Science and Technology RMUTT Journal

Vol.7 No.2 (2017) : 200–205

www.sci.rmutt.ac.th/stj

Online ISSN 2229-1547

Possible Solutions of the Diophantine Equation $x^2 + ky^2 = z^2$

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Abstract

This paper is to identify the Diophantine equation $x^2 + ky^2 = z^2$ where k, x, y and z are integers satisfies; case 1: $k = 4m + 2$, has no integer solution if y is odd, and have integer solutions (x, y, z) is $\left(\pm(ka - b), \pm 2\sqrt{ab}, \pm(ka + b)\right)$ where m is an integer, ab is a square number if y is even, case 2: $k = 2m + 1$, have integer solutions (x, y, z) is $\left(\pm\frac{ka - b}{2}, \pm\sqrt{ab}, \pm\frac{ka + b}{2}\right)$ where m is an integer, ab is an odd square number if y is odd, and $\left(\pm(ka - b), \pm 2\sqrt{ab}, \pm(ka + b)\right)$ where m is an integer, ab is a square number if y is even, case 3: $k = 4m$, have integer solutions (x, y, z) is $\left(\pm\left(\frac{k}{4}a - b\right), \pm\sqrt{ab}, \pm\left(\frac{k}{4}a + b\right)\right)$ where m is an integer, ab is a square number,

Keywords: Diophantine equation, Congruence, Integer solutions, Divisibility

1. Introduction

Nowadays, there are many studies in the literature that concern the Diophantine equation. In 1995, Wiles showed that the Diophantine equation $x^n + y^n = z^n, xyz \neq 0$ has no integer solution when $n \geq 3$ [1]. In 1999, Bruin studied the Diophantine equation $x^2 \pm y^4 = \pm z^6$ and $x^2 + y^8 = z^3$ [2]. In 2004, Bennett found the solution of the Diophantine equation $x^{2n} + y^{2n} = z^5$ [3]. In 2014, Abdelalim and Dyani searched the Diophantine equation $x^2 + 3y^2 = z^2$ have integer solutions (x, y, z) is

Received: May 09, 2017

Revised: August 21, 2017

Accepted: September 22, 2017

$\left(\pm\frac{3y_1^2-y_2^2}{2},\pm y_1 y_2,\pm\frac{3y_1^2+y_2^2}{2}\right)$ if y is odd, and have integer solutions (x,y,z) is $\left(\pm\left(3y_1^2-y_2^2\right),\pm 2y_1 y_2,\pm\left(3y_1^2+y_2^2\right)\right)$ if y is even [4]. In 2015, Abdelalim and Diany characterized the solutions of the Diophantine equation $x^2+y^2=2z^2$ [5].

Thus, this paper aims to study the Diophantine equation $x^2+ky^2=z^2$ where k,x,y and z are integers.

2. Materials and Experiment

Proposition 1: The Diophantine equation $x^2+ky^2=z^2$ has no integer solution where $k=4m+2,m,x,y$ and z are integers with y is odd.

Proof:

Suppose that (x,y,z) is a solution of the Diophantine equation $x^2+(4m+2)y^2=z^2$ with y is odd. Since $y^2\equiv 1 \pmod 4$ then $(4m+2)y^2\equiv 2 \pmod 4$, these consider into 2 cases as follow:

Case 1: If x is odd then z is odd.
Thus, $x^2\equiv 1 \pmod 4$ then
$x^2+(4m+2)y^2\equiv 3 \pmod 4$. This is a contradiction with $z^2\equiv 1 \pmod 4$.

Case 2: If x is even then z is even.
Thus, $x^2\equiv 0 \pmod 4$ then
$x^2+(4m+2)y^2\equiv 2 \pmod 4$. This is a contradiction with $z^2\equiv 0 \pmod 4$.

Therefore, by case 1 and 2, the Diophantine equation $x^2+ky^2=z^2$ has no integer solution where $k=4m+2,m,x,y$ and z are integers with y is odd.

Proposition 2: The Diophantine equation $x^2+ky^2=z^2$ has the solutions in the form $(x,y,z)=\left(\pm(ka-b),\pm 2\sqrt{ab},\pm(ka+b)\right)$ where $k=4m+2,m,x,y$ and z are integers with y is even.

Proof:

Let (x,y,z) be a solution of the Diophantine equation $x^2+ky^2=z^2$ where $k=4m+2$ and y is even.

We have,

$$x^2+ky^2=z^2$$
$$ky^2=z^2-x^2$$
$$ky^2=(|z|-|x|)(|z|+|x|).$$

Since y is even, so that $2\mid(|z|-|x|)(|z|+|x|)$ hence, $2\mid(|z|-|x|)$ or $2\mid(|z|+|x|)$, we have x,z are even or x,z are odd, then $|z|-|x|$ and $|z|+|x|$ are even, we have $\frac{|z|-|x|}{2},\frac{|z|+|x|}{2}$ are integers. It follows that,

$$\frac{ky^2}{4}=\frac{(|z|-|x|)(|z|+|x|)}{4}$$
$$k\left(\frac{y}{2}\right)^2=\frac{|z|-|x|}{2}\frac{|z|+|x|}{2}.$$

Hence, $k\left|\frac{|z|-|x|}{2}\frac{|z|+|x|}{2}\right.$ then $k\left|\frac{|z|-|x|}{2}\right.$ or $k\left|\frac{|z|+|x|}{2}\right.$, these consider into 2 cases as follow:

Case 1: If $k\left|\frac{|z|-|x|}{2}\right.$ so that, there exists an integer a such that $ka=\frac{|z|-|x|}{2}$ and let $b=\frac{|z|+|x|}{2}$. It follows that,

$$k\left(\frac{y}{2}\right)^2 = (ka)b$$

$$\left(\frac{y}{2}\right)^2 = ab$$

$$\frac{y}{2} = \pm\sqrt{ab}$$ where ab is a square number.

$$y = \pm 2\sqrt{ab}$$

Since $ka = \frac{|z|-|x|}{2}$ and $b = \frac{|z|+|x|}{2}$ then $|x| = -(ka-b)$ and $|z| = ka+b$ that is, $x = \pm(ka-b)$ and $z = \pm(ka+b)$.

Case 2: If $k\left|\frac{|z|+|x|}{2}\right.$ so that, there exists an integer a such that $ka = \frac{|z|+|x|}{2}$ and let $b = \frac{|z|-|x|}{2}$. It follows that,

$$k\left(\frac{y}{2}\right)^2 = b(ka)$$

$$\left(\frac{y}{2}\right)^2 = ab$$

$$\frac{y}{2} = \pm\sqrt{ab}$$ where ab is a square number.

$$y = \pm 2\sqrt{ab}$$

Since $ka = \frac{|z|+|x|}{2}$ and $b = \frac{|z|-|x|}{2}$ then $|x| = ka-b$ and $|z| = ka+b$ that is, $x = \pm(ka-b)$ and $z = \pm(ka+b)$.

Therefore, by case 1 and 2, the solutions of this equation are in the form

$$(x,y,z) = \left(\pm(ka-b), \pm 2\sqrt{ab}, \pm(ka+b)\right)$$

where ab is a square number.

3. Results and Discussion

Theorem 3: The Diophantine equation $x^2 + ky^2 = z^2$ has the solutions in the form $(x,y,\mathrm{z}) = \left(\pm\frac{ka-b}{2}, \pm\sqrt{ab}, \pm\frac{ka+b}{2}\right)$ where k is odd, x, y and z are integers with y is odd and ab is an odd square number.

Proof:

Let (x,y,z) be a solution of the Diophantine equation $x^2 + ky^2 = z^2$ with y is odd.

We have,

$$x^2 + ky^2 = z^2$$

$$ky^2 = z^2 - x^2$$

$$ky^2 = (|z|-|x|)(|z|+|x|).$$

Hence, $k\,|\,(|z|-|x|)(|z|+|x|)$ then $k\,|\,(|z|-|x|)$ or $k\,|\,(|z|+|x|)$, these consider into 2 cases as follow:

Case 1: If $k\,|\,(|z|-|x|)$ so that, there exists an integer a such that $ka = |z|-|x|$ and let $b = |z|+|x|$. It follows that,

$$ky^2 = (ka)b$$

$$y^2 = ab$$

$$y = \pm\sqrt{ab}$$ where ab is an odd square number.

Since $ka = |z|-|x|$ and $b = |z|+|x|$ then $|x| = -\frac{ka-b}{2}$ and $|z| = \frac{ka+b}{2}$ that is, $x = \pm\frac{ka-b}{2}$ and $z = \pm\frac{ka+b}{2}$.

Case 2: If $k\,|\,(|z|+|x|)$ so that, there exists an integer a such that $ka = |z|+|x|$ and let $b = |z|-|x|$. It follows that,

$$ky^2 = b(ka)$$
$$y^2 = ab$$
$$y = \pm\sqrt{ab} \text{ where } ab \text{ is an odd square number.}$$

Since $ka = |z| + |x|$ and $b = |z| - |x|$ then $|x| = \frac{ka-b}{2}$ and $|z| = \frac{ka+b}{2}$ that is, $x = \pm\frac{ka-b}{2}$ and $z = \pm\frac{ka+b}{2}$.

Therefore, by case 1 and 2, the solutions of this equation are in the form

$$(x, y, z) = \left(\pm\frac{ka-b}{2}, \pm\sqrt{ab}, \pm\frac{ka+b}{2}\right)$$

where ab is an odd square number.

Theorem 4: The Diophantine equation $x^2 + ky^2 = z^2$ has the solutions in the form $(x, y, z) = \left(\pm(ka-b), \pm 2\sqrt{ab}, \pm(ka+b)\right)$ where k is odd, x, y and z are integers with y is even and ab is a square number.

Proof:

Let (x, y, z) be a solution of the Diophantine equation $x^2 + ky^2 = z^2$ with y is even and ab is a square number. We have,

$$x^2 + ky^2 = z^2$$
$$ky^2 = z^2 - x^2$$
$$ky^2 = (|z| - |x|)(|z| + |x|).$$

Since y is even, so that $2 \mid (|z| - |x|)(|z| + |x|)$ hence, $2 \mid (|z| - |x|)$ or $2 \mid (|z| + |x|)$, we have x, z are even or x, z are odd, then $|z| - |x|$ and $|z| + |x|$ are even, we have $\frac{|z|-|x|}{2}, \frac{|z|+|x|}{2}$ are integers. It follows that,

$$\frac{ky^2}{4} = \frac{(|z|-|x|)(|z|+|x|)}{4}$$

$$k\left(\frac{y}{2}\right)^2 = \frac{|z|-|x|}{2}\frac{|z|+|x|}{2}.$$

Hence, $k \,\Big|\, \frac{|z|-|x|}{2}\frac{|z|+|x|}{2}$ then $k \,\Big|\, \frac{|z|-|x|}{2}$ or $k \,\Big|\, \frac{|z|+|x|}{2}$, these consider into 2 cases as follow:

Case 1: If $k \,\Big|\, \frac{|z|-|x|}{2}$ so that, there exists an integer a such that $ka = \frac{|z|-|x|}{2}$ and let $b = \frac{|z|+|x|}{2}$. It follows that,

$$k\left(\frac{y}{2}\right)^2 = (ka)b$$
$$\left(\frac{y}{2}\right)^2 = ab$$
$$\frac{y}{2} = \pm\sqrt{ab} \text{ where } ab \text{ is a square number.}$$
$$y = \pm 2\sqrt{ab}$$

Since $ka = \frac{|z|-|x|}{2}$ and $b = \frac{|z|+|x|}{2}$ then $|x| = -(ka-b)$ and $|z| = ka+b$ that is, $x = \pm(ka-b)$ and $z = \pm(ka+b)$.

Case 2: If $k \,\Big|\, \frac{|z|+|x|}{2}$ so that, there exists an integer a such that $ka = \frac{|z|+|x|}{2}$ and let $b = \frac{|z|-|x|}{2}$. It follows that,

$$k\left(\frac{y}{2}\right)^2 = b(ka)$$
$$\left(\frac{y}{2}\right)^2 = ab$$
$$\frac{y}{2} = \pm\sqrt{ab} \text{ where } ab \text{ is a square number.}$$
$$y = \pm 2\sqrt{ab}$$

Since $ka=\frac{|z|+|x|}{2}$ and $b=\frac{|z|-|x|}{2}$ then $|x|=ka-b$ and $|z|=ka+b$ that is, $x=\pm(ka-b)$ and $z=\pm(ka+b)$.

Therefore, by case 1 and 2, the solutions of this equation are in the form

$$(x,y,z)=\left(\pm\frac{ka-b}{2},\pm\sqrt{ab},\pm\frac{ka+b}{2}\right)$$

where ab is a square number.

Theorem 5: The Diophantine equation $x^2+ky^2=z^2$ has the solutions in the form

$$(x,y,z)=\left(\pm\left(\frac{k}{4}a-b\right),\pm\sqrt{ab},\pm\left(\frac{k}{4}a+b\right)\right)$$

where $k=4m$, m,x,y and z are integers and ab is a square number.

Proof:

Let (x,y,z) be a solution of the Diophantine equation $x^2+ky^2=z^2$. We have,

$$x^2+ky^2=z^2$$
$$ky^2=z^2-x^2$$
$$ky^2=(|z|-|x|)(|z|+|x|)$$
$$4my^2=(|z|-|x|)(|z|+|x|).$$

So that $4\mid(|z|-|x|)(|z|+|x|)$ hence, $4\mid(|z|-|x|)$ or $4\mid(|z|+|x|)$, we have x,z are even or x,z are odd, then $|z|-|x|$ and $|z|+|x|$ are even, we have $\frac{|z|-|x|}{2},\frac{|z|+|x|}{2}$ are integers. It follows that,

$$\frac{4my^2}{4}=\frac{(|z|-|x|)(|z|+|x|)}{4}$$
$$my^2=\frac{|z|-|x|}{2}\frac{|z|+|x|}{2}.$$

Hence, $m\mid\frac{|z|-|x|}{2}\frac{|z|+|x|}{2}$ then $m\mid\frac{|z|-|x|}{2}$ or $m\mid\frac{|z|+|x|}{2}$, these consider into 2 cases as follow:

Case 1: If $m\mid\frac{|z|-|x|}{2}$ so that, there exists an integer a such that $ma=\frac{|z|-|x|}{2}$ and let $b=\frac{|z|+|x|}{2}$. It follows that,

$$my^2=(ma)b$$
$$y^2=ab$$
$$y=\pm\sqrt{ab} \text{ where } ab \text{ is a square number.}$$

Since $ma=\frac{|z|-|x|}{2}$ and $b=\frac{|z|+|x|}{2}$ then $|x|=-(ma-b)$ and $|z|=ma+b$ that is, $x=\pm\left(\frac{k}{4}a-b\right)$ and $z=\pm\left(\frac{k}{4}a+b\right)$.

Case 2: If $m\mid\frac{|z|+|x|}{2}$ so that, there exists an integer a such that $ma=\frac{|z|+|x|}{2}$ and let $b=\frac{|z|-|x|}{2}$. It follows that,

$$my^2=b(ma)$$
$$y^2=ab$$
$$y=\pm\sqrt{ab} \text{ where } ab \text{ is a square number.}$$

Since $ma=\frac{|z|+|x|}{2}$ and $b=\frac{|z|-|x|}{2}$ then $|x|=ma-b$ and $|z|=ma+b$ that is, $x=\pm\left(\frac{k}{4}a-b\right)$ and $z=\pm\left(\frac{k}{4}a+b\right)$.

Therefore, by case 1 and 2, the solutions of this equation are in the form

$$(x,y,z)=\left(\pm\left(\frac{k}{4}a-b\right),\pm\sqrt{ab},\pm\left(\frac{k}{4}a+b\right)\right)$$

where ab is a square number.

Example find all positive integer solutions of the Diophantine equation $x^2+75=z^2$.

Consider $x^2+75=z^2$ then $x^2+3(5)^2=z^2$

By theorem 3. So that $y=\sqrt{ab}=5$ then

$a=1,b=25$ so that $(x,z)=(11,14)$

or $a=5,b=5$ so that $(x,z)=(5,10)$

or $a=25,b=1$ so that $(x,z)=(37,38)$

This theorem have three solutions.

By theorem 2.2 [4]. So that $y=y_1y_2=5$ then

$y_1=1,y_2=5$ or $y_1=5,y_2=1$

This theorem have two solutions. (Incomplete)

4. Conclusions

It can be seen that this paper shown the solution of the Diophantine equation $x^2+ky^2=z^2$ has no integer solution where $k=4m+2,m$ is an integer, y is odd and have integer solutions

$$(x,y,z)=\left(\pm(ka-b),\pm2\sqrt{ab},\pm(ka+b)\right)$$

where $k=4m+2,2m+1,$ ab is a square number, m,x,y and z are integers with y is even, we have seen that the solution of the Diophantine equation $x^2+ky^2=z^2$ is $(x,y,z)=\left(\pm\frac{ka-b}{2},\pm\sqrt{ab},\pm\frac{ka+b}{2}\right)$ where $k=2m+1,$ ab is an odd square number, m,x,y and z are integers with y is odd. Also, and

$$(x,y,z)=\left(\pm\left(\frac{k}{4}a-b\right),\pm\sqrt{ab},\pm\left(\frac{k}{4}a+b\right)\right)$$

where $k=4m,ab$ is a square number, m,x,y and z are integers.

5. Acknowledgement

I highly appreciate the editors and referees for his helpful comments and suggestions to make this study are more valuable.

6. References

[1] A. Wiles. Modula elliptic curves and Fermat last theorem. *Annals of Mathematics*. **142** (1995): 443-551.

[2] N. Bruin. The Diophantine equation $x^2\pm y^4=\pm z^6$ and $x^2+y^8=z^3$. *Compositio Mathematica.* **118** (1999): 305-321.

[3] M.A. Bennett. The equation $x^{2n}+y^{2n}=z^5$. *Journal de Thé orie des Nombres de Bordeaux*. **18** (2006): 315-321.

[4] S. Abdelalim, H. Dyani. The solution of the Diophantine equation $x^2+3y^2=z^2$. *International Journal of Algebra.* **8** (2014): 729-732.

[5] S. Abdelalim, H. Diany. Characterization of the solution of the Diophantine equation $x^2+y^2=2z^2$. *Gulf Journal of Mathematics.* **3** (2015): 1-4.