+1)^{n-3}\omega(2\omega+1)}{9(2\omega+1)} + \frac{3(-2\omega-1)^{n-3}(\omega+1)(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^{n-1}(2\omega+1)}{27(2\omega+1)} - \frac{3(\omega-1)^{n-3}(2\omega+1)}{9(2\omega+1)} - \frac{3(-\omega-2)^{n-3}(2\omega+1)}{9(2\omega+1)} + \frac{3^n\omega^{2(n-1)}(2\omega+1)}{27(2\omega+1)} \quad (\text{by } (2.1), (2.2), \text{and } (2.3))$$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}\left(-\omega-(-1)^{n-1}\right) + \frac{(\omega-1)^{n-3}}{3}\left(-1+(-1)^{n-1}(\omega+1)\right)$$

If $n \equiv 4 \pmod 6$, then

$T_{112}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}(-\omega+1)}{3} + \frac{(\omega-1)^{n-3}(-\omega-2)}{3} + \frac{(2\omega+1)^{n-3}(-2\omega-1)}{3}$$
$$= 3^{n-2} + (\omega+2)^{n-4} + (\omega-1)^{n-4} + (2\omega+1)^{n-4} \quad (\text{by } (2.2))$$
$$= 3^{n-2} + (-1)^{\frac{n-4}{6}} 3^{\frac{n-4}{2}+1}. \quad (\text{by } (2.6))$$

If $n \equiv 1 \pmod 6$, then

$T_{112}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}(-\omega-1)}{3} + \frac{(\omega-1)^{n-3}\omega}{3} + \frac{(2\omega+1)^{n-3}}{3}$$
$$= 3^{n-2} - \frac{(\omega+2)^{n-1} + (\omega-1)^{n-1} + (2\omega+1)^{n-1}}{9} \quad (\text{by } (2.2), (2.4), \text{and } (2.5))$$
$$= 3^{n-2} - (-1)^{\frac{n-1}{6}} 3^{\frac{n-1}{2}-1} \quad (\text{by } (2.6))$$

Row 9 in (2.9): In this case, $n \geq 5$ and $n \equiv 2 \pmod 3$.

$$\begin{aligned}
&T_{122}(n)\\
=\;& \frac{3^n}{27}+\frac{(\omega+2)^n(-\omega-1)}{9(2\omega+1)}-\frac{(1-\omega)^n\omega}{9(2\omega+1)}\\
&+\frac{(2\omega+1)^n(\omega+1)}{9(2\omega+1)}+\frac{(-2\omega-1)^n\omega}{9(2\omega+1)}\\
&+\frac{(3\omega)^n(-\omega-2)}{27(2\omega+1)}+\frac{(\omega-1)^n}{9(2\omega+1)}-\frac{(-\omega-2)^n}{9(2\omega+1)}\\
&+\frac{(3\omega^2)^n(-\omega+1)}{27(2\omega+1)}\\
=\;& \frac{3^n}{27}+\frac{3(\omega+2)^{n-3}(-\omega-1)(2\omega+1)}{9(2\omega+1)}\\
&+\frac{3(1-\omega)^{n-3}\omega(2\omega+1)}{9(2\omega+1)}\\
&-\frac{3(2\omega+1)^{n-3}(\omega+1)(2\omega+1)}{9(2\omega+1)}\\
&+\frac{3(-2\omega-1)^{n-3}\omega(2\omega+1)}{9(2\omega+1)}+\frac{3^n\omega^{n-2}(2\omega+1)}{27(2\omega+1)}\\
&+\frac{3(\omega-1)^{n-3}(2\omega+1)}{9(2\omega+1)}+\frac{3(-\omega-2)^{n-3}(2\omega+1)}{9(2\omega+1)}\\
&+\frac{3^n\omega^{2(n-2)}(2\omega+1)}{27(2\omega+1)} \quad \big(\text{by (2.1), (2.2), and (2.3)}\big)\\
=\;& 3^{n-2}+\frac{(\omega+2)^{n-3}}{3}(-\omega-1-(-1)^{n-2})\\
&+\frac{(\omega-1)^{n-3}}{3}(1-(-1)^{n-2}\omega)\\
&+\frac{(2\omega+1)^{n-3}}{3}(-\omega-1-(-1)^{n-2}\omega).
\end{aligned}$$

If $n \equiv 2 \pmod 6$, then

$$\begin{aligned}
&T_{122}(n)\\
=\;& 3^{n-2}+\frac{(\omega+2)^{n-3}(-\omega-2)}{3}+\frac{(\omega-1)^{n-3}(1-\omega)}{3}\\
&+\frac{(2\omega+1)^{n-3}(-2\omega-1)}{3}\\
=\;& 3^{n-2}-(-1)^{\frac{n-2}{6}}3^{\frac{n-2}{2}} \quad \big(\text{by (2.6)}\big)
\end{aligned}$$

If $n \equiv 5 \pmod 6$, then

$$\begin{aligned}
&T_{122}(n)\\
=\;& 3^{n-2}+\frac{(\omega+2)^{n-3}(-\omega)}{3}+\frac{(\omega-1)^{n-3}(1+\omega)}{3}\\
&+\frac{(2\omega+1)^{n-3}(-1)}{3}\\
=\;& 3^{n-2}+\frac{3(\omega+2)^{n-5}+3(\omega-1)^{n-5}+3(2\omega+1)^{n-5}}{3}\\
&\big(\text{by (2.2)}\big)\\
=\;& 3^{n-2}+(-1)^{\frac{n-5}{6}}3^{\frac{n-5}{2}+1} \quad \big(\text{by (2.6)}\big)
\end{aligned}$$

Row 10 in (2.9): In this case, $n \geq 6$ and $n \equiv 0 \pmod 3$.

$$\begin{aligned}
&T_{222}(n)\\
=\;& \frac{3^n}{27}+\frac{(\omega+2)^n(-\omega-2)}{9(2\omega+1)}+\frac{(1-\omega)^n(-\omega+1)}{9(2\omega+1)}\\
&+\frac{(2\omega+1)^n(-\omega+1)}{9(2\omega+1)}+\frac{(-2\omega-1)^n(-\omega-2)}{9(2\omega+1)}+\frac{(3\omega)^n}{27}\\
&+\frac{(\omega-1)^n(-\omega-2)}{9(2\omega+1)}+\frac{(-\omega-2)^n(-\omega+1)}{9(2\omega+1)}+\frac{(3\omega^2)^n}{27}\\
=\;& \frac{3^n}{27}+\frac{3(\omega+2)^{n-3}(-\omega-2)(2\omega+1)}{9(2\omega+1)}\\
&-\frac{3(1-\omega)^{n-3}(-\omega+1)(2\omega+1)}{9(2\omega+1)}\\
&-\frac{3(2\omega+1)^{n-3}(-\omega+1)(2\omega+1)}{9(2\omega+1)}\\
&+\frac{3(-2\omega-1)^{n-3}(-\omega-2)(2\omega+1)}{9(2\omega+1)}+\frac{(3\omega)^n}{27}\\
&+\frac{3(\omega-1)^{n-3}(-\omega-2)(2\omega+1)}{9(2\omega+1)}\\
&-\frac{3(-\omega-2)^{n-3}(-\omega+1)(2\omega+1)}{9(2\omega+1)}+\frac{(3\omega^2)^n}{27}\\
&\big(\text{by (2.1) and (2.2)}\big)\\
=\;& 3^{n-2}+\frac{(\omega+2)^{n-3}}{3}\big(-\omega-2+(-1)^n(-\omega+1)\big)\\
&+\frac{(\omega-1)^{n-3}}{3}\big(-\omega-2+(-1)^n(-\omega+1)\big)\\
&+\frac{(2\omega+1)^{n-3}}{3}\big(\omega-1+(-1)^n(\omega+2)\big).
\end{aligned}$$

If $n \equiv 0 \pmod 6$, then

$$\begin{aligned}
&T_{222}(n)\\
=\;& 3^{n-2}+\frac{(\omega+2)^{n-3}(-2\omega-1)}{3}+\frac{(\omega-1)^{n-3}(-2\omega-1)}{3}\\
&+\frac{(2\omega+1)^{n-3}(2\omega+1)}{3}\\
=\;& 3^{n-2}+\frac{-(\omega+2)^n/3-(\omega-1)^n/3-(2\omega+1)^n/3}{3}\\
&\big(\text{by (2.1) and (2..2)}\big)\\
=\;& 3^{n-2}-(-1)^{\frac{n}{6}}3^{\frac{n}{2}-1} \quad \big(\text{by (2.6)}\big)
\end{aligned}$$

If $n \equiv 3 \pmod 6$, then

$$\begin{aligned}
&T_{222}(n)\\
=\;& 3^{n-2}+\\
&\frac{(\omega+2)^{n-3}(-3)+(\omega-1)^{n-3}(-3)+(2\omega+1)^{n-3}(-3)}{3}\\
=\;& 3^{n-2}-(-1)^{\frac{n-3}{6}}3^{\frac{n-3}{2}+1} \quad \big(\text{by (2.6)}\big)
\end{aligned}$$

3. Conclusions

For $a, b, c \in \{0,1,2\}, n \in \mathbb{N}, n \geq a+b+c$, and $n \equiv a+b+c \pmod 3$, define

$$T_{abc}(n) = \sum_{i\equiv a, j\equiv b, k\equiv c \pmod 3} (i,j,k),$$

where the sum is over all non-negative integers i, j, k such that $n = i+j+k$ and

$$(i,j,k) = \frac{(i+j+k)!}{i!\,j!\,k!} \quad (i,j,k \in \mathbb{N} \cup \{0\}).$$

Then all of the $T_{abc}(n)$ are:

$T_{000}(n)$,
$T_{001}(n) = T_{010}(n) = T_{100}(n)$,
$T_{002}(n) = T_{020}(n) = T_{200}(n)$,
$T_{011}(n) = T_{101}(n) = T_{110}(n)$,
$T_{012}(n) = T_{021}(n) = T_{102}(n)$
$\quad = T_{201}(n) = T_{120}(n)$
$\quad = T_{210}(n)$,
$T_{022}(n) = T_{202}(n) = T_{220}(n)$,
$T_{111}(n)$,
$T_{112}(n) = T_{121}(n) = T_{211}(n)$,
$T_{122}(n) = T_{212}(n) = T_{221}(n)$,
$T_{222}(n)$.

Carlitz [1] provided explicit formulae for $T_{abc}(n)$ by using properties of the cube root of unity.

In this paper, we present a different method to get these explicit formulae. The cube root of unity ω, is defined by

$$\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right).$$

Explicit formulae for $T_{abc}(n)$ are calculated from (2.9) and almost all $T_{abc}(n)$ are also expressed in terms of ω as follows:

$T_{000}(n)$
$$= 3^{n-2} + \frac{(1+(-1)^n)((\omega+2)^n + (\omega-1)^n + (2\omega+1)^n)}{9}$$
$$= \begin{cases} 3^{n-2} + 2(-1)^{\frac{n}{6}}3^{\frac{n}{2}-1} & \text{if } n \equiv 0 \pmod 6, \\ 3^{n-2} & \text{if } n \equiv 3 \pmod 6, \end{cases}$$

$T_{001}(n)$
$$= 3^{n-2} + \frac{(1+(-1)^{n-1})((\omega+2)^{n-1} + (\omega-1)^{n-1} + (2\omega+1)^{n-1})}{9}$$
$$= \begin{cases} 3^{n-2} & \text{if } n \equiv 4 \pmod 6, \\ 3^{n-2} + 2(-1)^{\frac{n-1}{6}}3^{\frac{n-1}{2}-1} & \text{if } n \equiv 1 \pmod 6, \end{cases}$$

$T_{002}(n)$
$$= 3^{n-2} - (1+(-1)^{n-5})((\omega+2)^{n-5} + (\omega-1)^{n-5} + (2\omega+1)^{n-5})$$
$$= \begin{cases} 3^{n-2} & \text{if } n \equiv 2 \pmod 6, \\ 3^{n-2} - 2(-1)^{\frac{n-5}{6}}3^{\frac{n-5}{2}+1} & \text{if } n \equiv 5 \pmod 6, \end{cases}$$

$T_{011}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(1+(\omega+1)(-1)^{n-2}) + \frac{(\omega-1)^{n-3}}{3}(\omega-(-1)^{n-2}) + \frac{(2\omega+1)^{n-3}}{3}(\omega+(\omega+1)(-1)^{n-2})$$
$$= \begin{cases} 3^{n-2} + (-1)^{\frac{n-2}{6}}3^{\frac{n-2}{2}} & \text{if } n \equiv 2 \pmod 6, \\ 3^{n-2} + (-1)^{\frac{n-5}{6}}3^{\frac{n-5}{2}+1} & \text{if } n \equiv 5 \pmod 6, \end{cases}$$

$T_{012}(n) = 3^{n-2}$

$T_{022}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(-1-(-1)^{n-1}\omega) + \frac{(\omega-1)^{n-3}}{3}(\omega+1-(-1)^{n-1}) + \frac{(2\omega+1)^{n-3}}{3}(\omega+1-(-1)^{n-1}\omega)$$
$$= \begin{cases} 3^{n-2} - (-1)^{\frac{n-4}{6}}3^{\frac{n-4}{2}+1} & \text{if } n \equiv 4 \pmod 6, \\ 3^{n-2} - (-1)^{\frac{n-1}{6}}3^{\frac{n-1}{2}-1} & \text{if } n \equiv 1 \pmod 6, \end{cases}$$

$T_{111}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}\big(-\omega+1+(-1)^n(-\omega-2)\big) + \frac{(\omega-1)^{n-3}}{3}\big(-\omega+1+(-1)^n(-\omega-2)\big) + \frac{(2\omega+1)^{n-3}}{3}\big(\omega+2-(-1)^n(-\omega+1)\big)$$
$$= \begin{cases} 3^{n-2} - (-1)^{\frac{n}{6}}3^{\frac{n}{2}-1} & \text{if } n \equiv 0 \pmod 6, \\ 3^{n-2} + (-1)^{\frac{n-3}{6}}3^{\frac{n-3}{2}+1} & \text{if } n \equiv 3 \pmod 6, \end{cases}$$

$T_{112}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(-\omega-(-1)^{n-1}) + \frac{(\omega-1)^{n-3}}{3}\big(-1+(-1)^{n-1}(\omega+1)\big) + \frac{(2\omega+1)^{n-3}}{3}\big(-\omega+(-1)^{n-1}(\omega+1)\big)$$
$$= \begin{cases} 3^{n-2} + (-1)^{\frac{n-4}{6}}3^{\frac{n-4}{2}+1} & \text{if } n \equiv 4 \pmod 6, \\ 3^{n-2} - (-1)^{\frac{n-1}{6}}3^{\frac{n-1}{2}-1} & \text{if } n \equiv 1 \pmod 6, \end{cases}$$

$T_{122}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}(-\omega-1-(-1)^{n-2}) + \frac{(\omega-1)^{n-3}}{3}(1-(-1)^{n-2}\omega) + \frac{(2\omega+1)^{n-3}}{3}(-\omega-1-(-1)^{n-2}\omega)$$
$$= \begin{cases} 3^{n-2} - (-1)^{\frac{n-2}{6}}3^{\frac{n-2}{2}} & \text{if } n \equiv 2 \pmod 6, \\ 3^{n-2} + (-1)^{\frac{n-5}{6}}3^{\frac{n-5}{2}+1} & \text{if } n \equiv 5 \pmod 6, \end{cases}$$

$T_{222}(n)$
$$= 3^{n-2} + \frac{(\omega+2)^{n-3}}{3}\big(-\omega-2+(-1)^n(-\omega+1)\big) + \frac{(\omega-1)^{n-3}}{3}\big(-\omega-2+(-1)^n(-\omega+1)\big) + \frac{(2\omega+1)^{n-3}}{3}\big(\omega-1+(-1)^n(\omega+2)\big)$$
$$= \begin{cases} 3^{n-2} - (-1)^{\frac{n}{6}}3^{\frac{n}{2}-1} & \text{if } n \equiv 0 \pmod 6, \\ 3^{n-2} - (-1)^{\frac{n-3}{6}}3^{\frac{n-3}{2}+1} & \text{if } n \equiv 3 \pmod 6. \end{cases}$$

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5. References

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