



Generalized Identities Related for The k-Fibonacci Number, The k-Lucas Number and k-Fibonacci-Like Number

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Abstract

In this paper, we present generalized identities for k-Fibonacci, k-Lucas and k-Fibonacci-Like sequence. We obtain the Binet's formula for related some identities.

Keywords: k-Fibonacci number, k-Lucas number, k-Fibonacci-Like number, Binet's formula

1. Introduction

The Fibonacci sequence is well-known and popular in the study and research. Because the sequence is a recurrence sequence of positive integers that apply to various aspects, such as Science, Engineering and Business.

The article discusses the Fibonacci sequence of positive integers and has studied some of the properties. The changed the first two conditions of the sequence, resulting in a recurring

relationship with the change slightly (see [3], [6], [10], [19]-[21]).

Previously, the article studied about generalized of a large number of Fibonacci, Lucas, and Fibonacci-Like numbers and its properties (see [4], [7], [12]-[18]).

Next, the studies Fibonacci, Lucas, and Fibonacci-Like sequences have been generalized for positive real number k , resulting in k-Fibonacci, k-Lucas, and k-Fibonacci-Like sequence respectively.

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Some properties were later studied and more interesting was received (see [8], [11]).

The inspiration for doing this research, since to the direction of this research and development. Researcher present generalized identities involving factors of k-Fibonacci, k-Lucas, and k-Fibonacci-Like sequences and Binet's formula used to find some related identities.

2. Preliminaries

In this section, we will introduce the previous article, which is well-known for use in our research.

Throughout this paper, let k be any positive real number.

The k-Fibonacci sequence [2], say $\{F_{n,k}\}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1, \quad (2.1)$$

with $F_{k,0} = 0, F_{k,1} = 1$.

The first few the k- Fibonacci numbers are

$$\begin{aligned} F_{k,2} &= k \\ F_{k,3} &= k^2 + 1 \\ F_{k,4} &= k^3 + 2k \\ F_{k,5} &= k^4 + 3k^2 + 1 \\ F_{k,6} &= k^5 + 4k^3 + 3k \\ F_{k,7} &= k^6 + 5k^4 + 6k^2 + 1 \\ F_{k,8} &= k^7 + 6k^5 + 10k^3 + 4k. \end{aligned}$$

It $k = 1$, k- Fibonacci sequence is obtained

$$F_0 = 0, F_1 = 1.$$

And

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1$$

$$\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \dots\}.$$

We will find the Binet's formula allows us to express the k-Fibonacci numbers in function of the roots R_1 & R_2 of the following characteristic equation, associated to the recurrence relation (2.1).

$$\begin{aligned} x^2 - kx - 1 &= 0 \text{ and } R_1 = \frac{k + \sqrt{k^2 + 4}}{2}, \\ R_2 &= \frac{k - \sqrt{k^2 + 4}}{2} \text{ will get} \\ R_1 + R_2 &= k, R_1 R_2 = -1 \\ , R_1 - R_2 &= \sqrt{k^2 + 4}, R_1^2 - 1 = kR_1, R_2^2 - 1 = kR_2. \end{aligned} \quad (2.2)$$

Thus the Binet's formula of k-Fibonacci numbers is given by

$$F_{k,n} = \frac{R_1^n - R_2^n}{R_1 - R_2}, \quad (2.3)$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

The k-Lucas sequence [1], say $\{L_{k,n}\}$ is defined recurrently by

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \text{ for } n \geq 1, \quad (2.4)$$

with initial conditions $L_{k,0} = 2, L_{k,1} = k$.

Expressions of first k-Lucas numbers are presented and from these expressions.

$$\begin{aligned} L_{k,2} &= k^2 + 2 \\ L_{k,3} &= k^3 + 3k \\ L_{k,4} &= k^4 + 4k^2 + 2 \\ L_{k,5} &= k^5 + 5k^3 + 5k \\ L_{k,6} &= k^6 + 6k^4 + 9k^2 + 2 \\ L_{k,7} &= k^7 + 7k^5 + 14k^3 + 7k \\ L_{k,8} &= k^8 + 8k^6 + 20k^4 + 16k^2 + 2. \end{aligned}$$

It $k = 1$, k-Lucas sequence is obtained

$$L_0 = 2, L_1 = 1.$$

And

$$L_{n+1} = L_n + L_{n-1} \text{ for } n \geq 1, \\ \{L_n\} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, \dots\}.$$

The Binet's formula allows us to express the k-Lucas numbers in function of the roots R_1 & R_2 of the characteristic equation as in (2.2).

Thus the Binet's formula of k-Lucas numbers is given by

$$L_{k,n} = R_1^n + R_2^n, \quad (2.5)$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

The k-Fibonacci-Like sequence [9], say $\{T_{k,n}\}$ is defined by

$$T_{k,n} = kT_{k,n-1} + T_{k,n-2} \text{ for } n \geq 2, \quad (2.6)$$

with $T_{k,0} = m, T_{k,1} = mk$, where m is positive integer.

The first few k-Fibonacci-Like numbers are

$$\begin{aligned} T_{k,2} &= mk^2 + m \\ T_{k,3} &= mk^3 + 2mk \\ T_{k,4} &= mk^4 + 3mk^2 + m \\ T_{k,5} &= mk^5 + 4mk^3 + 3mk \\ T_{k,6} &= mk^6 + 5mk^4 + 6mk^2 + m \\ T_{k,7} &= mk^7 + 6mk^5 + 10mk^3 + 4mk \\ T_{k,8} &= mk^8 + 7mk^6 + 15mk^4 + 10mk^2 + m. \end{aligned}$$

It $k = 1$, k-Fibonacci-Like sequence is obtained

$$T_0 = m, T_1 = m.$$

And

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} \text{ for } n \geq 2 \\ \{T_n\} &= \{m, m, 2m, 3m, 5m, 8m, 13m, 21m, \dots\}. \end{aligned}$$

The Binet's formula allows us to express the k-Fibonacci-Like sequence in function of the

roots R_1 & R_2 of the following characteristic equation as in (2.2).

Thus the Binet's formula of k-Fibonacci-Like numbers is given by

$$T_{k,n} = m \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \quad (2.7)$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

From equation (2.6), if $m = 2$ then we get the Fibonacci-Like sequence [5] as

$$S_{k,n} = kS_{k,n-1} + S_{k,n-2} \text{ for } n \geq 2, \quad (2.8)$$

with $S_{k,0} = 2, S_{k,1} = 2k$.

The first few k-Fibonacci-Like numbers are

$$\begin{aligned} S_{k,2} &= 2k^2 + 2 \\ S_{k,3} &= 2k^3 + 4k \\ S_{k,4} &= 2k^4 + 6k^2 + 2 \\ S_{k,5} &= 2k^5 + 8k^3 + 6k \\ S_{k,6} &= 2k^6 + 10k^4 + 12k^2 + 2 \\ S_{k,7} &= 2k^7 + 12k^5 + 20k^3 + 8k \\ S_{k,8} &= 2k^8 + 14k^6 + 30k^4 + 20k^2 + 2. \end{aligned}$$

It $k = 1$, k-Fibonacci-Like sequence is obtained

$$S_0 = 2, S_1 = 2.$$

And

$$\begin{aligned} S_n &= S_{n-1} + S_{n-2} \text{ for } n \geq 2 \\ \{S_n\} &= \{2, 2, 4, 6, 10, 16, 26, \dots\}. \end{aligned}$$

The Binet's formula allows us to express the k-Fibonacci-Like numbers in function of the roots R_1 & R_2 of the following characteristic equation as in (2.2).

Thus the Binet's formula of k-Fibonacci-Like sequence is given by

$$S_{k,n} = 2 \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \quad (2.9)$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

3. Main Results

In this section, we present generalized identities of the k-Fibonacci sequence $\{F_{k,n}\}$, the k-Lucas sequence $\{L_{k,n}\}$, the k-Fibonacci-Like sequence $\{S_{k,n}\}$, the k-Fibonacci-Like sequence $\{T_{k,n}\}$,

Lemma 3.1: Let $n \geq 0, r \geq 0$ and k is positive real number Then the following equalities hold:

$$\begin{aligned} \text{i) } T_{k,n} &= \frac{m}{2} S_{k,n} \\ \text{ii) } T_{k,n} &= m F_{k,n+1} \\ \text{iii) } T_{k,n} &= \frac{m}{\sqrt{k^2 + 4}} (L_{k,n+1} - 2R_2^{n+1}) \end{aligned} \quad (3.1)$$

Proof. By Binet's formula (2.3), (2.5), (2.7) and (2.9), it is easy to prove. \square

Theorem 3.2:

$$T_{k,4n+r+1} + T_{k,r+1} = T_{k,2n+r+1} L_{k,2n}, \quad (3.2)$$

where $n \geq 0, r \geq 0$ and k is positive real number.

Proof. By Binet's formula (2.5) and (2.7), we have

$$\begin{aligned} & T_{k,4n+r+1} + T_{k,r+1} \\ &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} + m \frac{R_1^{r+2} - R_2^{r+2}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} + (-1)^{2n} m \frac{R_1^{r+2} - R_2^{r+2}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} + (R_1 R_2)^{2n} m \frac{R_1^{r+2} - R_2^{r+2}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} + R_1^{2n} R_2^{2n} m \frac{R_1^{r+2} - R_2^{r+2}}{R_1 - R_2} \end{aligned}$$

$$\begin{aligned} &= \frac{m R_1^{4n+r+2} - m R_2^{4n+r+2} + m R_1^{2n+r+2} R_2^{2n} - m R_1^{2n} R_2^{2n+r+2}}{R_1 - R_2} \\ &= \frac{m R_1^{4n+r+2} + m R_1^{2n+r+2} R_2^{2n} - m R_1^{2n} R_2^{2n+r+2} - m R_2^{4n+r+2}}{R_1 - R_2} \\ &= \left(m \frac{R_1^{2n+r+2} - R_2^{2n+r+2}}{R_1 - R_2} \right) (R_1^{2n} + R_2^{2n}) \\ &= T_{k,2n+r+1} L_{k,2n}. \end{aligned}$$

Thus, this completes the Proof. \square

Corollary 3.3:

$$S_{k,4n+r+1} + S_{k,r+1} = \frac{2}{m} T_{k,2n+r+1} L_{k,2n}, \quad (3.3)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.4:

$$L_{k,4n+r+2} + L_{k,r+2} - 2R_2^{4n+r+2} - 2R_2^{r+2} = \frac{\sqrt{k^2 + 4}}{m} T_{k,2n+r+1} L_{k,2n}, \quad (3.4)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.5:

$$F_{k,4n+r+2} + F_{k,r+2} = \frac{1}{m} T_{k,2n+r+1} L_{k,2n}, \quad (3.5)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Theorem 3.6:

$$T_{k,4n+r+2} - T_{k,r} = T_{k,2n+r+1} L_{k,2n+1}, \quad (3.6)$$

where $n \geq 0, r \geq 0$ and k is positive real number.

Proof. By Binet's formula (2.5) and (2.7), we have

$$\begin{aligned} & T_{k,4n+r+2} - T_{k,r} \\ &= m \frac{R_1^{4n+r+3} - R_2^{4n+r+3}}{R_1 - R_2} - m \frac{R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+3} - R_2^{4n+r+3}}{R_1 - R_2} + (-1)^{2n+1} m \frac{m R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+3} - R_2^{4n+r+3}}{R_1 - R_2} + (R_1 R_2)^{2n+1} m \frac{m R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \end{aligned}$$

$$\begin{aligned}
 &= m \frac{R_1^{4n+r+3} - R_2^{4n+r+3}}{R_1 - R_2} + R_1^{2n+1} R_2^{2n+1} m \frac{m R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\
 &= \frac{m R_1^{4n+r+3} - m R_2^{4n+r+3} + m R_1^{2n+r+2} R_2^{2n+1} - m R_1^{2n+1} R_2^{2n+r+2}}{R_1 - R_2} \\
 &= \frac{m R_1^{4n+r+3} + m R_1^{2n+r+2} R_2^{2n+1} - m R_1^{2n+1} R_2^{2n+r+2} - m R_2^{4n+r+3}}{R_1 - R_2} \\
 &= \left(m \frac{R_1^{2n+r+2} - R_2^{2n+r+2}}{R_1 - R_2} \right) (R_1^{2n+1} + R_2^{2n+1}) \\
 &= T_{k,2n+r+1} L_{k,2n+1} .
 \end{aligned}$$

Thus, this completes the Proof. \square

Corollary 3.7:

$$S_{k,4n+r+2} - S_{k,r} = \frac{2}{m} T_{k,2n+r+1} L_{k,2n+1} , \quad (3.7)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number

Corollary 3.8:

$$L_{k,4n+r+3} + L_{k,r+1} - 2R_2^{4n+r+3} - 2R_1^{r+1} = \frac{\sqrt{k^2+4}}{m} T_{k,2n+r+1} L_{k,2n+1} , \quad (3.8)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.9:

$$F_{k,4n+r+3} - F_{k,r+1} = \frac{1}{m} T_{k,2n+r+1} L_{k,2n+1} , \quad (3.9)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Theorem 3.10:

$$T_{k,4n+r+1} - m F_{k,r} = T_{k,2n+r} L_{k,2n+1} , \quad (3.10)$$

where $n \geq 0, r \geq 0$ and k is positive real number .

Proof. By Binet's formula (2.3), (2.5) and (2.7), we have

$$\begin{aligned}
 &T_{k,4n+r+1} - m F_{k,r} \\
 &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} - m \frac{R_1^r - R_2^r}{R_1 - R_2} \\
 &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} + (-1)^{2n+1} m \frac{R_1^r - R_2^r}{R_1 - R_2} \\
 &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} + (R_1 R_2)^{2n+1} m \frac{R_1^r - R_2^r}{R_1 - R_2} \\
 &= m \frac{R_1^{4n+r+2} - R_2^{4n+r+2}}{R_1 - R_2} + R_1^{2n+1} R_2^{2n+1} m \frac{R_1^r - R_2^r}{R_1 - R_2} \\
 &= \frac{m R_1^{4n+r+2} - m R_2^{4n+r+2} + m R_1^{2n+r+1} R_2^{2n+1} - m R_1^{2n+1} R_2^{2n+r+1}}{R_1 - R_2} \\
 &= \frac{m R_1^{4n+r+2} + m R_1^{2n+r+1} R_2^{2n+1} - m R_1^{2n+1} R_2^{2n+r+1} - m R_2^{4n+r+2}}{R_1 - R_2} \\
 &= \left(m \frac{R_1^{2n+r+1} - R_2^{2n+r+1}}{R_1 - R_2} \right) (R_1^{2n+1} + R_2^{2n+1}) \\
 &= T_{k,2n+r} L_{k,2n+1} .
 \end{aligned}$$

Thus, this completes the Proof. \square

Corollary 3.11:

$$S_{k,4n+r+1} - 2F_{k,r} = \frac{2}{m} T_{k,2n+r} L_{k,2n+1} , \quad (3.11)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.12:

$$L_{k,4n+r+2} + L_{k,r} - 2R_2^{4n+r+2} - 2R_1^r = \frac{\sqrt{k^2+4}}{m} T_{k,2n+r} L_{k,2n+1} , \quad (3.12)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.13:

$$F_{k,4n+r+2} - F_{k,r} = \frac{1}{m} T_{k,2n+r} L_{k,2n+1} , \quad (3.13)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Theorem 3.14:

$$T_{k,4n+r} + T_{k,r} = T_{k,2n+r} L_{k,2n}, \quad (3.14)$$

where $n \geq 0, r \geq 0$ and k is positive real number.

Proof. By Binet's formula (2.5) and (2.7), we have

$$\begin{aligned} T_{k,4n+r} + T_{k,r} &= m \frac{R_1^{4n+r+1} - R_2^{4n+r+1}}{R_1 - R_2} + m \frac{R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+1} - R_2^{4n+r+1}}{R_1 - R_2} + (-1)^{2n} m \frac{R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+1} - R_2^{4n+r+1}}{R_1 - R_2} + (R_1 R_2)^{2n} m \frac{R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\ &= m \frac{R_1^{4n+r+1} - R_2^{4n+r+1}}{R_1 - R_2} + R_1^{2n} R_2^{2n} m \frac{R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\ &= \frac{m R_1^{4n+r+1} - m R_2^{4n+r+1} + m R_1^{2n+r+1} R_2^{2n} - m R_1^{2n} R_2^{2n+r+1}}{R_1 - R_2} \\ &= \frac{m R_1^{4n+r+1} + m R_1^{2n+r+1} R_2^{2n} - m R_1^{2n} R_2^{2n+r+1} - m R_2^{4n+r+1}}{R_1 - R_2} \\ &= \left(m \frac{R_1^{2n+r+1} - R_2^{2n+r+1}}{R_1 - R_2} \right) (R_1^{2n} + R_2^{2n}) \\ &= T_{k,2n+r} L_{k,2n}. \end{aligned}$$

Thus, this completes the Proof. \square

Corollary 3.15:

$$S_{k,4n+r} + S_{k,r} = \frac{2}{m} T_{k,2n+r} L_{k,2n}, \quad (3.15)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.16:

$$L_{k,4n+r+1} + L_{k,r+1} - 2R_2^{4n+r+1} - 2R_2^{r+1} = \frac{\sqrt{k^2+4}}{m} T_{k,2n+r} L_{k,2n}, \quad (3.16)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.17:

$$F_{k,4n+r+1} + F_{k,r+1} = \frac{1}{m} T_{k,2n+r} L_{k,2n}, \quad (3.17)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Theorem 3.18:

$$\frac{(k^2+4)}{2m} T_{k,2n+r} S_{k,2n+r} L_{k,2n} = L_{k,6n+2r+2} + L_{k,2n+2r+2} + 2(-1)^r L_{k,2n}, \quad (3.18)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Proof. By Binet's formula (2.5), (2.7), and (2.9), we have

$$\begin{aligned} &\frac{(k^2+4)}{2m} T_{k,2n+r} S_{k,2n+r} L_{k,2n} \\ &= \frac{(k^2+4)}{2m} \left(m \frac{R_1^{2n+r+1} - R_2^{2n+r+1}}{R_1 - R_2} \right) \left(2 \frac{R_1^{2n+r+1} - R_2^{2n+r+1}}{R_1 - R_2} \right) \\ &\quad (R_1^{2n} + R_2^{2n}) \\ &= (R_1^{2n+r+1} - R_2^{2n+r+1})^2 (R_1^{2n} + R_2^{2n}) \\ &= (R_1^{4n+2r+2} - 2R_1^{2n+r+1} R_2^{2n+r+1} + R_2^{4n+2r+2}) (R_1^{2n} + R_2^{2n}) \\ &= R_1^{6n+2r+2} - 2R_1^{4n+r+1} R_2^{2n+r+1} + R_1^{2n} R_2^{4n+2r+2} + R_1^{4n+2r+2} R_2^{2n} \\ &\quad - 2R_1^{2n+r+1} R_2^{4n+r+1} + R_2^{6n+2r+2} \\ &= R_1^{6n+2r+2} + R_2^{6n+2r+2} + R_1^{2n} R_2^{4n+2r+2} + R_1^{4n+2r+2} R_2^{2n} \\ &\quad - 2R_1^{4n+r+1} R_2^{2n+r+1} - 2R_1^{2n+r+1} R_2^{4n+r+1} \\ &= R_1^{6n+2r+2} + R_2^{6n+2r+2} + R_1^{2n} R_2^{2n} (R_1^{2n+2r+2} + R_2^{2n+2r+2}) \\ &\quad - 2R_1^{2n+r+1} R_2^{2n+r+1} (R_1^{2n} + R_2^{2n}) \\ &= R_1^{6n+2r+2} + R_2^{6n+2r+2} + (R_1 R_2)^{2n} (R_1^{2n+2r+2} + R_2^{2n+2r+2}) \\ &\quad - 2(R_1 R_2)^{2n+r+1} (R_1^{2n} + R_2^{2n}) \\ &= R_1^{6n+2r+2} + R_2^{6n+2r+2} + (-1)^{2n} (R_1^{2n+2r+2} + R_2^{2n+2r+2}) \\ &\quad - 2(-1)^{2n+r+1} (R_1^{2n} + R_2^{2n}) \\ &= R_1^{6n+2r+2} + R_2^{6n+2r+2} + R_1^{2n+2r+2} + R_2^{2n+2r+2} \\ &\quad - 2(-1)^{2n+1} (-1)^r (R_1^{2n} + R_2^{2n}) \end{aligned}$$

$$\begin{aligned}
&= R_1^{6n+2r+2} + R_2^{6n+2r+2} + R_1^{2n+2r+2} + R_2^{2n+2r+2} \\
&+ 2(-1)^r (R_1^{2n} + R_2^{2n}) \\
&= L_{k,6n+2r+2} + L_{k,2n+2r+2} + 2(-1)^r L_{k,2n}.
\end{aligned}$$

Thus, this completes the Proof. \square

Corollary 3.19:

$$\frac{(k^2+4)}{2} S_{k,2n+r} F_{k,2n+r+1} L_{k,2n} = L_{k,6n+2r+2} + L_{k,2n+2r+2} + 2(-1)^r L_{k,2n}, \quad (3.19)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.20:

$$\frac{(k^2+4)}{m} T_{k,2n+r} F_{k,2n+r+1} L_{k,2n} = L_{k,6n+2r+2} + L_{k,2n+2r+2} + 2(-1)^r L_{k,2n}, \quad (3.20)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Theorem 3.21:

$$\frac{(k^2+4)}{2m} F_{k,2n} L_{k,2n} T_{k,2n+r-1} S_{k,2n+r-1} = F_{k,8n+2r} + 2(-1)^{r+1} F_{k,4n} - F_{k,2r}, \quad (3.21)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Proof. By Binet's formula (2.3), (2.5), (2.7), and (2.9), we have

$$\begin{aligned}
&\frac{(k^2+4)}{2m} F_{k,2n} L_{k,2n} T_{k,2n+r-1} S_{k,2n+r-1} \\
&= \frac{(k^2+4)}{2m} \left(\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} \right) \left(R_1^{2n} + R_2^{2n} \right) \left(m \frac{R_1^{2n+r} - R_2^{2n+r}}{R_1 - R_2} \right) \\
&\quad \left(2 \frac{R_1^{2n+r} - R_2^{2n+r}}{R_1 - R_2} \right) \\
&= \frac{1}{R_1 - R_2} (R_1^{4n} - R_2^{4n}) (R_1^{4n+2r} - 2R_1^{2n+r} R_2^{2n+r} + R_2^{4n+2r})
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{R_1 - R_2} (R_1^{8n+2r} - 2R_1^{6n+r} R_2^{2n+r} + R_1^{4n} R_2^{4n+2r} \\
&\quad - R_1^{4n+2r} R_2^{4n} + 2R_1^{2n+r} R_2^{6n+r} - R_2^{8n+2r}) \\
&= \frac{1}{R_1 - R_2} (R_1^{8n+2r} - R_2^{8n+2r} - 2R_1^{6n+r} R_2^{2n+r} + 2R_1^{2n+r} R_2^{6n+r} \\
&\quad + R_1^{4n} R_2^{4n+2r} - R_1^{4n+2r} R_2^{4n}) \\
&= \frac{R_1^{8n+2r} - R_2^{8n+2r}}{R_1 - R_2} + \frac{-2R_1^{6n+r} R_2^{2n+r} + 2R_1^{2n+r} R_2^{6n+r}}{R_1 - R_2} + \frac{R_1^{4n} R_2^{4n+2r} - R_1^{4n+2r} R_2^{4n}}{R_1 - R_2} \\
&= \frac{R_1^{8n+2r} - R_2^{8n+2r}}{R_1 - R_2} - 2R_1^{2n+r} R_2^{2n+r} \frac{R_1^{4n} - R_2^{4n}}{R_1 - R_2} - R_1^{4n} R_2^{4n} \frac{R_1^{2r} - R_2^{2r}}{R_1 - R_2} \\
&= \frac{R_1^{8n+2r} - R_2^{8n+2r}}{R_1 - R_2} - 2R_1^{2n+r} R_2^{2n+r} \frac{R_1^{4n} - R_2^{4n}}{R_1 - R_2} - R_1^{4n} R_2^{4n} \frac{R_1^{2r} - R_2^{2r}}{R_1 - R_2} \\
&= \frac{R_1^{8n+2r} - R_2^{8n+2r}}{R_1 - R_2} - 2(R_1 R_2)^{2n+r} \frac{R_1^{4n} - R_2^{4n}}{R_1 - R_2} - (R_1 R_2)^{4n} \frac{R_1^{2r} - R_2^{2r}}{R_1 - R_2} \\
&= \frac{R_1^{8n+2r} - R_2^{8n+2r}}{R_1 - R_2} - 2(-1)^{2n+r} \frac{R_1^{4n} - R_2^{4n}}{R_1 - R_2} - (-1)^{4n} \frac{R_1^{2r} - R_2^{2r}}{R_1 - R_2} \\
&= \frac{R_1^{8n+2r} - R_2^{8n+2r}}{R_1 - R_2} - 2(-1)^{2n} (-1)^r \frac{R_1^{4n} - R_2^{4n}}{R_1 - R_2} - (-1)^{4n} \frac{R_1^{2r} - R_2^{2r}}{R_1 - R_2} \\
&= \frac{R_1^{8n+2r} - R_2^{8n+2r}}{R_1 - R_2} - 2(-1)^r \frac{R_1^{4n} - R_2^{4n}}{R_1 - R_2} - \frac{R_1^{2r} - R_2^{2r}}{R_1 - R_2} \\
&= F_{k,8n+2r} + 2(-1)^{r+1} F_{k,4n} - F_{k,2r}.
\end{aligned}$$

Thus, this completes the Proof. \square

Corollary 3.22:

$$\frac{(k^2+4)}{2} F_{k,2n} L_{k,2n} T_{k,2n+r} S_{k,2n+r} = T_{k,8n+2r+1} + 2m(-1)^r F_{k,4n} - T_{k,2r+1}, \quad (3.22)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.23:

$$\frac{(k^2+4)}{m} F_{k,2n} L_{k,2n} T_{k,2n+r} S_{k,2n+r} = S_{k,8n+2r+1} + 4(-1)^r F_{k,4n} - S_{k,2r+1}, \quad (3.23)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

Corollary 3.24:

$$\frac{(k^2 + 4)^{\frac{3}{2}}}{2m} F_{k,2n} L_{k,2n} T_{k,2n+r-1} S_{k,2n+r-1} = L_{k,8n+2r} + 2(-1)^{r+1} L_{k,4n} - 2R_2^{8n+2r} - 4(-1)^{r+1} R_2^{4n} + 2R_2^{2r}, \quad (3.24)$$

where $n \geq 0, r \geq 0$, m is positive integer and k is positive real number.

4. Conclusions

In this paper, the properties of number are proved by Binet's formula. We obtain some properties and related some identities for k -Fibonacci sequence $\{F_{k,n}\}$, k -Lucas sequence $\{L_{k,n}\}$, k -Fibonacci-Like sequence $\{S_{k,n}\}$ and k -Fibonacci-Like sequence $\{T_{k,n}\}$.

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