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New Fixed Point Theorems for $\theta - \phi$ Suzuki Contraction on Partial Metric Spaces

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Abstract

In this paper, we establish new fixed point theorems for $\theta - \phi$ Suzuki contraction on complete partial metric spaces. The results presented in the paper improve and extend some previous results.

Keywords: Fixed points, Partial metric space, $\theta - \phi$ Suzuki contraction

1. Introduction

Fixed point theorem is considered a very important theory in applied in the branch. Mathematics and other disciplines, especially in the fields of spatial analysis function.

In 1992, S.Banach [11] introduce the notion fixed point theorem for contraction on complete metric space which is the beginning of the study, it is The Banach Contraction Principle following

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Theorem 1.1 [11]. Let (X, d) be a complete metric space and let T be a contraction on X, there exists $r \in [0,1)$ such that

$$d(Tx, Ty) \leq rd(x, y), \text{for all } x, y \in X.$$

Then T has a unique fixed point.

In 2014, Jleli and Samet [3] introduce type contraction that is called $\theta -$ contraction and establish fixed point theorem for $\theta -$ contraction on metric space.

In 2017, D.W. Zheng, Z.Y. Cai and P. Wang [9] introduce the notion of $\theta - \phi$

contraction and $\theta - \phi$ Suzuki contraction and establish new fixed point theorem for $\theta - \phi$ contraction on complete metric space following

Theorem 1.2 [9] Suppose (X, d) is a complete metric space and $T: X \to X$ is a $\theta - \phi$ Suzuki contraction, there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that for any x , $y \in X$, $Tx \neq Ty$. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

From the above results, they obtain the following fixed point theorems for $\theta - \phi$ contraction and $\theta - \phi$ Kannan-type contraction.

In 2018, T. Hu, D.W. Zheng and J. Zhou [10] introduce the notion of $\theta - \phi$ contraction, $\theta - \phi$ Kannan-type contraction and establish new fixed point theorem on complete partial metric space following

Theorem 1.3 [10] Suppose (X, p) is a complete partial metric space and $T: X \to X$ is a $\theta - \phi$ contraction, then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Theorem 1.4 [10] Let (X, p) be a complete partial metric space and suppose $T: X \to X$ is a $\theta - \phi$ Kannan-type contraction. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

In this paper, we establish new fixed point theorems for $\theta - \phi$ Suzuki contraction on complete partial metric spaces. The results presented in the paper improve and extend some previous results.

2. Preliminaries

Definition 2.1 A partial metric on a nonempty set X is a mapping $p : X \times X \to [0, +\infty)$ such that for all

$x, y, z \in X$;

$(P1)$ $x = y \leftrightarrow 0 \leq p(x,x) = p(x,y) = p(y,y)$;

$(P2)$ $p(x,x) \leq p(x,y)$;

$(P3)$ $p(x,y) = p(y,x)$;

$(P4)$ $p_b(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

For a partial metric p on X, the function $d_p: X \times X \to [0, \infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \quad (2.1)$$

is a metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p −balls $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$, where

$B_p(x, \varepsilon) = \{y \in X: p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Lemma 2.2 [3], [4], [5], [8]

(1) A sequence $\{x_n\}$ is Cauchy in a partial metric space (X, p) if and only if $\{x_n\}$ is Cauchy in a metric space (X, d_p);

(2) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete.

Moreover,

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_m). \quad (2.2)$$

Lemma 2.3 [6], [7]. Assume $\{x_n\} \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(x, x) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

Definition 2.4 Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x_n, x)$.

(ii) A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if $\lim_{n,m\to+\infty} p(x_n, x_m)$ exists.

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subset B_p(f(x_0), \varepsilon)$.

Definition 2.5 [2] Let (X, p) be a metric space. A mapping $T : X \to X$ is said to be an $\theta -$ contraction if there exist $\theta \in \Theta$ and $k \in (0,1)$ such that for any $x, y \in X$,

$d(Tx, Ty) \neq 0 \Rightarrow \theta(d(Tx, Ty)) \leq \theta(d(x, y))^k$ (2.3)

where $\theta : (0, \infty) \to (1, \infty)$ satisfies the following conditions:

$(\Theta 1)$ θ is non-decreasing;

$(\Theta 2)$ for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;

$(\Theta 3)$ θ is continuous on $(0, \infty)$

Definition 2.6 [1] Denote by ϕ the set of functions $\phi : [1, \infty) \to [1, \infty)$ satisfying the following conditions:

$(\Phi 1)$ $\phi : [1, \infty) \to [1, \infty)$ is non-decreasing;

$(\Phi 2)$ for each $t > 1$, $\lim_{n\to\infty} \phi^n(t) = 1$;

$(\Phi 3)$ ϕ is continuous on $[1, \infty)$.

Lemma 2.7 [1] If $\phi \in \Phi$ then $\phi(1) = 1$ and $\phi(t) < t$ for each $t > 1$.

Definition 2.8 Let (X, p) be a partial metric space and let $T : X \to X$ be a self-mapping;

(1) T is said to be a $\theta - \phi$ contraction if exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$\theta(p(Tx, Ty)) \leq \phi[\theta(p(x, y))] \qquad (2.4)$$

(2) T is said to be a $\theta - \phi$ Kannan-type contraction if exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X, Tx \neq Ty$,

$$\theta\big(p(Tx, Ty)\big) \leq \phi\left[\theta\left(\frac{p(x, Tx) + p(y, Ty)}{2}\right)\right] \qquad (2.5)$$

(3) T is said to be a $\theta - \phi$ Suzuki contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X, Tx \neq Ty$,

if $\frac{1}{2} p(x, Tx) < p(x, y)$,

then $\theta(p(Tx, Ty)) \leq \phi[\theta(N(x, y))]$ (2.6)

where $N(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty)\}$.

3. Main results

In this section, we obtain new fixed point theorem defined on complete partial metric space.

Theorem 3.1 Suppose (X, p) is a complete partial metric space and $T : X \to X$ is a $\theta - \phi$ Suzuki contraction. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x^* \in X$.

Proof. Fix $x_0 \in X$ and construct the sequence $\{x_n\}$ by $x_{n+1} = Tx_n,\ n = 1,2,3,\ldots$

Case 1. If $x_{n-1} = x_n$ for some $n \in N$, then $x^* = x_n$ is a fixed point for T.

Case 2. If $x_{n-1} \neq x_n$ for each $n \in N$, then $p(x_{n+1}, x_n) > 0$ for all $n \in N$.

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.6). To show that $\frac{1}{2}p(x_{n-1}, x_n) = \frac{1}{2}p(x_{n-1}, Tx_{n-1}) < p(x_{n-1}, x_n)$.

Hence $\theta(p(Tx_{n-1}, Tx_n)) \leq$

$$\phi[\theta(N(x_{n-1}, x_n))] \qquad (3.1)$$

where

$$\begin{aligned} N(x_{n-1}, x_n) &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n)\} \\ &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}. \end{aligned}$$

If $N(x_{n-1}, x_n) = p(x_n, x_{n+1})$ and using (3.1),

then $\theta(p(x_n, x_{n+1})) = \theta(p(Tx_{n-1}, Tx_n)) \leq \phi[\theta(p(x_n, x_{n+1}))]$

By the definition of θ and Lemma 2.7, we have

$$\begin{aligned} \theta(p(x_n, x_{n+1})) &= \theta(p(Tx_{n-1}, Tx_n)) \\ &\leq \phi[\theta(p(x_n, x_{n+1}))] \\ &< \theta(p(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction. Thus

$N(x_{n-1}, x_n) = p(x_{n-1}, x_n)$ that by (3.1),

we have $\theta(p(x_n, x_{n+1})) = \theta(p(Tx_{n-1}, Tx_n)) \leq \phi[\theta(p(x_{n-1}, x_n))]$.

Repeating this step, we conclude that

$$\begin{aligned} \theta(p(x_n, x_{n+1})) &= \theta(p(Tx_{n-1}, Tx_n)) \\ &\leq \phi[\theta(p(x_{n-1}, x_n))] \\ &\leq \phi[\phi[\theta(p(x_{n-2}, x_{n-1}))]] \\ &\leq \phi^2[\theta(p(x_{n-2}, x_{n-1}))] \\ &\leq \phi^3[\theta(p(x_{n-3}, x_{n-2}))] \\ &\vdots \\ &\leq \phi^n[\theta(p(x_0, x_1))]. \end{aligned}$$

By the definition of θ and property $(\Phi 2)$, we have $\lim_{n\to+\infty} \phi^n[p(x_0, x_1)] = 1$.

Letting $n \to \infty$, we obtain

$1 \leq \lim_{n\to+\infty} \theta(p(x_n, x_{n+1})) \leq \lim_{n\to+\infty} \phi^n[p(x_0, x_1)] = 1$.

By Sandwich theorem, $\lim_{n\to+\infty} \theta(p(x_n, x_{n+1})) = 1$.

And by $(\Theta 2)$, we have $\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0$. (3.2)

Similarly, setting $x = x_m$ and $y = x_m$. We obtain that

$$\frac{1}{2}p(x_m, x_{m+1}) = \frac{1}{2}p(x_m, Tx_m) < p(x_m, x_m).$$

Letting $n \to \infty$ and (3.2), we get

$$0 = \lim_{m\to+\infty} \frac{1}{2}p(x_m, x_{m+1}) < \lim_{m\to+\infty} p(x_m, x_m) = p(x_m, x_m).$$

Hence

$$\theta(p(Tx_m, Tx_m)) \leq \phi[\theta(N(x_m, x_m))] \qquad (3.3)$$

where

$$\begin{aligned} N(x_m, x_m) &= \max\{p(x_m, x_m), p(x_m, Tx_m), p(x_m, Tx_m)\} \\ &= \max\{p(x_m, x_m), p(x_m, x_{m+1}), p(x_m, x_{m+1})\} \\ &= \max\{p(x_m, x_m), 0, 0\}\ (as\ n \to \infty). \end{aligned}$$

Thus $N(x_m, x_m) = p(x_m, x_m)$ from (3.3), we have

$$\begin{aligned} \theta(p(x_{m+1}, x_{m+1})) &= \theta(p(Tx_m, Tx_m)) \\ &\leq \phi[\theta(p(x_m, x_m))] \\ &\leq \phi[\phi[\theta(p(x_{m-1}, x_{m-1}))]] \\ &\leq \phi^2[\theta(p(x_{m-1}, x_{m-1}))] \\ &\leq \phi^3[\theta(p(x_{m-2}, x_{m-2}))] \\ &\vdots \\ &\leq \phi^m[\theta(p(x_1, x_1))]. \end{aligned}$$

By the definition of θ and property $(\Phi 2)$, we have $\lim_{m\to+\infty} \phi^m[p(x_1, x_1)] = 1$.

Letting $n \to \infty$, we have

$1 \leq \lim_{m\to+\infty} \theta(p(x_{m+1}, x_{m+1})) \leq \lim_{n\to+\infty} \phi^m[p(x_1, x_1)] = 1$.

By Sandwich theorem, we have

$$\lim_{m\to+\infty} \theta(p(x_{m+1}, x_{m+1})) = 1.$$

And by ($\Theta 2$), we have $\lim_{m\to+\infty} p(x_{m+1}, x_{m+1}) = 0.$ (3.4)

Next, we prove that $\{x_n\}$ is a Cauchy sequence in the metric space (x, d_p). Otherwise, there exists some $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d_p(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \quad (3.5)$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way it is the smallest integer with $n(k) > m(k)$ and satisfying (3.5). Hence,

$$d_p(x_{m(k)}, x_{n(k)-1}) < \varepsilon.$$

Then we have

$$\begin{aligned}\varepsilon &\leq d_p(x_{m(k)}, x_{n(k)})\\ &\leq d_p(x_{m(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)})\\ &< \varepsilon + d_p(x_{n(k)-1}, x_{n(k)}).\end{aligned}$$

Noting that

$$d_p(x_{n(k)-1}, x_{n(k)}) = 2p(x_{n(k)-1}, x_{n(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) - p(x_{n(k)}, x_{n(k)}).$$

Let $k \to \infty$ from (3.2), (3.4) and the above inequality, we can conclude that

$$\begin{aligned}\varepsilon &\leq \lim_{k\to+\infty} d_p(x_{m(k)}, x_{n(k)})\\ &< \epsilon + 2\lim_{k\to+\infty} p(x_{n(k)-1}, x_{n(k)}) - \lim_{k\to+\infty} p(x_{n(k)-1}, x_{n(k)-1}) - \lim_{k\to+\infty} p(x_{n(k)}, x_{n(k)})\\ &= \varepsilon.\end{aligned}$$

By Sandwich theorem, we have

$$\lim_{k\to+\infty} d_p(x_{m(k)}, x_{n(k)}) = \varepsilon.$$

From (2.1), we have

$$\begin{aligned}\lim_{k\to+\infty} d_p(x_{m(k)}, x_{n(k)}) &= 2\lim_{k\to+\infty} p(x_{m(k)}, x_{n(k)}) - \lim_{k\to+\infty} p(x_{m(k)}, x_{m(k)})\\ &\quad - \lim_{k\to+\infty} p(x_{n(k)}, x_{n(k)})\end{aligned}$$

and from (3.4), we have

$$\begin{aligned}\varepsilon &= \lim_{k\to+\infty} d_p(x_{m(k)}, x_{n(k)})\\ &= 2\lim_{k\to+\infty} p(x_{m(k)}, x_{n(k)}). \quad (3.6)\end{aligned}$$

Again

$$\begin{aligned}&d_p(x_{n(k)}, x_{m(k)})\\ &\leq d_p(x_{n(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{m(k)})\\ &\leq d_p(x_{n(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{m(k)-1}) + d_p(x_{m(k)-1}, x_{m(k)})\end{aligned}$$

and

$$\begin{aligned}&d_p(x_{n(k)-1}, x_{m(k)-1})\\ &\leq d_p(x_{n(k)-1}, x_{n(k)}) + d_p(x_{n(k)}, x_{m(k)-1})\\ &\leq d_p(x_{n(k)-1}, x_{n(k)}) + d_p(x_{n(k)}, x_{m(k)}) + d_p(x_{m(k)}, x_{m(k)-1}).\end{aligned}$$

Consider

$$\begin{aligned}&d_p(x_{n(k)}, x_{m(k)})\\ &\leq d_p(x_{n(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{m(k)-1}) + d_p(x_{m(k)-1}, x_{m(k)})\\ &= [2p(x_{n(k)}, x_{n(k)-1}) - p(x_{n(k)}, x_{n(k)}) - p(x_{n(k)-1}, x_{n(k)-1})] + d_p(x_{n(k)-1}, x_{m(k)-1})\\ &\quad + [2p(x_{m(k)-1}, x_{m(k)}) - p(x_{m(k)-1}, x_{m(k)-1}) - p(x_{m(k)}, x_{m(k)})].\end{aligned}$$

Letting $k \to \infty$ and follow from (3.2) and (3.4) and the above in equations, we obtain

$$\lim_{k\to+\infty} d_p(x_{n(k)}, x_{m(k)}) \leq \lim_{k\to+\infty} d_p(x_{n(k)-1}, x_{m(k)-1}).$$

Consider

$$\begin{aligned}&d_p(x_{n(k)-1}, x_{m(k)-1})\\ &\leq d_p(x_{n(k)-1}, x_{n(k)}) + d_p(x_{n(k)}, x_{m(k)}) + d_p(x_{m(k)}, x_{m(k)-1})\\ &= [2p(x_{n(k)-1}, x_{n(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) - p(x_{n(k)}, x_{n(k)})] + d_p(x_{n(k)}, x_{m(k)})\\ &\quad + [2p(x_{m(k)}, x_{m(k)-1}) - p(x_{m(k)}, x_{m(k)}) - p(x_{m(k)-1}, x_{m(k)-1})].\end{aligned}$$

Letting $k \to \infty$ and follow from (3.2) and (3.4) and the above in equations, we get

$$\lim_{k\to+\infty} d_p(x_{n(k)-1}, x_{m(k)-1}) \leq \lim_{k\to+\infty} d_p(x_{n(k)}, x_{m(k)}).$$

Hence

$$\lim_{k\to+\infty} d_p(x_{n(k)-1}, x_{m(k)-1}) \leq \lim_{k\to+\infty} d_p(x_{n(k)}, x_{m(k)}) \leq \lim_{k\to+\infty} d_p(x_{n(k)-1}, x_{m(k)-1}).$$

By Sandwich theorem, we have

$$\lim_{k\to\infty} d_p\,(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d_p\,(x_{n(k)-1}, x_{m(k)-1}) = 2\lim_{k\to\infty} p\,(x_{n(k)-1}, x_{m(k)-1}) - \lim_{k\to\infty} p\,(x_{n(k)-1}, x_{n(k)-1}) - \lim_{k\to\infty} p\,(x_{m(k)-1}, x_{m(k)-1}).$$

From (3.4) and (3.6), we obtain

$$\varepsilon = \lim_{k\to\infty} d_p(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d_p(x_{n(k)-1}, x_{m(k)-1}) = 2\lim_{k\to\infty} p(x_{n(k)-1}, x_{m(k)-1}). \quad (3.7)$$

Let $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (2.6).

To show that

$$\frac{1}{2}p(x_{m(k)-1}, x_{m(k)}) = \frac{1}{2}p(x_{m(k)-1}, Tx_{m(k)-1}) < p(x_{m(k)-1}, x_{n(k)-1}).$$

Let $k \to \infty$ and from (3.2) and (3.7), we have

$$0 = \lim_{k\to\infty} \frac{1}{2}p(x_{m(k)-1}, x_{m(k)}) < \lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \frac{\varepsilon}{2}.$$

Hence

$$\theta[p(Tx_{m(k)-1}, Tx_{n(k)-1}) \leq \phi[\theta(N(x_{m(k)-1}, x_{n(k)-1}))] \quad (3.8)$$

where

$$N(x_{m(k)-1}, x_{n(k)-1}) = \max\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, Tx_{m(k)-1}), p(x_{n(k)-1}, Tx_{n(k)-1})\} = \max\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), p(x_{n(k)-1}, x_{n(k)})\} = \max\{p(x_{m(k)-1}, x_{n(k)-1}), 0, 0\}\ (as\ k \to \infty).$$

Since $N(x_{m(k)-1}, x_{n(k)-1}) = p(x_{m(k)-1}, x_{n(k)-1})$ from (3.8), we have

$$\theta[p(Tx_{m(k)-1}, Tx_{n(k)-1})] \leq \phi[\theta(p(x_{m(k)-1}, x_{n(k)-1}))].$$

Let $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \theta[p(x_{m(k)}, x_{n(k)})] = \lim_{k\to\infty} \theta[p(Tx_{m(k)-1}, Tx_{n(k)-1})] \leq \lim_{k\to\infty} \phi[\theta(p(x_{m(k)-1}, x_{n(k)-1}))].$$

From (3.6), we have

$$\frac{\lim_{k\to\infty} d_p(x_{m(k)}, x_{n(k)})}{2} = \lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}).$$

And from (3.5),

$$\frac{\varepsilon}{2} \leq \frac{\lim_{k\to\infty} d_p(x_{m(k)}, x_{n(k)})}{2} = \lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}).$$

Thus

$$\theta\left(\frac{\varepsilon}{2}\right) \leq \lim_{k\to\infty} \theta[p(x_{m(k)}, x_{n(k)})] \leq \lim_{k\to\infty} \phi[\theta(p(x_{m(k)-1}, x_{n(k)-1}))].$$

From (3.7),

$$\frac{\varepsilon}{2} = \frac{\lim_{k\to\infty} d_p(x_{m(k)-1}, x_{n(k)-1})}{2} = \lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}).$$

Therefore

$$\theta\left(\frac{\varepsilon}{2}\right) \leq \lim_{k\to\infty} \theta\left[p\left(x_{m(k)}, x_{n(k)}\right)\right] \leq \phi\left[\theta\left(\frac{\varepsilon}{2}\right)\right].$$

By Lemma 2.7, we have

$$\theta\left(\frac{\varepsilon}{2}\right) \leq \phi\left[\theta\left(\frac{\varepsilon}{2}\right)\right] < \theta\left(\frac{\varepsilon}{2}\right),$$

which it is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in (X, d_p).

The above show that $\{x_n\}$ must be a Cauchy sequence in the complete metric space (X, d_p). Thus, there exists some x^* in X that by (2.2) and (3.4), we have

$$\lim_{n\to\infty} d_p(x_n, x^*) = 0.$$

Hence

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n,m\to\infty} p(x_n, x_m) = \frac{1}{2}\lim_{n,m\to\infty} d_p(x_n, x_m) = 0.$$

To show that this x^* is a fixed point. By means of $(P2)$, to prove that

$$p(Tx^*,x^*) = p(x^*,x^*) = p(Tx^*,Tx^*) = 0.$$

From above $p(x^*,x^*) = 0$,

let $x = x^*$ and $y = x^*$ in (2.6), we obtain that

$$\frac{1}{2}\,p(x^*,Tx^*) < p(x^*,x^*).$$

Since $p(x^*,Tx^*) = p(x^*,x^*)$, we have

$$\frac{1}{2}\,p(x^*,x^*) < p(x^*,x^*).$$

Hence $\theta[p(Tx^*,Tx^*)] \leq \phi\big[\theta\big(N(x^*,x^*)\big)\big]$, (3.9)

where

$$\begin{aligned} &N(x^*,x^*)\\ &= \max\{p(x^*,x^*),p(x^*,Tx^*),p(x^*,Tx^*)\}\\ &= \max\{p(x^*,x^*),p(x^*,x^*),p(x^*,x^*)\}\\ &= \max\{p(x^*,x^*)\}. \end{aligned}$$

Thus $N(x^*,x^*) = p(x^*,x^*)$ from (3.9), we get

$$\theta[p(Tx^*,Tx^*)] \leq \phi\big[\theta\big(p(x^*,x^*)\big)\big].$$

Let $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \theta[p(Tx^*,Tx^*)] \leq \lim_{n\to\infty} \phi\big[\theta\big(p(x^*,x^*)\big)\big].$$

And by $(\Theta2)$ and Lemma 2.7, we have

$$\begin{aligned} 1 \leq \lim_{n\to\infty} &\theta[p(Tx^*,Tx^*)]\\ &\leq \lim_{n\to\infty} \phi\big[\theta\big(p(x^*,x^*)\big)\big] = 1. \end{aligned}$$

By Sandwich theorem, we obtain

$$\lim_{n\to\infty} \theta[p(Tx^*,Tx^*)] = 1.$$

And from $(\Theta2)$, we have

$$p(Tx^*,Tx^*) = \lim_{n\to\infty} p(Tx^*,Tx^*) = 0.$$

Therefore

$$p(Tx^*,Tx^*) = 0.$$

Let $x = x_{n-1}$ and $y = x^*$ in (2.6). To show that

$$\frac{1}{2}p(x_{n-1},Tx_{n-1}) < p(x_{n-1},x^*).$$

Let $n \to \infty$ and from $\{x_n\}$ is a Cauchy sequence, we get $x_n \to x^*$

$$\lim_{n\to\infty}\frac{1}{2}p(x_{n-1},Tx_{n-1}) < \lim_{n\to\infty} p(x_{n-1},x^*),$$

$$\tfrac{1}{2}p(x^*,x^*) < p(x^*,x^*).$$

That is,

$$\theta[p(Tx_{n-1},Tx^*)] \leq \phi\big[\theta\big(N(x_{n-1},x^*)\big)\big]. \quad (3.10)$$

where

$$\begin{aligned} &N(x_{n-1},x^*)\\ &= \max\{p(x_{n-1},x^*),p(x_{n-1},Tx_{n-1}),p(x^*,Tx^*)\}\\ &= \max\{p(x_{n-1},x^*),p(x_{n-1},x_n),p(x^*,x^*)\}\\ &= \max\{p(x^*,x^*),0,0\}(as\ n \to \infty). \end{aligned}$$

Thus $N(x_{n-1},x^*) = p(x^*,x^*)$ from (3.10), we have

$$\begin{aligned} \theta[p(x_n,Tx^*)] &= \theta[p(Tx_{n-1},Tx^*)]\\ &\leq \phi\big[\theta\big(p(x^*,x^*)\big)\big]. \end{aligned}$$

Let $n \to \infty$ and by Lemma 2.3, we obtain

$$\lim_{n\to\infty} \theta[p(x_n,Tx^*)] \leq \lim_{n\to\infty} \phi\big[\theta\big(p(x^*,x^*)\big)\big],$$

$$\theta[p(x^*,Tx^*)] \leq \phi\big[\theta\big(p(x^*,x^*)\big)\big].$$

By the definition of θ and Lemma 2.7, we have

$$1 \leq \theta[p(x^*,Tx^*)] \leq 1.$$

Let $n \to \infty$ by Sandwich theorem, we get

$$\lim_{n\to\infty} \theta[p(x^*,Tx^*)] = 1.$$

And from $(\Theta2)$, we get

$$p(x^*,Tx^*) = \lim_{n\to\infty} p(x^*,Tx^*) = 0.$$

Therefore

$$p(x^*,Tx^*) = 0.$$

Now, we shall show that T has a unique fixed point. Suppose there exists another fixed point y^* of T such that $Tx^* = x^* \neq Ty^* = y^*$.

Let $x = x^*$ and $y = y^*$ in (2.6). To show that

$$0 = \tfrac{1}{2}p(x^*,x^*) = \tfrac{1}{2}p(x^*,Tx^*) < p(x^*,y^*).$$

Hence

$$\theta[p(Tx^*,Ty^*)] \leq \phi\big[\theta\big(N(x^*,y^*)\big)\big], \quad (3.11)$$

where

$$\begin{aligned} &N(x^*,y^*)\\ &= \max\{p(x^*,y^*),p(x^*,Tx^*),p(y^*,Ty^*)\}\\ &= \max\{p(x^*,y^*),p(x^*,x^*),p(y^*,y^*)\}\\ &= \max\{p(x^*,y^*),0,0\}(as\ n \to \infty). \end{aligned}$$

Thus $N(x^*,y^*) = p(x^*,y^*)$ and from (3.11), we conclude that

$$\begin{aligned} \theta[p(x^*,y^*)] &= \theta[p(Tx^*,Ty^*)]\\ &\leq \phi\big[\theta\big(p(x^*,y^*)\big)\big]\\ &< \theta\big(p(x^*,y^*)\big), \end{aligned}$$

which is a contradiction. Therefore T has a unique fixed point.

Remark 3.2 Theorem 3.1 improves the main results [9] and [10].

It follows from Theorem 3.1 and [9], we obtain the following fixed point results for $\theta - \phi$ contraction and $\theta - \phi$ Kannan-type contraction.

Corollary 3.3 Suppose (X, p) is a complete partial metric space and $T: X \to X$ is a $\theta - \phi$ contraction. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Corollary 3.4 Let (X, p) be a complete partial metric space and suppose $T: X \to X$ is a $\theta - \phi$ Kannan-type contraction. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Remark 3.5 Corollary 3.3 and 3.4 improves the some main results in [10].

It follows from Theorem 3.1, we obtain the following fixed point results for $\theta - \phi$ Suzuki contraction.

Corollary 3.6 Suppose (X, d) is a complete metric space and suppose $T: X \to X$ is a $\theta - \phi$ Suzuki contraction, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$Tx \neq Ty,$$
$$\frac{1}{2} d(x, Tx) < d(x, y) \to \theta\big(d(Tx, Ty)\big) \leq \phi\big[\theta\big(N(x, y)\big)\big]$$

where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Remark 3.7 Corollary 3.6 improves the some main results in [9].

4. Conclusions

We prove a new fixed point theorems for $\theta - \phi$ Suzuki contraction on complete partial metric spaces. The results presented in the paper improve and extend some previous results.

5. References

[1] M.P. Schellekens, A characterization of partial metrizability: domains are quantifiaple, Theor. Computer Science, 305 (2003), 409-432. https://doi.org/10.1016/s0304-3975(02)00705-3

[2] S. Romaquera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl., 2010 (2010) Article ID 493298, 1-7. https://doi.org/10.1155/2010/493298

[3] M. Jleli, B. Samet, A new generalization of the Banach contraction principle, J. Inequal. Appl., 2014 (2014), 38. https://doi.org/10.1186/1029-242x-2014-38

[4] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal., 71 (2009), 5313-5317. 1,2.4

[5] R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Struc-tures, 7(1999), 71-83.

[6] S. Oltra, O. Valero, Banach's fixed point theorem for partial metric spaces, Rend. Istit. Mat. Univ.Trieste, Spanish Ministry of Science and Technology, (2004), 17-26.

[7] T. Abdeljawad, Fixed points for generalized weakly contractive mappings in partial metric spaces,Mathematical and computer Modelling, 54 (2011) no. 11-12, 2923-2927. http://doi.org/10.1016/j.mcm.2011.07.013

[8] I. Altun, A. Erduran, Fixed point theorems for monotone mappings on partial metric spaces, Fixed Point Theory and Applications, 2011 (2011), 1155-1165. http://doi.org/10.1155/2011/508730

[9] Dingwei Zheng, Zhangyong Cai, Pei Qang, New fixed point theorems for $\theta-\phi$ contraction in complete metric spaces. Journal of Nonlinear Sciences and Applications, 10 (2017), 2662-2670. http://doi.org/10.22436/jnsa.010.05.32

[10] T.Tao Hu, Dingwei Zheng and Jingren Zhou[1] Some New Fixed Point Theorems on Partial Metric Spaces International Journal of Mathematical Analysis Vol. 12, 2018, no. 7, 343-352 HIKARI Ltd,www.m-hikari.com http://doi.org/10.12988/ijma.2018.8538

[11] S.Banach, Sur les opérations dans les ensembles abseraits et leur application aux équations intégrales, Fund. Math., 3 (1922), 133-181.