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Some Pell and Pell-Lucas identities by matrix methods and their applications

Somnuk Srisawat[1*] and Wanna Sriprad[1]

[1]Devision of Mathematics, Faculty of Science and Technology,
Rajamangala University of Technology Thanyaburi, Pathum Thani 12110, Thailand
*E-mail: somnuk_s@rmutt.ac.th

Abstract

In this paper, we establish some identities involving Pell and Pell-Lucas numbers by using matrix methods. Moreover, we present the solution of some Diophantine equations by applying these identities.

Keywords: Pell numbers, Pell-Lucas numbers, Diophantine equations.

1. Introduction

The Pell sequence $\{P_n\}$ is defined by $P_0 = 0, P_1 = 1,$ and $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. The first few terms of $\{P_n\}$ are $0, 1, 2, 5, 12$ and so on. The terms of this sequence are called Pell numbers and we denoted the n^{th} Pell numbers by P_n. The Pell numbers for negative subscripts are defined as $P_{-n} = (-1)^{n+1} P_n$, for $n \geq 1$. Then it is known that $P_n = 2P_{n-1} + P_{n-2}$, for $n \in \mathbb{Z}$. Also, the Pell-Lucas sequence $\{Q_n\}$ is defined by $Q_0 = 2, Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$. The first few terms of $\{Q_n\}$ are $2, 2, 6, 14, 34$ and so on. The terms of this sequence are called Pell-Lucas numbers and we denoted the n^{th} Pell-Lucas numbers by Q_n. The Pell-Lucas numbers for negative subscripts are defined as $Q_{-n} = (-1)^n Q_n$, for $n \geq 1$. It can be seen that $Q_n = 2P_n + 2P_{n-1}$ and $Q_n = P_{n+1} + P_{n-1}$ for all $n \in \mathbb{Z}$. The Binet's formula for $\{P_n\}$ and $\{Q_n\}$ are $P_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, and $Q_n = \alpha^n + \beta^n$, for $n \geq 0$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 = 2x + 1$. It is well-known that many identities of Pell and Pell-Lucas

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numbers are proved by using Binet's formula, induction or metrics (see [1-3]).

In this paper, we establish some identities for Pell and Pell-Lucas numbers by using matrix methods. Moreover, we present the solutions of some Diophantine equations by applying these identities.

2. Main results

In this section, we establish some identities for Pell and Pell-Lucas numbers by using matrix methods and we begin with the following Lemma.

Lemma 2.1 *If X is a square matrix with $X^2 = 2X + I,$ then $X^n = P_n X + P_{n-1} I,$ for all $n \in \mathbb{Z}$.*

Proof. If $n = 0$, then the proof is obvious. It can be shown by induction that $X^n = P_n X + P_{n-1} I$, for all $n \in \mathbb{N}$. Now, we will show that

$$X^{-n} = P_{-n} X + P_{-n-1} I \text{, for all } n \in \mathbb{N}.$$

Let $Y = 2I - X = -X^{-1}$. Then we have

$$\begin{aligned} Y^2 &= 4I - 4X + X^2 \\ &= 2(2I - X) + I \\ &= 2Y + I. \end{aligned}$$

It implies that $Y^n = P_n Y + P_{n-1} I$.

That is $(-X^{-1})^n = P_n(2I - X) + P_{n-1} I.$

Thus,
$$\begin{aligned} (-1)^n X^{-n} &= 2P_n I - P_n X + P_{n-1} I \\ &= -P_n X + (2P_n + P_{n-1}) I \\ &= -P_n X + P_{n+1} I. \end{aligned}$$

Therefore,
$$\begin{aligned} X^{-n} &= (-1)^{n+1} P_n X + (-1)^n P_{n+1} I \\ &= P_{-n} X + P_{-(n+1)} I \\ &= P_{-n} X + P_{-n-1} I. \end{aligned}$$

This complete the proof. □

Form Lemma 2.1, we can give Corollary 2.2 easily. Also one can consult [1] for more information about the matrices M.

Corollary 2.2 *Let* $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. *Then*

$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, for *all* $n \in \mathbb{Z}$.

Next, let us define the matrix W as in the following Lemma and by using this matrix, we obtain some identities for Pell and Pell-Lucas numbers.

Lemma 2.3 *Let* $W = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. *Then*

$W^n = \begin{bmatrix} \frac{1}{2}Q_n & P_n \\ 2P_n & \frac{1}{2}Q_n \end{bmatrix}$, for *all* $n \in \mathbb{Z}$.

Proof. Note that $W^2 = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} = 2W + I$.

By Lemma 2.1, we have $W^n = P_n W + P_{n-1} I$.

It follows that

$$\begin{aligned} W^n &= \begin{bmatrix} P_n + P_{n-1} & P_n \\ 2P_n & P_n + P_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}Q_n & P_n \\ 2P_n & \frac{1}{2}Q_n \end{bmatrix}. \end{aligned}$$

Therefore, we get the result. □

Now, by using the matrix W, we get Lemma 2.4 and Lemma 2.5, respectively.

Lemma 2.4 *Let n be any integer. Then the following equality holds:*

$$Q_n^2 - 8P_n^2 = 4(-1)^n.$$

Proof. Since $\det(W) = -1$ and $\det(W^n) = \frac{1}{4}Q_n^2 - 2P_n^2$, it follows that $Q_n^2 - 8P_n^2 = 4(-1)^n$. □

Lemma 2.5 *For any integers m and n, the following equality holds:*

$$2Q_{m+n} = Q_m Q_n + 8P_m P_n.$$

Proof. Using $W^{m+n} = W^m W^n$, we get the result. □

Lemma 2.6 *Let n be any integer. Then following equalities hold:*

(i) $\alpha^n = \alpha P_n + P_{n-1}$,

(ii) $\beta^n = \beta P_n + P_{n-1}$.

Proof. Take $X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, then $X^2 = 2X + I$.

By Lemma 2.1, we have $X^n = P_n X + P_{n-1} I$.

It follows that

$$\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha P_n + P_{n-1} & 0 \\ 0 & P_n + P_{n-1} \end{bmatrix}.$$

This implies that

$\alpha^n = \alpha P_n + P_{n-1}$ and $\beta^n = \beta P_n + P_{n-1}$. □

By Lemma 2.1 and Lemma 2.6, we get the following Theorem.

Theorem 2.7 *Let $A = \begin{bmatrix} \alpha & 0 \\ 1 & \beta \end{bmatrix}$, then*

$A^n = \begin{bmatrix} \alpha^n & 0 \\ P_n & \beta^n \end{bmatrix}$, *for all $n \in \mathbb{Z}$.*

Proof. Since

$$A^2 = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+\beta & \beta^2 \end{bmatrix} = \begin{bmatrix} 2\alpha+1 & 0 \\ 2 & 2\beta+1 \end{bmatrix} = 2A + I,$$

it follows from Lemma 2.1 and Lemma 2.6 that

$$A^n = P_n A + P_{n-1} I = \begin{bmatrix} \alpha P_n + P_{n-1} & 0 \\ P_n & \beta P_n + P_{n-1} \end{bmatrix} = \begin{bmatrix} \alpha^n & 0 \\ P_n & \beta^n \end{bmatrix}.$$

Therefore, the result is proved. □

By using Theorem 2.7, we get the following theorem.

Theorem 2.8 *For any integers m and n, the following equality holds:*

$$(-1)^{m+n} Q_{m+n}^2 - 8(-1)^m P_m^2 - 8(-1)^n P_n^2 = 8(-1)^{m+n} P_m P_n Q_{m+n} + 4.$$

Proof. Let a matrix A as in Theorem 2.7.

Then we have

$$A^{n+1} + A^{n-1} = \begin{bmatrix} 2\sqrt{2}\,\alpha^n & 0 \\ Q_n & -2\sqrt{2}\,\beta^n \end{bmatrix}$$

Since

$$(A^{m+1} + A^{m-1})(A^{n+1} + A^{n-1}) = A^{m+n+2} + 2A^{m+n} + A^{m+n-2},$$

we get that $2\sqrt{2}\,P_{m+n} = \alpha^n Q_m - \beta^m Q_n$. Thus,

$$\begin{aligned} 8P_{m+n}^2 &= (2\sqrt{2}\,P_{m+n})(2\sqrt{2}\,P_{n+m}) \\ &= (\alpha^n Q_m - \beta^m Q_n)(\alpha^m Q_n - \beta^n Q_m) \\ &= Q_m Q_n Q_{m+n} - (-1)^n Q_m^2 - (-1)^m Q_n^2. \end{aligned} \quad (2.1)$$

From Lemma 2.4 and (2.1), we obtain

$$(-1)^{m+n} Q_{m+n}^2 + (-1)^m Q_m^2 + (-1)^n Q_n^2 = (-1)^{m+n} Q_m Q_n Q_{m+n} + 4 \quad (2.2)$$

By Lemma 2.4, Lemma 2.5 and (2.2), we get that

$$(-1)^{m+n} Q_{m+n}^2 - 8(-1)^m P_m^2 - 8(-1)^n P_n^2 = 8(-1)^{m+n} P_m P_n Q_{m+n} + 4. \quad (2.3)$$

This complete the proof. □

Theorem 2.9 *For any integers m and n, the following equality holds:*

$$8(-1)^{m+n} P_{m+n}^2 - (-1)^n Q_n^2 + 8(-1)^m P_m^2 = 8(-1)^{m+n} Q_n P_m P_{m+n} - 4.$$

Proof. By similar argument as in Theorem 2.8 and using the properties

$$(A^{n+1} + A^{n-1})A^m = A^m (A^{n+1} + A^{n-1}) = A^{m+n+1} + A^{m+n-1},$$

we get that $Q_{m+n} = \alpha^m Q_n - 2\sqrt{2}\beta^n P_m$

and $Q_{m+n} = 2\sqrt{2}\alpha^n P_m + \beta^m Q_n$.

Therefore, we have

$$Q_{m+n}^2 = (\alpha^m Q_n - 2\sqrt{2}\beta^n P_m)(2\sqrt{2}\alpha^n P_m + \beta^m Q_n)$$

$$= 8Q_n P_m P_{m+n} + (-1)^m Q_n^2 - 8(-1)^n P_m^2. \qquad (2.4)$$

By Lemma 2.4 and (2.4), we obtain

$$8(-1)^{m+n} P_{m+n}^2 - (-1)^n Q_n^2 + 8(-1)^m P_m^2$$

$$= 8(-1)^{m+n} Q_n P_m P_{m+n} - 4. \qquad (2.5)$$

Therefore, the proof is completed. □

3. Applications

In this section we give the solutions of some Diophantine equations by applying Theorem 2.8 and Theorem 2.9.

Theorem 3.1 *If m and n are even integers, then $(x, y, z) = (P_m, P_n, Q_{m+n})$ is a solution of the equation $z^2 - 8x^2 - 8y^2 = 8xyz + 4$. If m and n are odd integers, then $(x, y, z) = (P_m, P_n, Q_{m+n})$ is a solution of the equation $z^2 + 8x^2 + 8y^2 = 8xyz + 4$, and if m is an odd integer and n is an even integer, then $(x, y, z) = (P_m, P_n, Q_{m+n})$ is a solution of the equation $z^2 - 8x^2 + 8y^2 = 8xyz - 4$.*

Proof. The result follows immediately from Theorem 2.8. □

Theorem 3.2 *If m and n are even integers, then $(x, y, z) = (Q_m, Q_n, Q_{m+n})$ is a solution of the equation $z^2 + x^2 + y^2 = xyz + 4$. If m and n are odd integers, then $(x, y, z) = (Q_m, Q_n, Q_{m+n})$ is a solution of the equation $z^2 - x^2 - y^2 = xyz + 4$, and if m is an odd integer and n is an even integer, then $(x, y, z) = (Q_m, Q_n, Q_{m+n})$ is a solution of the equation $z^2 + x^2 - y^2 = xyz - 4$.*

Proof. The result follows directly from (2.1). □

Theorem 3.3 *If m and n are even integers, then $(x, y, z) = (Q_n, P_m, P_{m+n})$ is a solution of the equation $8z^2 - x^2 + 8y^2 = 8xyz - 4$. If m and n are odd integers, then $(x, y, z) = (Q_n, P_m, P_{m+n})$ is a solution of the equation $8z^2 + x^2 - 8y^2 = 8xyz - 4$, and if m is an odd integer and n is an even integer, then $(x, y, z) = (Q_n, P_m, P_{m+n})$ is a solution of the equation $8z^2 + x^2 + 8y^2 = 8xyz + 4$*

Proof. The result follows immediately from Theorem 2.9. □

4. Conclusion

In this paper, some identities for Pell and Pell-Lucas numbers are established by using matrix methods and the solutions of some Diophantine equations are presented by applying these identities.

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