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On the Probability of Stopping time of the Target Equity Allocation Management Strategy

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Abstract

This research focuses on the probability of the target equity allocation management (TEAM) strategy which has been stopped at one time unit. We approximate the probability that the TEAM strategy will be stopped by considering its upper bound.

Keywords: TEAM strategy, Probability, upper bound, Laplace distribution

1. Introduction

Investment is an important tool in order to increase the value of investor's wealth. In finance, investor will hold financial assets of value with an expectation of favorite returns. Typically, there are two types of assets, i.e., risk-free and risky assets. For example, for the stock price of one share at time n (denoted by S_n), the rate of return of the stock price at time n is defined by

$$R_n = \frac{S_n - S_{n-1}}{S_{n-1}}$$

Since the stock price S_n is uncertain, the rate of return R_n is also uncertain. Thus stocks are risky assets. On the other hand, if the future value of the asset is known some of risk-free assets include the cash fund, government bond, and so on, then the asset is risk-free and its rate of return is called the interest rate.

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Buy-and-hold (BH) is an investment strategy that an investor buys assets and holds them for a long period of time without the volatility of stock prices on the market in a short time. Thus we divide an initial wealth (W_0) into two parts, i.e., $W_0 = V_0 + U_0$, where V_0 is the part invested in a risk-free asset and U_0 is the part invested in a risky asset. It follows that

$$U_n = U_0(1+R_1)(1+R_2)\ldots(1+R_n)$$

is risky asset at time n and $V_n = V_0(1+c)^n$ is risk-free asset at time n, where R_n is the rate of return of the risky asset at time n and c is a constant interest rate. The total wealth at time n is given by

$$W_n = V_n + U_n = V_0(1+c)^n + U_0\prod_{i=1}^{n}(1+R_i)$$

Gerth [1] introduced the *target equity allocation management* (TEAM) strategy. For the TEAM strategy, we have an initial portfolio value W_0 with initial risk-free asset V_0 and initial risky asset U_0, (i.e., $W_0 = V_0 + U_0, V_0 > 0$ and $U_0 > 0$).

The TEAM strategy incorporates sweep feature that makes $U_i = U_{i-1}(1+c)$ for all $i = 1,2,\ldots,n$. From Robert, we can be seen that the risk-free asset at time n is

$$V_0 = W_0 - U_0,$$

$$V_n = V_0(1+c)^n + U_0(1+c)^{n-1}\sum_{i=1}^{n}(R_i - c) \qquad (1)$$

where R_i is the rate of return at time i. Throughout this paper, we assume that the rate of return process $\{R_n : n = 1,2,3,\ldots,N\}$ is independent and identically distributed (i.i.d.) random variables where N is natural number.

Robert [2] compared the risk between BH and TEAM strategies by using variance and coefficient of variation. He found that the TEAM strategy involved lower risk than the BH strategy when the BH and TEAM strategies have the same initial allocations.

In this paper, we are interested in the probability of TEAM strategy that a risk-free asset will be depleted at the first time, but it cannot be obtained directly thus we will approximate the probability that the TEAM strategy will stop by considering its upper bound.

2. Materials and Results

We introduce the stopping time of TEAM strategy with respect to a sequence of random variables $V_1, V_2, \ldots, V_n$ as a random variable τ_{v_0} with values in $\{1,2,3,\ldots\}$ given by

$$\tau_{v_0} = \inf\{k : V_k < 0, v_0 = V_0\}.$$

Therefore, the probability that the TEAM strategy will stop at one of the time $1,2,3,\ldots,n$ which is denoted by

$$\Phi_n(v_0) = \Pr(\tau_{v_0} \le n) \qquad (2)$$

$$= \Pr(V_k < 0\ \exists k = 0,1,2,\ldots,n \mid v_0 = V_0). \qquad (3)$$

Observe that $\Phi_0(v_0) = 0$ for all $v_0 > 0$ and $\Phi_0(v_0) = 1$ when $v_0 < 0$. Since (2) and (3) are equivalent, we can focus on (3) only.

Theorem 1. If $V_n = V_0(1+c)^n + U_0(1+c)^{n-1}\sum_{i=1}^{n}(R_i - c)$ for some positive integer n, then

$$\Pr(V_n < 0) = \Pr\left(\sum_{i=1}^{n} R_i < nc - \frac{V_0}{W_0 - V_0}(1+c)\right).$$

Proof. Let $\omega_0 \in \{\omega: V_n(\omega) < 0\}$

$$V_0(1+c)^n + U_0(1+c)^{n-1}\sum_{i=1}^{n}(R_i(\omega_0) - c) < 0$$

$$U_0(1+c)^{n-1}\sum_{i=1}^{n}(R_i(\omega_0) - c) < -V_0(1+c)^n$$

$$\sum_{i=1}^{n}(R_i(\omega_0) - c) < -\frac{V_0}{U_0}(1+c).$$

Thus,

$$\sum_{i=1}^{n} R_i(\omega_0) < nc - \frac{V_0}{W_0 - V_0}(1+c).$$

In other hand,

$$\omega_0 \in \left\{\omega: \textstyle\sum_{i=1}^{n} R_i(\omega_0) < nc - \frac{V_0}{W_0 - V_0}(1+c)\right\}.$$

Hence,

$$\{\omega: V_n(\omega) < 0\} \subset \left\{\omega: \textstyle\sum_{i=1}^{n} R_i(\omega_0) < nc - \frac{V_0}{W_0 - V_0}(1+c)\right\}.$$

Conversely, it can be seen easily from the assumption of V_n that

$$\{\omega: V_n(\omega) < 0\} \supset \left\{\omega: \textstyle\sum_{i=1}^{n} R_i(\omega_0) < nc - \frac{V_0}{W_0 - V_0}(1+c)\right\}.$$

Hence,

$$\{\omega: V_n(\omega) < 0\} = \left\{\omega: \textstyle\sum_{i=1}^{n} R_i(\omega_0) < nc - \frac{V_0}{W_0 - V_0}(1+c)\right\}.$$

That is,

$$\Pr(V_n < 0) = \Pr\left(\sum_{i=1}^{n} R_i < nc - \frac{V_0}{W_0 - V_0}(1+c)\right).$$

This completes the proof.

In order to circumvent difficulty occurring when calculating the probability that the TEAM strategy will stop by (3) directly, we will approximate such probability by considering its upper bound. For convenience, we let

$$\Phi_n(v_0) = \Psi_n(h(V_0))$$

where $h(v) = -\frac{v}{W_0 - v}(1+c)$ By using Theorem 1, we obtain that

$$\Psi_n(h(V_0)) = \Pr\left(\sum_{i=1}^{k} R_i < kc + h(V_0)\ \exists k = 1,2,\ldots,n\right)$$

In order to prove the Theorem 3, we shall use the equivalent definition of $\Psi_n(h(V_0))$ which is given by

$$\Psi_n(h(V_0)) = \Pr\left(\max_{1\le k\le n}\left(kc - \sum_{i=1}^{k} R_i\right) > -h(V_0)\right) \quad (4)$$

where $V_0 \in (0, W_0)$.

Theorem 2. Given $N \in \{1,2,3,\ldots\}$, $c > 0$ and $h(V_0) \le 0$, if $\{R_i: i \ge 1\}$ is an i.i.d. return process, then the probability that the TEAM strategy will stop at one of the times $1,2,3,\ldots,N$ satisfies the following the equation

$$\Psi_N(h(V_0)) = \Psi_1(h(V_0)) + \int_{h(V_0)+c}^{\infty} \Psi_{N-1}(h(V_0) + c - x)\ dF_{R_1}(x) \quad (5)$$

where $\Psi_0(h(V_0)) = 0$.

Proof. We will prove (5) by mathematical induction. We start with $n = 1$. Since $\Psi_0(h(V_0)) = 0$ for all $V_0 \ge 0$, we have

$$\int_{h(V_0)+C}^{\infty} \Psi_{N-1}(h(V_0) + c - x)\ dF_{R_1}(x) = 0 \quad (6)$$

This proves (6) for $n = 1$. Next we assume that (5) holds for $n = k \ge 2$. We have

$$\begin{aligned}
\Psi_{k+1}(h(V_0)) &= \Pr\left(\max_{1\le n\le k+1}\left(nc - \sum_{i=1}^{n} R_i\right) > -h(V_0)\right) \\
&= \Pr(c - R_1 > -h(V_0)) \\
&\quad + \Pr\left(\max_{2\le n\le k+1}\left(nc - R_1 - \sum_{i=2}^{n} R_i\right) > -h(V_0), R_1 > h(V_0 + c)\right) \\
&= \Psi_1(h(V_0)) \\
&\quad + \int_{h(V_0)+c}^{\infty} \Pr\left(\max_{2\le n\le k+1}\left(nc - x - \sum_{i=2}^{n} R_i\right) > -h(V_0)\right) dF_{R_1}(x) \\
&= \Psi_1(h(V_0)) \\
&\quad + \int_{h(V_0)+c}^{\infty} \Pr\left(\max_{2\le n\le k+1}\left(c(n-1) - \sum_{i=2}^{n} R_i\right) > -(h(V_0) + c) + x\right) dF_{R_1}(x) \\
&= \Psi_1(h(V_0)) \\
&\quad + \int_{h(V_0)+c}^{\infty} \Pr\left(\max_{1\le n\le k}\left(cn - \sum_{i=2}^{n} R_i\right) > -(h(V_0) + c) + x\right) dF_{R_1}(x) \\
&= \Psi_1(h(V_0)) + \int_{h(V_0)+c}^{\infty} \Psi_k(h(V_0) + c - x)\, dF_{R_1}(x)
\end{aligned}$$

which proves (5) for $n = k + 1$, and concludes the proof.

Theorem 3. Given $N \in \{1,2,3,\dots\}, c > 0$ be an interest rate, and $\{R_i : i \geq 1\}$ be i.i.d. and let $E[R_1] = \mu > c$. If $\delta > 0$ satisfies the following condition

$$E\left[e^{-\delta R_1}\right] \leq e^{-\delta c} \tag{7}$$

then

$$\Phi_N(v_0) \leq e^{\delta h(v_0)} \tag{8}$$

where $h(v_0) = -\frac{v_0}{W_0 - v_0}(1 + c)$

Proof. Assume that $E[R_1] \geq c$. We shall prove this theorem by induction. We start with $N = 1$. By Markov's inequality, we obtain

$$\begin{aligned} \Phi_1(v_0) &= \Psi_1(h(v_0)) \\ &= \Pr(R_1 \leq h(v_0) + c) \\ &= \Pr(e^{-\delta R_1} \geq e^{-\delta(h(v_0)+c)}) \\ &= \Pr(e^{-\delta(R_1 - h(v_0) - C)} \geq 1) \\ &\leq E\left[e^{-\delta(R_1 - h(v_0) - c)}\right] \\ &= e^{\delta(h(v_0)+c)} E\left[e^{-\delta R_1}\right] \\ &\leq e^{\delta(h(v_0)+c)} e^{-\delta R_1} \\ &\leq e^{\delta h(v_0)} \end{aligned} \tag{9}$$

Assume that (9) holds for some $N = k \geq 2$. By Theorem 2, we have

$$\begin{aligned} \Phi_{k+1}(v_0) &= \Psi_{k+1}(h(V_0)) \\ &= \Psi_1(h(V_0)) \\ &+ \int_{h(V_0)+c}^{\infty} \Psi_k(h(V_0) + c - x)\ dF_{R_1}(x). \end{aligned} \tag{10}$$

Firstly, we consider the second term on the right-hand side of (10). By using the inductive assumption, we have

$$\begin{aligned} &\int_{h(V_0)+c}^{\infty} \Psi_k(h(V_0) + c - x)\ dF_{R_1}(x) \\ &\leq \int_{h(V_0)+c}^{\infty} e^{\delta(h(v_0)+c-x)}\ dF_{R_1}(x) \end{aligned} \tag{11}$$

Now, we consider the first term on the right-hand side of (10). By using Markov's inequality, we obtain

$$\begin{aligned} &\Pr(R_1 \leq h(_0) + c) \\ &= \Pr\left(-R_1 \geq -(h(v_0) + c)\right) \\ &= \Pr\left(e^{-\delta R_1} \mathbb{I}_{(-(h(v_0)+c),\infty)}(-R_1) \geq e^{-\delta(h(v_0)+c)}\right) \\ &= \Pr\left(e^{-\delta R_1} \mathbb{I}_{(-\infty,h(v_0)+c)}(R_1) \geq e^{-\delta(h(v_0)+c)}\right) \\ &\leq \frac{E\left[e^{-\delta R_1} \mathbb{I}_{(-\infty,h(V_0)+c)}(R_1)\right]}{e^{-\delta(h(v_0)+c)}} \\ &= \int_{-\infty}^{h(V_0)+c} e^{\delta(h(v_0)+c-x)}\ dF_{R_1}(x). \end{aligned} \tag{12}$$

Therefore, we have

$$\begin{aligned} \Phi_{k+1}(v_0) &= \Psi_{k+1}(h(v_0)) \\ &\leq \int_{-\infty}^{h(V_0)+c} e^{\delta(h(v_0)+c-x)}\ dF_{R_1}(x) \\ &+ \int_{h(V_0)+c}^{\infty} e^{\delta(h(v_0)+c-x)}\ dF_{R_1}(x) \\ &= \int_{-\infty}^{\infty} e^{\delta(h(v_0)+c-x)}\ dF_{R_1}(x) \\ &= \frac{e^{\delta h(v_0)}}{e^{-\delta c}} \int_{-\infty}^{\infty} e^{-\delta x}\ dF_{R_1}(x) \\ &\leq \frac{e^{\delta h(v_0)}}{e^{-\delta c}}\ E\left[e^{\delta R_1}\right] \\ &\leq e^{\delta h(v_0)}. \end{aligned} \tag{13}$$

This proves (8) for $N = k + 1$, and completes the proof.

Next, we shall describe sub-adjustment coefficient as follows: variable. If there exists $h_0 > 0$ such that

$$E[e^{h_0 X}] \leq e^{h_0 c} \tag{14}$$

then h_0 is called a sub-adjustment coefficient of (c, X). In particular, if (14) is an equality, then h_0 is called an adjustment coefficient of (c, X).

Remark 4. Under the assumption of Theorem 3, we obtain

that

$$\Phi_n(v_0) \leq e^{\delta h(v_0)} \tag{15}$$

where n is positive integer and $\delta > 0$ is a sub-adjustment coefficient of (c, R_1).

3. Application

We use 235 data of monthly gold prices in USD from London market since January 1995 transform gold prices to rate of return $(R_i, i = 1,2,3,\dots,235)$, as illustrated in Figure 1. The goodness of fit for the rate of return is done by Kolmogorov-Smirnov test, and we accept the Laplace distribution of all with p-value 0.04125 with $\alpha = 0.02$. The rate of return $\{R_i\colon i = 1,2,3,\dots,n\}$ is assumed to be independent and identically distributed. Using the maximum likelihood estimation (MLE) method, we find that the estimated parameter of the parameters is $(\mu, b) = (0.0059, 0.0269)$ for Laplace distribution. The simulation result is carried out with 100,000 paths for the probability of TEAM strategy to stop at one time unit as cited in the equation (3). We assume that $R_i \sim Laplace(0.0059, 0.0269)$ for all n. The simulation results of the probability of TEAM strategy with initial risk free are shown in Figure 2.

Figure 1 shows the approximation of the probability of TEAM strategy with Laplace distribution and Figure 2 shows the upper bound of the probability of TEAM strategy from Example 1.

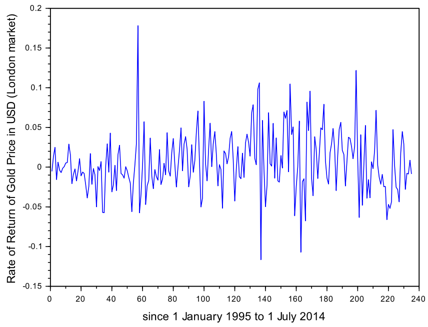

Figure 1 Rate of return of gold prices

Example 1. Let $\{R_n : n \in N\}$ be an i.i.d. process such that R_1 has a Laplace distribution with $E[R_1] = 0.5, c = 0.1$ and $b = 0.8$.

Consider

$$f(\delta) \triangleq E\left[e^{-\delta R_1}\right] - e^{-c\delta}$$

$$= \frac{e^{-E[R_1]\delta}}{1 - b^2\delta^2} - e^{-c\delta}$$

Since $f(0.0001) = -0.0269 < 0$ and $f(0.9) = 0.41 > 0$, by Bolzano's Theorem, there exists $\delta \in (0,1)$ such that $f(\delta) = 0$, i.e.,

$$E[e^{-\delta R_1}] = e^{-\delta c}, \tag{16}$$

We find that $\delta = 0.49752$ satisfies (16). By Theorem 3, we have

$$\Phi_N\big(h(v_0)\big) \leq e^{0.49752 h(v_0)}.$$

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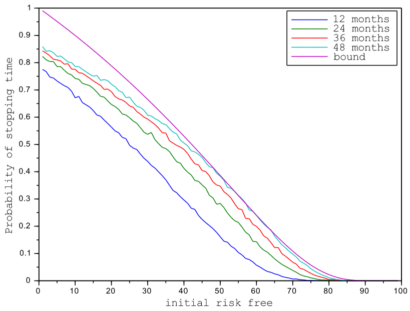

Figure 2 The results of Example 1 and simulation which is the relation between the initial risk-free asset (V_0) and probability $\Phi_n(V_0)$

4. References

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