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Generalized Identities Related for the Fibonacci Number, Lucas Number, Fibonacci-Like Number and Generalized Fibonacci-Like Number By Matrix Method

Mongkol Tatong

Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi, Pathum Thani 12110, Thailand

E-mail : mongkol_t@rmutt.ac.th

Abstract

In this paper, we present generalized identities for Fibonacci, Lucas, Fibonacci-Like and Generalized Fibonacci-Like sequence. We obtain some identity relations by using the matrix method and Binet's formula.

Keywords: Fibonacci number, Lucas number, Fibonacci-Like number, Generalized Fibonacci-Like number Binet's formula, Matrix method

1. Introduction

The Fibonacci sequence is a very important research in the study and research in the past decade. This sequence can be applied to engineering business as well as to science.

Researcher study about the generalized Fibonacci sequence by changing the initials conditions $F_0 = a$ and $F_1 = b$. Moreover the coefficients p and q of the recurrence sequence are changing by $F_n = pF_{n-1} + qF_{n-2}$ for $n \geq 2$ (see [1]-[15]).

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In addition to studying the Fibonacci numbers from the recurrence sequence, we also want to study the Fibonacci numbers using matrix operations to find some properties (see [16]-[18]).

Sequence can be used with many aspects and this direction is very interesting. For this reason, researchers are motivated to study of Fibonacci, Lucas, and Fibonacci-Like sequences. In addition, the sequence can be proved using the matrix method and Binet's formula. In this paper, we use matrix method to show some identity.

2. Preliminaries

In this section, we will introduce the previous article, which is well-known for use in our research.

The Fibonacci sequence $\{F_n\}$ [3] is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 3 \tag{2.1}$$

with initial conditions $F_1 = F_2 = 1$. The first few terms of the sequence $\{F_n\}$ are $1,1,2,3,5,8,13,21$ and so on.

Binet's formula allows us to show the Fibonacci numbers in the function of the root $R_1 \,\&\, R_2$ from the recurrence relation (2.1), which is related to the following characteristic equations

$$x^2 - x - 1 = 0, \tag{2.2}$$

and $R_1 = \dfrac{1+\sqrt{5}}{2}$, $R_2 = \dfrac{1-\sqrt{5}}{2}$ so that $R_1 + R_2 = 1$, $R_1^2 - 1 = R_1$, $R_2^2 - 1 = R_2$, $R_1 R_2 = -1$.

Thus the Binet's formula of Fibonacci numbers is given by

$$F_n = \frac{R_1^n - R_2^n}{R_1 - R_2}, \tag{2.3}$$

where $R_1 \,\&\, R_2$ are the root of the characteristic equation and $R_1 > R_2$.

The Lucas sequence $\{L_n\}$ [1] is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 2 \tag{2.4}$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. The first few terms of the sequence $\{L_n\}$ are $2,1,3,4,7,11,18,29$ and so on.

The Binet's formula allows us to express the Lucas numbers in function of the roots $R_1 \,\&\, R_2$ of the following characteristic equation as in (2.2).

Thus the Binet's formula of Lucas numbers is given by

$$L_n = R_1^n + R_2^n, \tag{2.5}$$

where $R_1 \,\&\, R_2$ are the root of the characteristic equation and $R_1 > R_2$.

The Generalized Fibonacci-Like sequence $\{T_n\}$ [4] is defined by the recurrence relation

$$T_n = T_{n-1} + T_{n-2}, \text{ for } n \geq 2. \tag{2.6}$$

with initial conditions $T_0 = m$ and $T_1 = m$, where m is positive integer. The first few terms of the sequence $\{T_n\}$ are $m, m, 2m, 3m, 5m, 8m, 13m, 21m$ and so on.

The Binet's formula allows us to express the Generalized Fibonacci-Like numbers in function of the roots R_1 & R_2 of the following characteristic equation as in (2.2).

Thus the Binet's formula of Fibonacci-Like numbers is given by

$$T_n = m\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \tag{2.7}$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

Particular cases of (2.6), if $m = 2$ then we call the Fibonacci-Like sequence $\{S_n\}$ [12] is defined by

$$S_n = S_{n-1} + S_{n-2}, \text{ for } n \geq 2 \tag{2.8}$$

with initial conditions $S_0 = 2$ and $S_1 = 2$. The first few terms of the sequence $\{S_n\}$ are $2, 2, 4, 6, 10, 16, 26$ and so on.

The Binet's formula allows us to express the Fibonacci-Like numbers in function of the roots R_1 & R_2 of the following characteristic equation as in (2.2).

Thus the Binet's formula of Fibonacci-Like sequence is given by

$$S_n = 2\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \tag{2.9}$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

In 1960, Charles H. King [22] studied on the following Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

He showed that $\det(Q) = -1$ and

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \text{ for } n \geq 1$$

Moreover, it is clearly shown below that $\det(Q^n) = (-1)^n$ then

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

3. Main Results

In this section, we establish some identity relations of the Fibonacci sequence $\{F_n\}$, the Lucas sequence $\{L_n\}$, the Fibonacci-Like sequence $\{S_n\}$, the Generalized Fibonacci-Like sequence $\{T_n\}$ by using matrix methods. We begin with the following Lemma.

Lemma 3.1: $T_{2n+1} = 2T_{2n-1} + T_{2n-2}$, (3.1)

where n is positive integer.

Proof. By Binet's formula (2.7), we have

$$\begin{aligned} T_{2n+1} &= m\frac{R_1^{2n+2} - R_2^{2n+2}}{R_1 - R_2} \\ &= m\frac{R_1^{2n}R_1^2 - R_2^{2n}R_2^2}{R_1 - R_2} \\ &= m\frac{R_1^{2n}\left(2 + R_1^{-1}\right) - R_2^{2n}\left(2 + R_2^{-1}\right)}{R_1 - R_2} \\ &= m\frac{2R_1^{2n} + R_1^{2n-1} - 2R_2^{2n} - R_2^{2n-1}}{R_1 - R_2} \\ &= m\frac{2R_1^{2n} - 2R_2^{2n} + R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\ &= 2m\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} + m\frac{R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\ &= 2T_{2n-1} + T_{2n-2}. \end{aligned}$$

Thus, this completes the Proof. □

Theorem 3.2: Let $Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$. Then

$$Q_T^n = m^{n-1}\begin{pmatrix} T_{2n} & T_{2n-1} \\ T_{2n-1} & T_{2n-2} \end{pmatrix}, \tag{3.2}$$

where m and n are positive integer.

Proof. We proof that $Q_T^n = m^{n-1}\begin{pmatrix} T_{2n} & T_{2n-1} \\ T_{2n-1} & T_{2n-2} \end{pmatrix}$ for every $n \in N$, by using the Principle of Mathematical Induction on n. Because

$$Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix} = \begin{pmatrix} T_2 & T_1 \\ T_1 & T_0 \end{pmatrix}.$$

Thus $n=1$ is true. We assume the result is true for a positive integer $n=k$, then

$$Q_T^k = m^{k-1}\begin{pmatrix} T_{2k} & T_{2k-1} \\ T_{2k-1} & T_{2k-2} \end{pmatrix}.$$

Since Lemma 3.1, we consider the positive integer $n=k+1$. Then

$$\begin{aligned} Q_T^{k+1} &= Q_T^k Q_T \\ &= m^{k-1}\begin{pmatrix} T_{2k} & T_{2k-1} \\ T_{2k-1} & T_{2k-2} \end{pmatrix}\begin{pmatrix} 2m & m \\ m & m \end{pmatrix} \\ &= m^{k-1}\begin{pmatrix} 2mT_{2k} + mT_{2k-1} & mT_{2k} + mT_{2k-1} \\ 2mT_{2k-1} + mT_{2k-2} & mT_{2k-1} + mT_{2k-2} \end{pmatrix} \\ &= m^{k}\begin{pmatrix} 2T_{2k} + T_{2k-1} & T_{2k} + T_{2k-1} \\ 2T_{2k-1} + T_{2k-2} & T_{2k-1} + T_{2k-2} \end{pmatrix} \\ &= m^{k}\begin{pmatrix} T_{2k+2} & T_{2k+1} \\ T_{2k+1} & T_{2k} \end{pmatrix}. \end{aligned}$$

Thus $n=k+1$ is true, this completes the Proof.

□

Corollary 3.3: Let $Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$. Then

$$Q_T^n = \frac{m^{n-1}}{2}\begin{pmatrix} S_{2n} & S_{2n-1} \\ S_{2n-1} & S_{2n-2} \end{pmatrix}, \tag{3.3}$$

where m and n are positive integer.

Corollary 3.4: Let $Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$. Then

$$Q_T^n = m^{n-1}\begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}, \tag{3.4}$$

where m and n are positive integer.

Corollary 3.5: Let $Q_L = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Then

$$Q_L^n = \begin{cases} \dfrac{5^{\frac{n}{2}}}{m}\begin{pmatrix} T_n & T_{n-1} \\ T_{n-1} & T_{n-2} \end{pmatrix}, & \text{for } n \text{ is even}, \\ 5^{\frac{n-1}{2}}\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}, & \text{for } n \text{ is odd}, \end{cases} \tag{3.5}$$

where m and n are positive integer.

From Corollary 3.5, so that if $Q_L = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ then

$$Q_L^n = \begin{cases} \dfrac{5^{\frac{n}{2}}}{2}\begin{pmatrix} S_n & S_{n-1} \\ S_{n-1} & S_{n-2} \end{pmatrix}, & \text{for } n \text{ is even}, \\ 5^{\frac{n-1}{2}}\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}, & \text{for } n \text{ is odd}, \end{cases}$$

where m and n are positive integer.

Theorem 3.6: For a positive integer n, following equalities hold:

i) $\det\left(Q_T^n\right) = m^{2n}$,

ii) $T_{2n}T_{2n-2} - T_{2n-1}^2 = m^2$. (3.6)

Proof. By $\det\left(Q_T\right) = m^2$. Thus $\det\left(Q_T^n\right) = \left(\det\left(Q_T\right)\right)^n = \left(m^2\right)^n = m^{2n}$, and the determinant for Q_T^n in (3.2) will be ii), we get $m^{2n-2}\left(T_{2n}T_{2n-2} - T_{2n-1}^2\right) = m^{2n}$, thus $T_{2n}T_{2n-2} - T_{2n-1}^2 = m^2$.

□

Corollary 3.7: For a positive integer n, following equalities hold:

i) $\det\left(Q_T^n\right) = m^{2n}$,

ii) $S_{2n}S_{2n-2} - S_{2n-1}^2 = 4.$ (3.7)

Corollary 3.8: For a positive integer n, following equalities hold:

i) $\det\left(Q_T^n\right) = m^{2n}$,

ii) $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1.$ (3.8)

Corollary 3.9: For a positive integer n, following equalities hold:

i) $\det\left(Q_L^n\right) = 5^n$,

ii) $T_n T_{n-2} - T_{n-1}^2 = m^2$,

iii) $S_n S_{n-2} - S_{n-1}^2 = 4$,

iv) $L_{n+1}L_{n-1} - L_n^2 = 5.$ (3.9)

Theorem 3.10: Let n is positive integer. Then the Binet's formula of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$T_n = m\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \quad (3.10)$$

where $R_1 = \frac{1+\sqrt{5}}{2}$ and $R_2 = \frac{1-\sqrt{5}}{2}$.

Proof. Let Q_T is matrix in Theorem 3.2, $\lambda_1 = \frac{3m+\sqrt{5}m}{2}$ and $\lambda_2 = \frac{3m-\sqrt{5}m}{2}$ are the eigenvalues of matrix Q_T, $v_1 = \begin{pmatrix} R_1 & 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} R_2 & 1 \end{pmatrix}$ are eigenvectors that is correspond to a eigenvalues. Then we find diagonalizable of matrix Q_T by

$$D = P^{-1}Q_T P = \begin{pmatrix} \frac{3m+\sqrt{5}m}{2} & 0 \\ 0 & \frac{3m-\sqrt{5}m}{2} \end{pmatrix},$$

where

$$P = \left(v_1^T, v_2^T\right) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \\ 1 & 1 \end{pmatrix}$$

and

$$D = \operatorname{diag}\left(\lambda_1, \lambda_2\right) = \begin{pmatrix} \frac{3m+\sqrt{5}m}{2} & 0 \\ 0 & \frac{3m-\sqrt{5}m}{2} \end{pmatrix}.$$

Thus $Q_T = PDP^{-1} = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$.

Since properties of the Diagonal Matrix, we obtain

$$Q_T^n = PD^nP^{-1} = \begin{pmatrix} m^n\frac{R_1^{2n+1} - R_2^{2n+1}}{R_1 - R_2} & m^n\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} \\ m^n\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} & m^n\frac{R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \end{pmatrix},$$

where n is positive integer. We get

$$m^{n-1}\begin{pmatrix} T_{2n} & T_{2n-1} \\ T_{2n-1} & T_{2n-2} \end{pmatrix} = m^{n-1}\begin{pmatrix} m\frac{R_1^{2n+1} - R_2^{2n+1}}{R_1 - R_2} & m\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} \\ m\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} & m\frac{R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \end{pmatrix}.$$

This completes the Proof. □

Corollary 3.11: Let m and n are positive integer. Then the Binet's formula of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$T_n = \frac{m}{2}S_n = m\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \quad (3.11)$$

where $R_1 = \frac{1+\sqrt{5}}{2}$ and $R_2 = \frac{1-\sqrt{5}}{2}$.

Corollary 3.12: Let m and n are positive integer. Then the Binet's formula of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$T_n = mF_{n+1} = m\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \quad (3.12)$$

where $R_1 = \frac{1+\sqrt{5}}{2}$ and $R_2 = \frac{1-\sqrt{5}}{2}$.

Lemma 3.13: $T_{2n} + T_{2n-2} = mL_{2n}$, (3.13)

where m and n are positive integer.

Proof. By Binet's formula (2.5) and (2.7), we have

$$\begin{aligned} T_{2n} + T_{2n-2} &= m\frac{R_1^{2n+1} - R_2^{2n+1}}{R_1 - R_2} + m\frac{R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\ &= m\frac{R_1^{2n+1} - R_2^{2n+1} + R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\ &= m\frac{R_1^{2n+1} + R_1^{2n-1} - R_2^{2n+1} - R_2^{2n-1}}{R_1 - R_2} \\ &= m\frac{R_1^{2n}\left(R_1 + R_1^{-1}\right) - R_2^{2n}\left(R_2 + R_2^{-1}\right)}{R_1 - R_2} \\ &= m\sqrt{5}\frac{R_1^{2n} + R_2^{2n}}{R_1 - R_2} \\ &= m\left(R_1^{2n} + R_2^{2n}\right) \\ &= mL_{2n}. \end{aligned}$$

Thus, this completes the Proof. □

Lemma 3.14: $L_{2n}^2 - 4 = 5F_{2n}^2$, (3.14)

where m and n are positive integer.

Proof. By Binet's formula (2.3) and (2.5), we have

$$\begin{aligned} L_{2n}^2 - 4 &= \left(R_1^{2n} + R_2^{2n}\right)^2 - 4 \\ &= R_1^{4n} + 2R_1^{2n}R_2^{2n} + R_2^{4n} - 4 \\ &= R_1^{4n} + 2\left(R_1R_2\right)^{2n} + R_2^{4n} - 4 \\ &= R_1^{4n} + 2 + R_2^{4n} - 4 \\ &= R_1^{4n} - 2 + R_2^{4n} \\ &= R_1^{4n} - 2R_1^{2n}R_2^{2n} + R_2^{4n} \\ &= \left(R_1^{2n} - R_2^{2n}\right)^2 \\ &= \frac{5\left(R_1^{2n} - R_2^{2n}\right)^2}{\left(R_1 - R_2\right)^2} \\ &= 5F_{2n}^2. \end{aligned}$$

Thus, this completes the Proof. □

Theorem 3.15: The generalized two roots of Q_T^n are

$$\lambda_1 = m\frac{\left(L_{2n} + \sqrt{5}F_{2n}\right)}{2} \text{ and}$$

$$\lambda_2 = m\frac{\left(L_{2n} - \sqrt{5}F_{2n}\right)}{2}. \quad (3.15)$$

Where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $F_{2n} = \frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Proof. The characteristic equation of Q_T^n. By Lemma 3.13 and Theorem 3.6, we get

$$\begin{aligned} \det\left(Q_T^n - \lambda I\right) &= \begin{vmatrix} m^{n-1}T_{2n} - \lambda & m^{n-1}T_{2n-1} \\ m^{n-1}T_{2n-1} & m^{n-1}T_{2n-2} - \lambda \end{vmatrix} \\ &= m^{2n-2}\left(T_{2n} - \lambda\right)\left(T_{2n-2} - \lambda\right) - m^{2n-2}T_{2n-1}^2 \\ &= m^{2n-2}\lambda^2 - m^{2n-2}\left(T_{2n} + T_{2n-2}\right)\lambda \\ &\quad - m^{2n-2}\left(T_{2n}T_{2n-2} - T_{2n-1}^2\right) \\ &= m^{2n-2}\lambda^2 - m^{2n-2}mL_{2n}\lambda - m^{2n-2}m^2 \\ &= m^{2n-2}\lambda^2 - m^{2n-1}L_{2n}\lambda - m^{2n}. \end{aligned}$$

Thus, the characteristic equation of Q_T^n is

$$\lambda^2 - mL_{2n}\lambda - m^2 = 0,$$

and we get the generalized characteristic roots as following:

$$\lambda_1, \lambda_1 = \frac{mL_{2n} \pm m\sqrt{L_{2n}^2 - 4}}{2}.$$

By Lemma 3.14, it can be writing

$$\lambda_1, \lambda_2 = m\frac{\left(L_{2n} \pm \sqrt{5}F_{2n}\right)}{2}.$$

Therefore,

$$R_1^{2n} = \frac{L_{2n} + \sqrt{5}F_{2n}}{2} \text{ and } R_2^{2n} = \frac{L_{2n} - \sqrt{5}F_{2n}}{2}.$$

Thus, we give the Binet's formula by matrix method for the Fibonacci numbers and Lucas numbers given in (2.3) and (2.5) by

$$F_{2n} = \frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} \text{ and } L_{2n} = R_1^{2n} + R_2^{2n} .$$

This completes the Proof. □

Corollary 3.16: The generalized two roots of Q_T^n are

$$\lambda_1 = \frac{mL_{2n} + \sqrt{\left(\sqrt{5}T_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}$$

and

$$\lambda_2 = \frac{mL_{2n} - \sqrt{\left(\sqrt{5}T_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}, \tag{3.16}$$

where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $T_{2n-1} = m\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Corollary 3.17: The generalized two roots of Q_T^n are

$$\lambda_1 = \frac{mL_{2n} + \sqrt{\left(\frac{\sqrt{5}m}{2}S_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}$$

and

$$\lambda_2 = \frac{mL_{2n} - \sqrt{\left(\frac{\sqrt{5}m}{2}S_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}, \tag{3.17}$$

where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $S_{2n-1} = 2\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Corollary 3.18: The generalized two roots of Q_T^n are

$$\lambda_1 = \frac{mL_{2n} + \sqrt{\left(\sqrt{5}mF_{2n} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}$$

and

$$\lambda_2 = \frac{mL_{2n} - \sqrt{\left(\sqrt{5}mF_{2n} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}, \tag{3.18}$$

where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $F_{2n} = \frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Lemma 3.19: $\lim_{n\to\infty} \frac{T_{2n-1}}{T_{2n-2}} = R_1$. (3.19)

Proof. By Binet's formula (2.7), we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{T_{2n-1}}{T_{2n-2}} &= \lim_{n\to\infty} \frac{R_1^{2n} - R_2^{2n}}{R_1^{2n-1} - R_2^{2n-1}} \\
&= \lim_{n\to\infty} \frac{1 - \left(\frac{R_2}{R_1}\right)^{2n}}{\frac{1}{R_1} - \left(\frac{R_2}{R_1}\right)^{2n} \frac{1}{R_2}} \\
&= \frac{\lim_{n\to\infty} 1 - \lim_{n\to\infty}\left(\frac{R_2}{R_1}\right)^{2n}}{\lim_{n\to\infty}\frac{1}{R_1} - \lim_{n\to\infty}\left(\left(\frac{R_2}{R_1}\right)^{2n}\frac{1}{R_2}\right)} \\
&= \frac{\lim_{n\to\infty} 1 - \lim_{n\to\infty}\left(\frac{R_2}{R_1}\right)^{2n}}{\lim_{n\to\infty}\frac{1}{R_1} - \left(\lim_{n\to\infty}\left(\frac{R_2}{R_1}\right)^{2n} \lim_{n\to\infty}\frac{1}{R_2}\right)} \\
&= R_1 .
\end{aligned}$$

Thus, this completes the proof. □

Theorem 3.20: Let

$$\frac{Q_T^n}{m^{n-1}T_{2n-2}} = \begin{pmatrix} \frac{T_{2n}}{T_{2n-2}} & \frac{T_{2n-1}}{T_{2n-2}} \\ \frac{T_{2n-1}}{T_{2n-2}} & 1 \end{pmatrix}$$. Then the

determinant of $\lim_{n\to\infty} \frac{Q_T^n}{m^{n-1}T_{2n-2}}$ as follows

characteristic equations of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$R_1^2 - R_1 - 1 = 0. \quad (3.20)$$

Proof. Since the matrix Q_T^n in (3.2), we can write

$$\frac{Q_T^n}{m^{n-1}T_{2n-2}} = \begin{pmatrix} \frac{T_{2n}}{T_{2n-2}} & \frac{T_{2n-1}}{T_{2n-2}} \\ \frac{T_{2n-1}}{T_{2n-2}} & 1 \end{pmatrix},$$

and by Lemma 3.19 , thus

$$\lim_{n\to\infty} \frac{Q_T^n}{m^{n-1}T_{2n-2}} = \begin{pmatrix} \lim\limits_{n\to\infty} \frac{T_{2n}}{T_{2n-2}} & \lim\limits_{n\to\infty} \frac{T_{2n-1}}{T_{2n-2}} \\ \lim\limits_{n\to\infty} \frac{T_{2n-1}}{T_{2n-2}} & \lim\limits_{n\to\infty} 1 \end{pmatrix} = \begin{pmatrix} R_1^2 & R_1 \\ R_1 & 1 \end{pmatrix}.$$

But $R_1^2 = R_1 + 1$, thus

$$\lim_{n\to\infty} \frac{Q_T^n}{m^{n-1}T_{2n-2}} = \begin{pmatrix} R_1+1 & R_1 \\ R_1 & 1 \end{pmatrix}.$$

Therefore

$$\det\left(\lim_{n\to\infty} \frac{Q_T^n}{m^{n-1}T_{2n-2}}\right) = \begin{vmatrix} R_1+1 & R_1 \\ R_1 & 1 \end{vmatrix}$$
$$= R_1^2 - R_1 - 1,$$

and thus $R_1^2 - R_1 - 1 = 0$. This completes the Proof. □

Corollary 3.21: Let

$$\frac{2Q_T^n}{m^n S_{2n-2}} = \begin{pmatrix} \frac{S_{2n}}{S_{2n-2}} & \frac{S_{2n-1}}{S_{2n-2}} \\ \frac{S_{2n-1}}{S_{2n-2}} & 1 \end{pmatrix}$$. Then the

determinant of $\lim_{n\to\infty} \frac{2Q_T^n}{m^n S_{2n-2}}$ as follows

characteristic equations of the Fibonacci-Like sequence $\{S_n\}$ is given by

$$R_1^2 - R_1 - 1 = 0.$$

Corollary 3.22: Let $\frac{Q_T^n}{m^n F_{2n-1}} = \begin{pmatrix} \frac{F_{2n+1}}{F_{2n-1}} & \frac{F_{2n}}{F_{2n-1}} \\ \frac{F_{2n}}{F_{2n-1}} & 1 \end{pmatrix}$.

Then the determinant of $\lim_{n\to\infty} \frac{Q_T^n}{m^n F_{2n-1}}$ as follows characteristic equations of Fibonacci sequence $\{F_n\}$ is given by

$$R_1^2 - R_1 - 1 = 0.$$

Theorem 3.23: Let n and k are positive integer. Then the following relation between $\{S_n\}$ and $\{T_n\}$ is given by

$$mS_{2n+2k} = T_{2k+1}S_{2n-1} + T_{2k}S_{2n-2}. \quad (3.21)$$

Proof. By relation between Fibonacci-Like sequence $\{S_n\}$ and Fibonacci-Like sequence $\{T_n\}$

$$m\begin{pmatrix} S_{2n+1} \\ S_{2n} \end{pmatrix} = Q_T \begin{pmatrix} S_{2n-1} \\ S_{2n-2} \end{pmatrix}.$$

And we multiply with Q_T^k, we get

$$mQ_T^k \begin{pmatrix} S_{2n+1} \\ S_{2n} \end{pmatrix} = Q_T^{k+1} \begin{pmatrix} S_{2n-1} \\ S_{2n-2} \end{pmatrix}.$$

Thus

$$m\begin{pmatrix} S_{2n+2k+1} \\ S_{2n+2k} \end{pmatrix} = \begin{pmatrix} T_{2k+2}S_{2n-1} + T_{2k+1}S_{2n-2} \\ T_{2k+1}S_{2n-1} + T_{2k}S_{2n-2} \end{pmatrix}.$$

This completes the Proof. □

Corollary 3.24: Let n and k are positive integer. Then the following relation between $\{F_n\}$ and $\{T_n\}$ is given by

$$mF_{2n+2k+1} = T_{2k-1}F_{2n+2} + T_{2k-2}F_{2n+1}. \quad (3.22)$$

Theorem 3.25: Let n and r are positive integers and $n \geq r$. Then the following equalities are hold:

i) $mT_{2n+2r} = T_{2n}T_{2r} + T_{2n-1}T_{2r-1}$,

ii) $mT_{4n} = T_{2n}^2 + T_{2n-1}^2$,

iii) $mT_{4n+1} = T_{2n}\left(T_{2n+1} + T_{2n-1}\right)$,

iv) $m\ T_{2n-2r} = T_{2n}T_{2r-2} - T_{2n-1}T_{2r-1}$. (3.23)

Proof. Let the Q_T^n-matrix in (3.2). By $Q_T^{n+r} = Q_T^n Q_T^r$, we get

$$m^{n+r-1}\begin{pmatrix} T_{2n+2r} & T_{2n+2r-1} \\ T_{2n+2r-1} & T_{2n+2r-2} \end{pmatrix}$$

$$= m^{n+r-2}\begin{pmatrix} T_{2n}T_{2r} + T_{2n-1}T_{2r-1} & T_{2n}T_{2r-1} + T_{2n-1}T_{2r-2} \\ T_{2n-1}T_{2r} + T_{2n-2}T_{2r-1} & T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r-2} \end{pmatrix}$$

Thus

$$m\begin{pmatrix} T_{2n+2r} & T_{2n+2r-1} \\ T_{2n+2r-1} & T_{2n+2r-2} \end{pmatrix}$$

$$= \begin{pmatrix} T_{2n}T_{2r} + T_{2n-1}T_{2r-1} & T_{2n}T_{2r-1} + T_{2n-1}T_{2r-2} \\ T_{2n-1}T_{2r} + T_{2n-2}T_{2r-1} & T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r-2} \end{pmatrix}$$

Therefore, equalities i), ii), and iii). And we get

$$Q_T^{-r} = m^{-r-1}\begin{pmatrix} T_{2r-2} & -T_{2r-1} \\ -T_{2r-1} & T_{2r} \end{pmatrix}.$$

Since $Q_T^{n-r} = Q_T^n Q_T^{-r}$. We have

$$m^{n-r-1}\begin{pmatrix} T_{2n-2r} & T_{2n-2r-1} \\ T_{2n-2r-1} & T_{2n-2r-2} \end{pmatrix}$$

$$= m^{n-r-2}\begin{pmatrix} T_{2n}T_{2r-2} - T_{2n-1}T_{2r-1} & -T_{2n}T_{2r-1} + T_{2n-1}T_{2r} \\ T_{2n-1}T_{2r-2} - T_{2n-2}T_{2r-1} & -T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r} \end{pmatrix}$$

Thus

$$m\begin{pmatrix} T_{2n-2r} & T_{2n-2r-1} \\ T_{2n-2r-1} & T_{2n-2r-2} \end{pmatrix}$$

$$= \begin{pmatrix} T_{2n}T_{2r-2} - T_{2n-1}T_{2r-1} & -T_{2n}T_{2r-1} + T_{2n-1}T_{2r} \\ T_{2n-1}T_{2r-2} - T_{2n-2}T_{2r-1} & -T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r} \end{pmatrix},$$

and iv) immediately seen. This completes the Proof. □

Corollary 3.26: Let n and r are positive integers and $n \geq r$. Then the following equalities are hold:

i) $2T_{2n+2r} = T_{2n}S_{2r} + T_{2n-1}S_{2r-1}$,

ii) $2T_{4n} = T_{2n}S_{2n} + T_{2n-1}S_{2n-1}$,

iii) $2T_{4n+2} = T_{2n}S_{2n+2} + T_{2n-1}S_{2n+1}$,

iv) $2T_{2n-2r} = T_{2n}S_{2r-2} - T_{2n-1}S_{2r-1}$. (3.24)

Corollary 3.27: Let n and r are positive integers and $n \geq r$. Then the following equalities are hold:

i) $T_{2n+2r} = T_{2n}F_{2r+1} + T_{2n-1}F_{2r}$,

ii) $T_{4n} = T_{2n}F_{2n+1} + T_{2n-1}F_{2n}$,

iii) $T_{4n+2} = T_{2n}F_{2n+3} + T_{2n-1}F_{2n+2}$,

iv) $T_{2n-2r} = T_{2n}F_{2r-1} - T_{2n-1}F_{2r}$. (3.25)

4. Conclusions

In this paper, the properties of number are proved by Binet's formula and matrix representation. We obtain some properties and related some identities for Fibonacci sequence $\{F_n\}$, Lucas sequence $\{L_n\}$, Fibonacci-Like sequence $\{S_n\}$ and Generalized Fibonacci-Like sequence $\{T_n\}$.

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