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**On Some Identities and Generating Functions for**

**$(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas Numbers**

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**Abstract**

In this paper, we obtain the Binet's formula for  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas numbers and then we get some identities for these numbers by using the Binet's formula. Moreover, we obtain the generating functions for  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas sequences and another expression for the general term of the sequences by using the ordinary generating functions.

**Keywords:** Pell number; Pell-Lucas number;  $(s, t)$ -Pell number;  $(s, t)$ -Pell-Lucas number; Binet's formula; Generating function.

**1. Introduction**

It is well-known that the Fibonacci and Lucas numbers are the most famous of the recursive sequences that have been studied in the literature over several years. They are widely used in a variety of research areas such as Engineering, Architecture, Nature and Art. (see: [6-10]). For  $n \geq 2$ , the

classical Fibonacci  $\{F_n\}$  and Lucas  $\{L_n\}$  sequences are defined by the recurrence relation:  $F_n = F_{n-1} + F_{n-2}$  and  $L_n = L_{n-1} + L_{n-2}$ , with the initial conditions  $F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1$  respectively. On the other hand, other sequences that also important are Pell and Pell-Lucas sequences. The Pell and Pell-Lucas sequences are

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defined by  $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ , for  $n \geq 2$  and  $Q_0 = 0, Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ , for  $n \geq 2$ , respectively. For more detailed information about Pell and Pell-Lucas sequences can be found in [4, 6]. Recently, Pell and Pell-Lucas numbers were generalized and studied by many authors in the different ways to derive many identities. For a lot of identities of Pell, Pell-Lucas numbers and their generalization can be found in [1, 2, 5-7] and the references therein.

In this paper we investigate the generalization of Pell and Pell-Lucas numbers, which is called  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas numbers and then we obtain the Binet's formula and some identities for these numbers. Also, we give the generating functions for the  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas sequences and another expression for the general term of the sequences, by using the ordinary generating functions.

**2.  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas**

**Numbers and some identities**

In this section, a new generalization of Pell and Pell-Lucas numbers are introduced and it's Binet's formula are obtained. After that, by using the Binet's formula, we obtain some identities for these numbers. We begin this section with the following definition.

**Definition 2.1** [2] Let  $s, t$  be any real number with  $s^2 + t > 0, s > 0$  and  $t \neq 0$ . Then the  $(s, t)$ -Pell sequences  $\{P_n(s, t)\}_{n=0}^\infty$  and the  $(s, t)$ -Pell-Lucas sequences  $\{Q_n(s, t)\}_{n=0}^\infty$  are defined respectively by

$$P_n(s, t) = 2sP_{n-1}(s, t) + tP_{n-2}(s, t), \text{ for } n \geq 2, \quad (2.1)$$

$$Q_n(s, t) = 2sQ_{n-1}(s, t) + tQ_{n-2}(s, t), \text{ for } n \geq 2, \quad (2.2)$$

with initial conditions  $P_0(s, t) = 0, P_1(s, t) = 1$  and  $Q_0(s, t) = 2, Q_1(s, t) = 2s$ .

The first few terms of  $\{P_n(s, t)\}_{n=0}^\infty$  are  $0, 1, 2s, 4s^2 + t, 8s^3 + 4st$  and so on. Also, the first few terms of  $\{Q_n(s, t)\}_{n=0}^\infty$  are  $2, 2s, 4s^2 + 2t, 8s^3 + 6st$  and so on. The terms of  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas sequences are called  $(s, t)$ -Pell-numbers and  $(s, t)$ -Pell-Lucas-numbers respectively.

Throughout this paper, for convenience we will use the symbol  $P_n$  and  $Q_n$  instead of  $P_n(s, t)$  and  $Q_n(s, t)$  respectively. Also, we denoted the set of whole numbers by  $\mathbb{N}_0$  (i. e.  $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$ ).

Particular case of the Definition 2.1 are :

- If  $s = \frac{1}{2}, t = 1$  then the classical Fibonacci and Lucas sequence are obtained.
- If  $s = t = 1$  then the classical Pell and Pell-Lucas sequence are obtained.
- If  $s = \frac{1}{2}, t = 2$  then the classical Jacobsthal and Jacobsthal-Lucas sequence are obtained.

Next, we give the explicit formula for the term of order  $n$  of  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas numbers. From the Definition 2.1, we have that the characteristic equation of (2.1) and (2.2) are in the form

$$x^2 = 2sx + t \quad (2.3)$$

and the root of equation (2.3) are  $\alpha = s + \sqrt{s^2 + t}$  and  $\beta = s - \sqrt{s^2 + t}$ . We note that  $\alpha + \beta = 2s$ ,  $\alpha - \beta = 2\sqrt{s^2 + t}$  and  $\alpha\beta = -t$  and we get the following theorem.

**Theorem 2.2 (Binet's formula)**

The  $n^{\text{th}}$   $(s, t)$ -Pell number and the  $n^{\text{th}}$   $(s, t)$ -Pell-Lucas number are given by

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (2.4)$$

and

$$Q_n = \alpha^n + \beta^n \quad (2.5)$$

respectively, where  $\alpha, \beta$  are the roots of the characteristic equation (2.3) and  $\alpha > \beta$ .

*Proof.* Since the characteristic equation (2.3) has two distinct roots, the closed form of  $\{P_n\}$  is given by

$$P_n = c_1 \alpha^n + c_2 \beta^n,$$

for some coefficients  $c_1$  and  $c_2$ . Giving to  $n$  the values  $n = 0$  and  $n = 1$  then solving this system of linear equations

we obtain  $c_1 = \frac{1}{\alpha - \beta} = -c_2$ , and therefore

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Similarly, the closed form of  $Q_n$  is given by

$$Q_n = c_1 \alpha^n + c_2 \beta^n,$$

for some coefficients  $c_1$  and  $c_2$ . By the same fashion as above, we obtain  $c_1 = c_2 = 1$ , and hence

$$Q_n = \alpha^n + \beta^n. \quad \square$$

**Theorem 2.3 (Catalan's identity)**

Let  $n, r \in \mathbb{N}_0$  with  $n > r$ . Then

$$P_{n+r} P_{n-r} - P_n^2 = -(-t)^{n-r} P_n^2 \quad (2.6)$$

and

$$Q_{n+r} Q_{n-r} - Q_n^2 = (-t)^{n-r} (Q_r^2 - 4(-t)^r) \quad (2.7)$$

*Proof.* Using the Binet's formula (2.4), we have

$$\begin{aligned} P_{n+r} P_{n-r} - P_n^2 &= \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha - \beta} \cdot \frac{\alpha^{n-r} - \beta^{n-r}}{\alpha - \beta} \\ &\quad - \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^2 \\ &= \frac{-(\alpha\beta)^{n-r} (\alpha^r - \beta^r)^2}{(\alpha - \beta)^2} \\ &= -(-t)^{n-r} P_r^2. \end{aligned}$$

Using the Binet's formula (2.5), we have

$$\begin{aligned} Q_{n+r} Q_{n-r} - Q_n^2 &= (\alpha^{n+r} + \beta^{n+r})(\alpha^{n-r} + \beta^{n-r}) \\ &\quad - (\alpha^n + \beta^n)^2 \\ &= (\alpha\beta)^{n-r} ((\alpha^r + \beta^r)^2 - 4(\alpha\beta)^r) \\ &= (-t)^{n-r} (Q_r^2 - 4(-t)^r). \quad \square \end{aligned}$$

Note that for  $r = 1$ , equation (2.6) and (2.7) give Cassini's identity for the  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas numbers respectively.

**Theorem 2.4 (Cassini's identity)**

Let  $n \in \mathbb{N}_0$ . Then

$$P_{n+1} P_{n-1} - P_n^2 = -(-t)^{n-1} \quad (2.8)$$

and

$$Q_{n+1} Q_{n-1} - Q_n^2 = 4(s^2 + t)(-t)^{n-1} \quad (2.9)$$

*Proof.* By letting  $r = 1$  in Theorem 2.3,

we get the results.  $\square$

**Theorem 2.5 (d'Ocagne's identity)**

Let  $m, n \in \mathbb{N}_0$  with  $m > n$ . Then

$$P_m P_{n+1} - P_{m+1} P_n = (-t)^n P_{m-n} \tag{2.10}$$

and

$$Q_m Q_{n+1} - Q_{m+1} Q_n = 2(-t)^n \sqrt{s^2 + t} \left( Q_{m-n} - 2(s + \sqrt{s^2 + t})^{m-n} \right). \tag{2.11}$$

*Proof.* Using the Binet's formula (2.4), we have

$$\begin{aligned} P_m P_{n+1} - P_{m+1} P_n &= \frac{\alpha^m - \beta^m}{\alpha - \beta} \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &\quad - \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} \cdot \frac{\alpha^n - \beta^n}{\alpha - \beta} \\ &= \frac{(\alpha\beta)^n (\alpha - \beta)(\alpha^{m-n} - \beta^{m-n})}{(\alpha - \beta)^2} \\ &= (\alpha\beta)^n \cdot \frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta} \\ &= (-t)^n P_{m-n}. \end{aligned}$$

Using the Binet's formula (2.5), we have

$$\begin{aligned} Q_m Q_{n+1} - Q_{m+1} Q_n &= (\alpha^m + \beta^m)(\alpha^{n+1} + \beta^{n+1}) - (\alpha^{m+1} + \beta^{m+1})(\alpha^n + \beta^n) \\ &= (\alpha\beta)^n (\alpha - \beta)(\alpha^{m-n} + \beta^{m-n} - 2\alpha^{m-n}). \end{aligned}$$

Since  $\alpha = s + \sqrt{s^2 + t}$  and  $\beta = s - \sqrt{s^2 + t}$ , we get

$$Q_m Q_{n+1} - Q_{m+1} Q_n = 2(-t)^n \sqrt{s^2 + t} (Q_{m-n} - 2(s + \sqrt{s^2 + t})^{m-n}). \square$$

**Theorem 2.6** Let  $\{P_n\}$  and  $\{Q_n\}$  be  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas sequences. Then

$$\lim_{n \rightarrow +\infty} \frac{P_n}{P_{n-1}} = \alpha \tag{2.12}$$

and

$$\lim_{n \rightarrow +\infty} \frac{Q_n}{Q_{n-1}} = \alpha. \tag{2.13}$$

*Proof.* By using the Binet's formula (2.4), we have

$$\lim_{n \rightarrow +\infty} \frac{P_n}{P_{n-1}} = \lim_{n \rightarrow +\infty} \frac{\alpha^n - \beta^n}{\alpha^{n-1} - \beta^{n-1}} = \lim_{n \rightarrow +\infty} \frac{1 - \left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha} - \frac{1}{\beta} \left(\frac{\beta}{\alpha}\right)^n}.$$

Since  $\left| \frac{\beta}{\alpha} \right| < 1$ ,  $\lim_{n \rightarrow +\infty} \left(\frac{\beta}{\alpha}\right)^n = 0$ , and therefore

$$\lim_{n \rightarrow +\infty} \frac{P_n}{P_{n-1}} = \alpha.$$

On the other hand, using the Binet's formula (2.5)

and using the same way as above, we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{Q_n}{Q_{n-1}} &= \lim_{n \rightarrow +\infty} \frac{\alpha^n + \beta^n}{\alpha^{n-1} + \beta^{n-1}} \\ &= \lim_{n \rightarrow +\infty} \frac{1 + \left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha} + \frac{1}{\beta} \left(\frac{\beta}{\alpha}\right)^n} \\ &= \alpha. \end{aligned} \quad \square$$

**3. Generating Functions for  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas Numbers**

In this section, the generating functions for the  $(s, t)$ -Pell and  $(s, t)$ -Pell-Lucas sequences are given. First, we shall give the generating functions for the  $(s, t)$ -Pell sequences. We shall write the  $(s, t)$ -Pell sequence as a power series where each term of the sequence correspond to coefficients of the series and from that fact, we find the generating function. Let us consider the  $(s, t)$ -Pell sequences for any positive integer  $s, t$ . By definition of ordinary generating function of some sequence, considering this sequence, the ordinary generating function associated is defined by

$$\begin{aligned} G(P_n; x) &= \sum_{n=0}^{\infty} P_n x^n \\ &= P_0 + P_1 x + P_2 x^2 + \dots + P_n x^n + \dots \end{aligned} \tag{3.1}$$

From (2.1) and  $P_0 = 0, P_1 = 1$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} P_n x^n &= x + \sum_{n=2}^{\infty} (2sP_{n-1} + tP_{n-2}) x^n \\ &= x + 2s \sum_{n=2}^{\infty} P_{n-1} x^n + t \sum_{n=2}^{\infty} P_{n-2} x^n \\ &= x + 2sx \sum_{n=2}^{\infty} P_{n-1} x^{n-1} + tx^2 \sum_{n=2}^{\infty} P_{n-2} x^{n-2}. \end{aligned} \quad (3.2)$$

Now, consider that  $k = n-1$  and  $j = n-2$ , the equation (3.2) can be written by

$$\begin{aligned} \sum_{n=0}^{\infty} P_n x^n &= x + 2sx \sum_{k=1}^{\infty} P_k x^k + tx^2 \sum_{j=0}^{\infty} P_j x^j \\ &= x + 2sx \sum_{k=0}^{\infty} P_k x^k + tx^2 \sum_{j=0}^{\infty} P_j x^j. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} P_n x^n = x + 2sx \sum_{n=0}^{\infty} P_n x^n + tx^2 \sum_{n=0}^{\infty} P_n x^n.$$

Therefore, the ordinary generating function of the  $(s, t)$ -Pell sequence can be written as

$$G(P_n; x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - 2sx - tx^2}. \quad (3.3)$$

Applying the ratio test for absolute convergence and using (2.12), we have

$$\lim_{n \rightarrow +\infty} \left| \frac{P_n x^n}{P_{n-1} x^{n-1}} \right| = |x| \lim_{n \rightarrow +\infty} \frac{P_n}{P_{n-1}} = \alpha |x|, \quad (3.4)$$

and so the series converges absolutely if  $|x| < \frac{1}{\alpha}$  and diverges if  $|x| > \frac{1}{\alpha}$ . Thus its radius of convergence  $R$  is equal to  $\frac{1}{\alpha}$ . Now, by the similar argument as above, we get that the ordinary generating function of  $(s, t)$ -Pell-Lucas sequence can be written as

$$G(Q_n; x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{2 - 2sx}{1 - 2sx - tx^2}, \quad (3.5)$$

and its radius of convergence  $R$  is equal to  $\frac{1}{\alpha}$ .

Finally, we give another expression for the general term of the  $(s, t)$ -Pell sequence using the ordinary generating function.

### Theorem 3.1

Let  $p(x) = G(P_n; x) = \sum_{n=0}^{\infty} P_n x^n$ , for  $x \in (-\frac{1}{\alpha}, \frac{1}{\alpha})$ .

$$\text{Then } P_n = \frac{p^{(n)}(0)}{n!}, \quad (3.6)$$

where  $p^{(n)}(x)$  denotes the  $n^{\text{th}}$  order derivative of the function  $p(x)$ .

*Proof.* Since  $p(x) = \sum_{n=0}^{\infty} P_n x^n$ , we have

$$p'(x) = \sum_{n=1}^{\infty} n P_n x^{n-1},$$

$$p''(x) = \sum_{n=2}^{\infty} n(n-1) P_n x^{n-2},$$

$$p'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) P_n x^{n-3},$$

⋮

$$\begin{aligned} p^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2) \cdots (n-(k-1)) P_n x^{n-k} \\ &= k(k-1)(k-2) \cdots 1 \cdot P_k \\ &\quad + \sum_{n=k+1}^{\infty} n(n-1)(n-2) \cdots (n-(k-1)) P_n x^{n-k} \\ &= k! P_k + \sum_{n=k+1}^{\infty} n(n-1)(n-2) \cdots (n-(k-1)) P_n x^{n-k}, \end{aligned}$$

then  $p^{(k)}(0) = k! P_k$ . Thus for all  $n \geq 1$ , we have

$$P_n = \frac{p^{(n)}(0)}{n!}. \quad \square$$

By using the same approximation as in Theorem 3.1, we obtain the following theorem.

**Theorem 3.2** Let

$$q(x) = G(Q_n; x) = \sum_{n=0}^{\infty} Q_n x^n, \text{ for } x \in \left(-\frac{1}{\alpha}, \frac{1}{\alpha}\right).$$

$$\text{Then } Q_n = \frac{q^{(n)}(n)}{n!}, \quad (3.7)$$

where  $q^{(n)}(x)$  denotes the  $n^{\text{th}}$  order derivative of the function  $q(x)$ .

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