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The proportional intermingling of two different exponential distribution terms

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บทคัดย่อ

บทความนี้นำเสนอการแจกแจงแบบใหม่ที่เรียกว่า การแจกแจงผลต่างเอกซ์โพเนนเชียลแบบสัดส่วน ซึ่งมีฟังก์ชันความหนาแน่นของความน่าจะเป็นคือ

$$f_X(x;\alpha,\lambda)=\frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}\right),\qquad x\geq 0,$$

เมื่อ $\lambda>0$ และ $\alpha\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$ โดยศึกษาสมบัติบางประการของการแจกแจงผลต่างเอกซ์โพเนนเชียลแบบสัดส่วน ได้แก่ ค่าคาดหวัง ความแปรปรวน ฟังก์ชันก่อกำเนิดโมเมนต์ และลิมิตของการแจกแจงที่จุด $\alpha=0,\frac{1}{2}$ และ 1

คำสำคัญ: การแจกแจงเอกซ์โพเนนเชียลและการแจกแจงผลต่างเอกซ์โพเนนเชียลแบบสัดส่วน

Abstract

In this paper we introduce a new distribution, called proportional exponential difference (PED) distribution, such that its probability density function is in the form:

$$f_X(x;\alpha,\lambda)=\frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}\right),\qquad x\geq 0,$$

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where $\lambda > 0$ และ $\alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. We study some properties of PED distribution such as expected value, variance, moment generating function and its limit at $\alpha = 0, \frac{1}{2}$ and 1.

Keywords: exponential distribution, PED distribution.

1. Introduction

The positive random variable X is called an exponential distribution with parameter $\lambda > 0$, denoted by $X \sim \exp(\lambda)$, and its probability density function is in the form

$$f(x;\lambda) = \lambda e^{-\lambda x}, \qquad x \ge 0. \qquad (1)$$

Expected value and variance of the exponential distribution with parameter λ are obtained by $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$, respectively. In addition, its moment generating function is satisfied

$$\mathrm{M(t)} = \frac{\lambda}{\lambda - t}, \qquad \mathrm{t} < \lambda,$$

(these proof see in [3,4]).

The exponential distribution is widely applied for data modeling in several phenomena such as actuarial science, life time data and waiting time problems. Therefore, the extension of exponential distribution is an interested problem because it maybe not suitable for some models. Thus, many researchers are interested in the extension of exponential distribution for describing the interested models. For example, Gupta and Kundu [1,2] introduced an extension of the exponential distribution called the generalized exponential (GE) distribution and exponentiated exponential (EE) distribution which is clarified in term of the probability density function:

$$f_{GE}(x;\alpha,\lambda,\mu) = \tfrac{\alpha}{\lambda} e^{-\frac{(x-\mu)}{\lambda}} (1 - e^{-\frac{(x-\mu)}{\lambda}})^{\alpha-1}, \quad x > \mu,$$

and

$$f_{EE}(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}, \ \ x > 0,$$

where $\alpha > 0$ and $\lambda > 0$.

In 2013, Oguntunde, et al.[6] described the sum of two independent exponentially distributed random variables, $Z = X + Y$ where $X \sim \exp(\lambda_1)$ and $Y \sim \exp(\lambda_2)$. The probability density function is obtained by

$$f_Z(z) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 z} - e^{-\lambda_2 z}), \qquad \mathrm{z} \ge 0.$$

Our goal is to introduce a new distribution which is constructed by the difference of exponential function and study its behaviors.

2. Definition and Notation

Firstly, we shall present the motivation of our research. A new distribution is constructed by the difference of exponential functions as following form

$$h(x) = (1-\alpha)e^{-\frac{\lambda x}{1-\alpha}} - \alpha e^{-\frac{\lambda x}{\alpha}},$$

where $\lambda > 0$ and $\alpha \in (0,1) - \{\frac{1}{2}\}$. We obtain that

$$\int_0^\infty h(x)dx = \tfrac{1-2\alpha}{\lambda}.$$

So that

$$\int_0^\infty \tfrac{\lambda}{1-2\alpha} h(x)dx = 1.$$

Next, we shall show $\frac{\lambda}{1-2\alpha} h(x) \ge 0$ for all $x \ge 0$.

Case 1: $0 < \alpha < \frac{1}{2}$. We have $1-\alpha > \frac{1}{2} > \alpha$ and we get $1 - 2\alpha > 0$. Since the negative exponential function is decreasing, so we obtain that $e^{-\frac{\lambda x}{1-\alpha}} \geq e^{-\frac{\lambda x}{\alpha}}$. This implies that $(1-\alpha)e^{-\frac{\lambda x}{1-\alpha}} - \alpha e^{-\frac{\lambda x}{\alpha}} > 0$. Therefore, $\frac{\lambda}{1-2\alpha} h(x) > 0.$

Case 2: $\frac{1}{2} < \alpha < 1$. We have $1-\alpha < \frac{1}{2} < \alpha$, we get $1-2\alpha < 0$. This implies that $(1-\alpha)e^{-\frac{\lambda x}{1-\alpha}} - \alpha e^{-\frac{\lambda x}{\alpha}} < 0$. Therefore, $\frac{\lambda}{1-2\alpha} h(x) > 0.$

This leads to the new distribution defined as follows:

Definition 1 Let $\lambda > 0$ and $\alpha \in (0,1) - \left\{\frac{1}{2}\right\}$. A positive random variable X is called a proportional exponential difference (PED) distribution with proportional parameter α and scale parameter λ, denoted by $X \sim \text{PED}(\alpha, \lambda)$, if its probability density function (pdf) satisfies

$$f_X(x;\alpha,\lambda) = \frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}} - \alpha e^{-\frac{\lambda x}{\alpha}}\right), \quad x \geq 0.$$

Theorem 1 Let $X \sim \text{PED}(\alpha, \lambda)$. Then the cumulative distribution function (cdf) of X is obtained by

$$F_X(x;\alpha,\lambda) = 1 - \frac{1}{1-2\alpha}\left((1-\alpha)^2 e^{-\frac{\lambda x}{1-\alpha}} - \alpha^2 e^{-\frac{\lambda x}{\alpha}}\right), \ x \geq 0.$$

Proof. Consider

$$\begin{aligned} F_X(x;\alpha,\lambda) &= \int_0^x f_X(x;\alpha,\lambda)dx \\ &= \int_0^x \frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}} - \alpha e^{-\frac{\lambda x}{\alpha}}\right)dx \\ &= \frac{1}{1-2\alpha}\left((1-2\alpha) - (1-\alpha)^2 e^{-\frac{\lambda x}{1-\alpha}} + \alpha^2 e^{-\frac{\lambda x}{\alpha}}\right) \\ &= 1 - \frac{1}{1-2\alpha}\left((1-\alpha)^2 e^{-\frac{\lambda x}{1-\alpha}} - \alpha^2 e^{-\frac{\lambda x}{\alpha}}\right). \end{aligned}$$

So we have done.

Fig. 1 and Fig. 2 show the pdf and cdf of PED on various values of parameters.

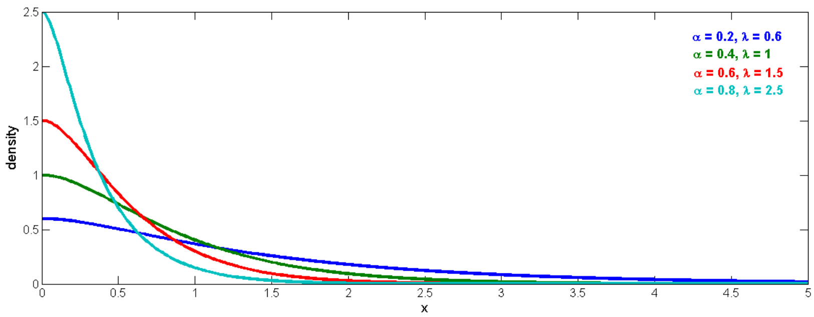

Fig. 1 The probability density function of the PED (0.2,0.6) (blue), PED (0.4,1) (green), PED (0.6,1.5) (red), PED (0.8,2.5) (sky blue).

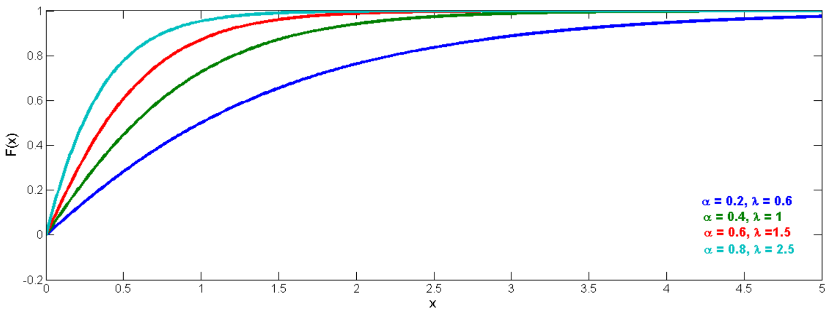

Fig. 2 The cumulative distribution function of the PED (0.2,0.6) (blue), PED (0.4,1) (green), PED (0.6,1.5) (red), PED (0.8,2.5) (sky blue).

3. Some Properties of PED distribution

In this section, we shall study the expected value, variance, moment generating function of PED distribution and limit properties of proportional parameter.

3.1 Moment generating function

Theorem 2 Let $X \sim \mathrm{PED}(\alpha,\lambda)$. Then the corresponding moment generating function of X is obtained by

$$M_X(t)=\frac{\lambda}{1-2\alpha}\left(\frac{(1-\alpha)^2}{\lambda-(1-\alpha)t}-\frac{\alpha^2}{\lambda-\alpha t}\right),$$

for all $t<\min\left\{\frac{\lambda}{1-\alpha},\frac{\lambda}{\alpha}\right\}$.

Proof. The result of Theorem 2 is derived as follows:

$$\begin{aligned}M_X(t)&=\int_0^\infty e^{tx}\left(\frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}\right)\right)dx\\&=\frac{\lambda}{1-2\alpha}\int_0^\infty(1-\alpha)e^{-\left(\frac{\lambda}{1-\alpha}-t\right)x}dx-\frac{\lambda}{1-2\alpha}\int_0^\infty\alpha e^{-\left(\frac{\lambda}{\alpha}-t\right)x}\,dx\\&=\frac{\lambda}{1-2\alpha}\left(\lim_{s\to\infty}\int_0^s(1-\alpha)e^{-\left(\frac{\lambda}{1-\alpha}-t\right)x}dx-\lim_{s\to\infty}\int_0^s\alpha e^{-\left(\frac{\lambda}{\alpha}-t\right)x}dx\right)\\&=\frac{\lambda}{1-2\alpha}\left(\lim_{s\to\infty}\left(-\frac{(1-\alpha)}{\frac{\lambda}{1-\alpha}-t}e^{-\left(\frac{\lambda}{1-\alpha}-t\right)x}\right)_{x=0}^{x=s}+\lim_{s\to\infty}\left(\frac{\alpha}{\frac{\lambda}{\alpha}-t}e^{-\left(\frac{\lambda}{\alpha}-t\right)x}\right)_{x=0}^{x=s}\right)\\&=\frac{\lambda}{1-2\alpha}\left(\frac{(1-\alpha)^2}{\lambda-(1-\alpha)t}-\frac{\alpha^2}{\lambda-\alpha t}\right).\end{aligned}$$

This completes the proof.

Theorem 3 Let $X \sim \mathrm{PED}(\alpha,\lambda)$. Then,

$$M_X^{(n)}(t)=\frac{n!\lambda}{1-2\alpha}\left(\frac{(1-\alpha)^{n+2}}{(\lambda-(1-\alpha)t)^{n+1}}-\frac{\alpha^{n+2}}{(\lambda-\alpha t)^{n+1}}\right),\qquad(2)$$

for all $t<\min\left\{\frac{\lambda}{1-\alpha},\frac{\lambda}{\alpha}\right\}$.

Proof. We will prove Theorem 3 by mathematical induction.

Consider $n=1$

$$M'_X(t)=\frac{\lambda}{1-2\alpha}\left(\frac{(1-\alpha)^3}{(\lambda-(1-\alpha)t)^2}-\frac{\alpha^3}{(\lambda-\alpha t)^2}\right).$$

Obviously, equation (2) holds for $n=1$.

Now assume that equation (2) is true for $n=k\ge 1$. Then,

$$\begin{aligned}M_X^{(n+1)}(t)&=\frac{d}{dt}M_X^{(n)}(t)\\&=\frac{n!\lambda}{1-2\alpha}\left((n+1)\frac{(1-\alpha)^{n+3}}{(\lambda-(1-\alpha)t)^{n+2}}-(n+1)\frac{\alpha^{n+3}}{(\lambda-\alpha t)^{n+2}}\right)\\&=\frac{(n+1)!\lambda}{1-2\alpha}\left(\frac{(1-\alpha)^{n+3}}{(\lambda-(1-\alpha)t)^{n+2}}-\frac{\alpha^{n+3}}{(\lambda-\alpha t)^{n+2}}\right),\end{aligned}$$

which proves equation (2) for $n=k+1$ and concludes the proof.

Corollary 4 If $X \sim \mathrm{PED}(\alpha,\lambda)$ and $Y \sim \exp(\lambda)$ then

$$M_X(t)=\frac{1}{1-2\alpha}\left((1-\alpha)^2M_Y((1-\alpha)t)-\alpha^2M_Y(\alpha t)\right),$$

and

$M_X^{(n)}(t)=\frac{1}{1-2\alpha}\left((1-\alpha)^2 M_Y^{(n)}((1-\alpha)t)-\alpha^2 M_Y^{(n)}(\alpha t)\right),$

for all $t<\min\left\{\frac{\lambda}{1-\alpha},\frac{\lambda}{\alpha}\right\}.$

3.2 Expected value and Variance

Theorem 5 The expected value of random variable $X\sim \text{PED}(\alpha,\lambda)$ is

$$E[X]=\frac{\alpha^2-\alpha+1}{\lambda}>0.$$

Proof. Since $E[X]=M'_X(0),$ we have

$$\begin{aligned}E[X]&=\frac{\lambda}{1-2\alpha}\left(\frac{(1-\alpha)^3}{\lambda^2}-\frac{\alpha^3}{\lambda^2}\right)\\&=\frac{\lambda}{1-2\alpha}\left(\frac{(1-2\alpha)(\alpha^2-\alpha+1)}{\lambda^2}\right)\\&=\frac{\alpha^2-\alpha+1}{\lambda}\\&=\frac{1}{\lambda}\left((\alpha-\tfrac{1}{2})^2+\tfrac{3}{4}\right)>0.\end{aligned}$$

Theorem 6 The variance of random variable $X\sim \text{PED}(\alpha,\lambda)$ is

$$Var(X)=\frac{2\alpha^5-5\alpha^4+5\alpha^2-4\alpha+1}{(1-2\alpha)\lambda^2}.$$

Proof. From $E[X^2]=M''_X(0),$ we have

$$\begin{aligned}E[X^2]&=\frac{2!\lambda}{1-2\alpha}\left(\frac{(1-\alpha)^4}{(\lambda-(1-\alpha)t)^3}-\frac{\alpha^4}{(\lambda-\alpha t)^3}\right)\\&=\frac{2(-4\alpha^3+6\alpha^2-4\alpha+1)}{(1-2\alpha)\lambda^2}.\end{aligned}$$

Thus

$$\begin{aligned}Var(X)&=E[X^2]-(E[X])^2\\&=\frac{2(-4\alpha^3+6\alpha^2-4\alpha+1)}{(1-2\alpha)\lambda^2}-\frac{(\alpha^2-\alpha+1)^2}{\lambda^2}\\&=\frac{2\alpha^5-5\alpha^4+5\alpha^2-4\alpha+1}{(1-2\alpha)\lambda^2}.\end{aligned}$$

3.3 Limit properties of proportional parameter

From the definition of PED distribution as mentioned in section 2, we found that the set of proportional parameter α is quite strange because the points $\alpha=0,\frac{1}{2}$ and 1 are not defined for our distribution. However, we can treat these points as removable point of continuity of function.

Theorem 7 Let $X\sim \text{PED}(\alpha,\lambda)$. Then

$$\lim_{\alpha\to 0^+} f_X(x;\alpha,\lambda)=\lim_{\alpha\to 1^-} f_X(x;\alpha,\lambda)=\lambda e^{-\lambda x},$$

for all $x\geq 0$.

Proof. Since $e^{-\frac{\lambda x}{\alpha}}\to 0$ as $\alpha\to 0^+,$ this leads to the following:

$$\begin{aligned}\lim_{\alpha\to 0^+} f_X(x;\alpha,\lambda)&=\lim_{\alpha\to 0^+}\frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}\right)\\&=\lambda e^{-\lambda x}.\end{aligned}$$

Similarly, since $e^{-\frac{\lambda x}{1-\alpha}}\to 0$ as $\alpha\to 1^-,$

$$\lim_{\alpha\to 1^-} f_X(x;\alpha,\lambda)=\lim_{\alpha\to 1^-}\frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}\right)=\lambda e^{-\lambda x}.$$

Theorem 8 Let $\lambda>0$ and f be a probability density function as mentioned in Section 2. For each $x\geq 0$. Then

$$\lim_{\alpha\to\frac{1}{2}} f(x;\alpha,\lambda)=\frac{1}{2}\left(2\lambda e^{-2\lambda x}+4\lambda^2 xe^{-2\lambda x}\right).$$

Proof. Let $x\geq 0$ be given.

Since $\lim_{\alpha\to\frac{1}{2}}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}\right)=0$ and $\lim_{\alpha\to\frac{1}{2}}1-2\alpha=0$, by using the L'Hospital's Rule (see [5]), we obtain that

$$\begin{aligned}\lim_{\alpha\to\frac{1}{2}} f(x;\alpha,\lambda)&=\lim_{\alpha\to\frac{1}{2}}\lambda\left(\frac{(1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}}{1-2\alpha}\right)\\&=\lim_{\alpha\to\frac{1}{2}}\lambda\left(\frac{(1-\alpha)\left(\frac{-\lambda x}{(1-\alpha)^2}\right)e^{-\frac{\lambda x}{1-\alpha}}-e^{-\frac{\lambda x}{1-\alpha}}}{-2}-\frac{\alpha\left(\frac{\lambda x}{\alpha^2}\right)e^{-\frac{\lambda x}{\alpha}}-e^{-\frac{\lambda x}{\alpha}}}{-2}\right)\\&=\lambda\left(\frac{(-2\lambda x-1-2\lambda x-1)e^{-2\lambda x}}{-2}\right)\\&=\lambda\left(\frac{(2+4\lambda x)e^{-2\lambda x}}{2}\right)\\&=\frac{1}{2}\left(2\lambda e^{-2\lambda x}+4\lambda^2 xe^{-2\lambda x}\right).\end{aligned}$$

This completes the proof.

From Theorem 7 and 8, we extend the set of proportional parameter of PED distribution from $(0,1)-\{\frac{1}{2}\}$ to $[0,1]$ as following form

$$f(x;\alpha,\lambda)=\begin{cases}\lambda e^{-\lambda x} & \text{if } \alpha=0 \text{ or } \alpha=1,\\ \frac{\lambda}{1-2\alpha}\left((1-\alpha)e^{-\frac{\lambda x}{1-\alpha}}-\alpha e^{-\frac{\lambda x}{\alpha}}\right) & \text{if } 0<\alpha<\frac{1}{2} \\ & \quad\text{or } \frac{1}{2}<\alpha<1,\\ \frac{1}{2}\left(2\lambda e^{-2\lambda x}+4\lambda^2 xe^{-2\lambda x}\right) & \text{if } \alpha=\frac{1}{2},\end{cases}$$

for all $x\geq 0$ and $\lambda>0$.

4. Conclusion

We propose a new distribution which is more complex distribution than exponential distribution because $\text{PED}(\alpha,\lambda)$ is closed to exponential distribution as mentioned in Theorem 7 when parameter α approaches 0 or 1. Thus PED distribution is a generalized exponential distribution. Moreover, we found that the limit of $\text{PED}(\alpha,\lambda)$ has the probability density function in the term of average of exponential and gamma distributions as shown in Theorem 8. Finally, we expect that the PED distribution shall be attracted to modeling and some analysis in actuarial science, finance and related fields for statistical inferences.

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